L.8 Cross Products - continuation

Important Properties of Cross Product:

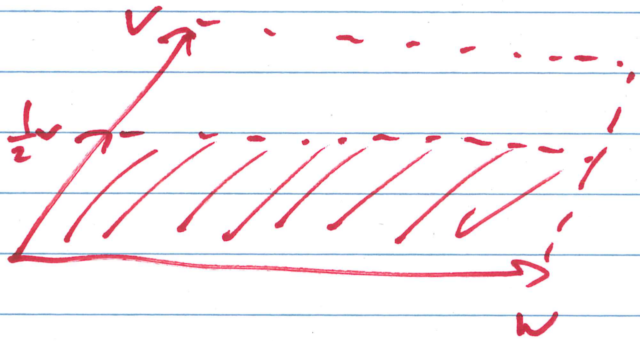
1) $\vec{v} \times \vec{w}$ is a vector

2) $(\vec{v} \times \vec{w}) \cdot \vec{v} = (\vec{v} \times \vec{w}) \cdot \vec{w} = 0$ &

$\vec{v} \times \vec{v} = 0$

3) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ (follows from Right Hand Rule)

4) $(\lambda \vec{v}) \times \vec{w} = \lambda \cdot (\vec{v} \times \vec{w}) = \vec{v} \times (\lambda \vec{w})$



5) Distributive Law:

$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

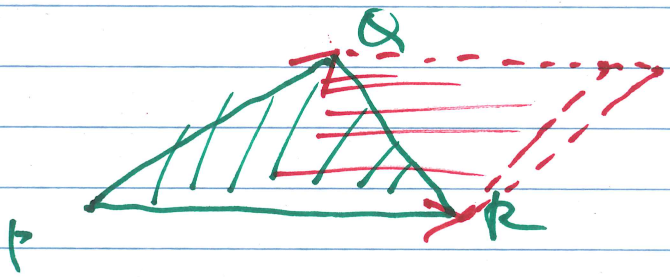
Hard to verify by use of the geometric def. of cross ~~product~~ product.

Ex: Consider the points in the space:

$P(1,1,1)$, $Q(-1,1,0)$, $R(0,1,2)$

Find the area of the triangle $\triangle PQR$.

Sol:



We use that $\text{Area}(\triangle PQR) =$

$$\frac{1}{2} \cdot \text{Area}\left(\begin{array}{c} \text{▱ formed} \\ \text{by } \overline{PQ} \ \& \ \overline{PR} \end{array}\right) =$$

$$\frac{1}{2} \cdot |\overline{PQ} \times \overline{PR}|$$

$$\overline{PQ} = (-1-1,\ 1-1,\ 0-1) = (-2, 0, -1)$$

$$\overline{PR} = (0-1,\ 1-1,\ 2-1) = (-1, 0, 1)$$

$$\begin{array}{ccccc} i & j & k & i & j \\ -2 & 0 & -1 & -2 & 0 \\ -1 & 0 & 1 & -1 & 0 \end{array}$$

$$\overline{PQ} \times \overline{PR} = (1 \cdot 0 - (-1) \cdot 0)\,\hat{i} + (1 - (-2))\,\hat{j} + (0 - 0)\,\hat{k}$$

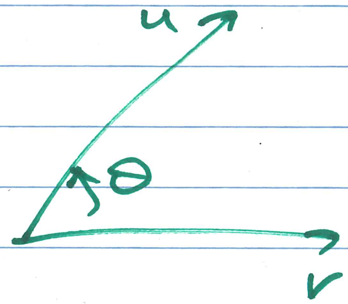
$$= 3\hat{j} \implies |\overline{PQ} \times \overline{PR}| = 3 \implies A(\triangle PQR) = 3/2$$

Connection of $\times$-product to $\cdot$-product

$$\boxed{|\vec{u}\times\vec{v}|^2 + |\vec{u}\cdot\vec{v}|^2 = |\vec{u}|^2\cdot|\vec{v}|^2}$$

Pf:

Recall,

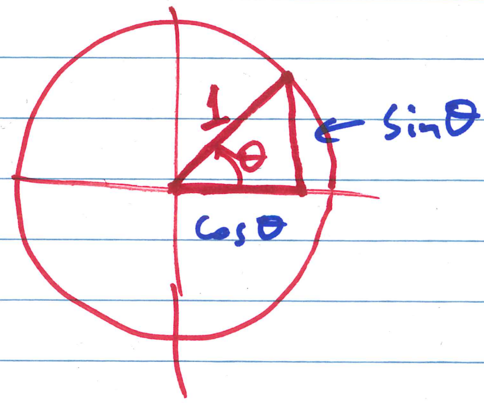


$$|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin(\theta)$$

$$|\vec{u}\cdot\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\cos(\theta)$$

Remember that for any θ:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$



$$\Rightarrow \quad |\bar{u}\times\bar{v}|^2 + |\bar{u}\cdot\bar{v}|^2 = |\bar{u}|^2|\bar{v}|^2\sin^2(\theta) +$$

$$|\bar{u}|^2\cdot|\bar{v}|^2\cdot\cos^2(\theta) = |\bar{u}|^2\cdot|\bar{v}|^2$$

Exercise: Cacalate $|\bar{u}\times\bar{v}|$ given that $|\bar{u}|=3$, $|\bar{v}|=4$, $\bar{u}\cdot\bar{v}=10$.

Sol: We know:

$$|\bar{u}\times\bar{v}|^2 + |\bar{u}\cdot\bar{v}|^2 = |\bar{u}|^2\cdot|\bar{v}|^2$$

$$|\bar{u}\times\bar{v}|^2 + 10^2 = 3^2\cdot 4^2 = 144$$

$$\Rightarrow \quad |\bar{u}\times\bar{v}|^2 = 144-100 = 44$$

$$\Rightarrow \quad |\bar{u}\times\bar{v}| = \sqrt{44} = 2\sqrt{11}$$

Another example of easy exercise where x-product is useful:

Exercise: Find a vector perpendicular to both

$$\bar{u} = \bar{i} - \bar{j} + \bar{k} = (1,-1,1)$$

$$\bar{v} = -3\bar{i} + \bar{j} - 7\bar{k} = (-3,1,-7)$$

Sol: Of course, $\bar{u}\times\bar{v} \perp \bar{u}$ & $\bar{v}$.

perpendicular

and we need to find $\bar{u}\times\bar{v}$ . . . .

Less trivial exercise:

Prove $\quad \bar{u}\cdot(\bar{v}\times\bar{w}) = (\bar{u}\times\bar{v})\cdot\bar{w}$

Pf:

~~Approach 1:~~ $\quad \bar{u} = (x_1, x_2, x_3)$

$$\bar{v} = (y_1, y_2, y_3)$$

$$\bar{w} = (z_1, z_2, z_3)$$

And perform the computation on LHS & RHS, and compare results.

~~Approach 2: By use the identity connecting dot-product w/ cross-product:~~

~~(for vectors $\vec{u}$ & $\bar{v}\times\bar{w}$)~~

$$\cancel{|\vec{u}\times(\vec{v}\times w)|^2 + |\vec{u}\cdot(\bar{v}\times\bar{w})|^2 = |\bar{u}|^2\cdot|v\times w|^2}$$

~~use it for vectors $\bar{u}\times\bar{v}$ & $\bar{w}$:~~

~~$\bar{u}\times$ ... less promising...~~

~~Come back to Approach 1.~~

Compute LHS:

$$\bar{v} \times \bar{w} = \begin{matrix} i & j & k & i & j \\ y_1 & y_2 & y_3 & y_1 & y_2 \\ z_1 & z_2 & z_3 & z_1 & z_2 \end{matrix}$$

$$= (y_2 z_3 - y_3 z_2)\bar{i} + (y_3 z_1 - y_1 z_3)\bar{j} + (y_1 z_2 - y_2 z_1)\bar{k}$$

$$\bar{u} \cdot (\bar{v} \times \bar{w}) = (x_1, x_2, x_3) \cdot)$$

$$= x_1 (y_2 z_3 - y_3 z_2) + x_2 (y_3 z_1 - y_1 z_3) +$$

$$x_3 \cdot (y_1 z_2 - y_2 z_1)$$