APPROXIMATE INVARIANCE FOR ERGODIC ACTIONS OF AMENABLE GROUPS

MICHAEL BJÖRKLUND AND ALEXANDER FISH

ABSTRACT. We develop in this paper some general techniques to analyze action sets of small doubling for probability measure-preserving actions of amenable groups.

As an application of these techniques, we prove a dynamical generalization of Kneser's celebrated density theorem for subsets in $(\mathbb{Z}, +)$, valid for any countable amenable group, and we show how it can be used to establish a plethora of new inverse product set theorems for upper and lower asymptotic densities. We provide several examples demonstrating that our results are optimal for the settings under study.

CONTENTS

1. INTRODUCTION

1.1. **Motivation**

Aperiodic subsets of locally compact groups which are "almost closed" under multiplication are classical objects of study in harmonic analysis, number theory and geometry, and continually appear in new applications. Despite the fact that these objects are often "far" from being actual subgroups, they nevertheless seem to obey some form of "approximate" group theory, whose foundation is still in its early infancy. With this paper we wish to take the first steps towards what could be called "approximate" dynamics or "approximate" ergodic theory, a line of research concerned with the interplay between "expansion" and "approximate" invariance of so called "action sets", dynamical analogues of product sets in groups.

Definition 1.1. Let G be a group and let Y be a set upon which G acts. Given $A \subset G$ and $B \subset Y$, their *action set* AB is defined by

$$AB = \bigcup_{a \in A} aB = \big\{ ab \, : \, a \in A, \;\; b \in B \big\} \subset Y.$$

Let $k \geqslant 1$ be an integer. We say that (A, B) is k-*doubling* if there exists a finite subset $F \subset G$ of cardinality at most k such that $AB \subset FB$; in this case, we also say that B is k-*approximately invariant* under A. To avoid trivialities, one usually insists that k is "small" compared to the "sizes" of the sets A and B.

In the case when $Y = G$, endowed with the natural G-action on itself from the left, and $A, B \subset G$, then the action set AB is called the *product set* of A and B, and $A \subset G$ is called k-*doubling* if (A, A) is a k-doubling pair. We say that $A \subset G$ is a k-*approximate subgroup* if it is k-doubling, symmetric and contains the identity element e_G.

Let us begin by making a trivial observation: If $G \curvearrowright Y$, then k-doubling sets in G naturally give rise to k-approximately invariant subsets in Y as follows. Suppose that $A \subset G$ is k-doubling and $B_o \subset Y$ is any subset; then $B = AB_o$ is k-approximately invariant under A - indeed, since $A^2 \subset FA$ for some subset $F \subset G$ with $|F| \leqslant k$, we have

$$AB = A^2 B_o \subset FAB_o = FB.$$

A fundamental line of research is to investigate to which extent the converse holds, that is to say, if a set $B \subset Y$ is k-approximately invariant under a subset $A \subset G$, must then A be "nicely" contained in a k'-approximate subgroup of G where k' is not "much larger" than k. If $k = 1$, i.e. if B is invariant (on the nose) under A, then B is clearly also invariant under the subgroup of G which is generated by A. The case $k = 2$ is already much harder to deal with, and will be the focus of our investigations here.

Before we outline the theme of this paper, and state our main results, we say a few words about history. The term "k-approximate subgroup" was coined by Tao in [27], but implicit uses of the notion can be traced back much further. For instance, the study of 2-approximate subgroups was initiated in the work by Mann [24] on Schnirelmann densities of sumsets in $(\mathbb{Z}, +)$, which Khintchine [11] later referred to as one of the "Three Pearls of Number Theory", and it was continued in the subsequent work by Kneser [21], as well as in many important works by Kneser [22], Kempermann [20] and others on product sets in compact groups.

However, the impetus to Tao's work was the early works of Freiman [13] on general *finite* k-approximate subgroups in $(\mathbb{Z}, +)$, later extended by Ruzsa; as well as in the more recent works by Bourgain-Gamburd [9] and Helfgott [17] on finite k-approximate subgroups in finite simple

groups. A very general theorem in this direction was recently established by Breuillard, Green and Tao [10].

From a very different point of view, Yves Meyer [25] began in the sixties his very influential study of "large" and discrete k-approximate subgroups in Euclidean spaces, which are today mostly known under the names *quasicrystals*, *Meyer sets* or *approximate lattices*. Extensions to non-abelian groups were recently developed by the first author and Tobias Hartnick [7, 6, 8].

As a warm-up for the things to come, we provide a classification of 2-doubling pairs (A, B), where A is a "large" and "aperiodic" (or "spread-out") subset of a countable (infinite) abelian group G, for instance $(\mathbb{Z}, +)$, and B is a Borel set with positive measure in some ergodic Borel G-space (Y, ν). In other words, we shall assume that there are $s, t \in G$ such that

$$AB \subset sB \cup tB.$$

To avoid trivialities, we wish to exclude the case when $sB \cup tB = Y$, which can be done by assuming that $\nu(B) < 1/2$. If one further assumes that A is "bigger" than B, then we wish to show that A must "essentially" coincide with a 2-approximate subgroup of a very special form. To make the terms in the quotation marks more precise, we shall need some definitions.

Definition 1.2 (Large/Spread-out)**.** Let G be a countable amenable group and let d^* denote the upper Banach density on G (see (A.3) for the definition). We say that $A \subset G$ is

- *large* if $d^*(A) > 0$.
- *spread-out* if A is large and there is no subset $A_o \subset A$ with $d^*(A_o) = d^*(A)$ and no finite-index subgroup $G_o < G$ such that $G_o A_o \neq G$.

In other words, A is spread-out if one cannot pass to a subset with the same upper Banach density which is contained in a proper periodic subset of G. In particular, A, as well as any of its subsets with the same upper Banach density, projects onto every finite quotient of G. We note that if G lacks proper finite-index subgroups, for instance if $G = (\mathbb{Q}, +)$, then every large set is automatically spread-out.

Remark 1.3. Our definition of a spread-out set might be somewhat hard to digest, and perhaps it seems to be a bit too strong of an assumption; we have chosen the formulation above to make certain parts of our arguments run smoother, but it will be clear from our proofs that one needs much less.

For instance, one objection to the phrasing above could be that the condition $d^*(A_o) = d^*(A)$ is not very informative; with some additional work, one could prove that all of our main results still hold if one in addition insists that this identity is realized along the *same* Følner sequence (see Subsection A.1 for definitions). However, to keep the exposition clean, we shall refrain from such technical indulgences.

The exact formulation of our classification now reads as follows. Definitions of the ergodic-theoretical notions used here can be found in Subsection 2.1.

Theorem 1.4 (Warm-up)**.** *Let G be a countable abelian group and let $G \curvearrowright (Y, \nu)$ be a totally ergodic Borel G-space. Suppose that $A \subset G$ is spread-out, and $B \subset Y$ is a Borel set with positive ν-measure such that*

$$\nu(B) \leqslant d^*(A) < 1/2 \quad and \quad AB \subset sB \cup tB, \quad modulo \ \nu\text{-null sets}, \tag{1.1}$$

for some $s, t \in G$. Then $d^*(A) = \nu(B)$, and there exist

(i) a torus compactification $(\mathbb{T}, \tau)$ of G and a closed interval $I_o \subset \mathbb{T}$ with $m_{\mathbb{T}}(I_o) = d^*(A)$,

(ii) a G-factor map $\sigma : (Y, \nu) \to (\mathbb{T}, m_{\mathbb{T}})$ and a closed interval $J_o \subset \mathbb{T}$ with $m_{\mathbb{T}}(J_o) = \nu(B)$,

such that $A \subset \tau^{-1}(I_o)$ *and* $B = \sigma^{-1}(J_o)$ *modulo* ν-*null sets.*

The theorem shows that the structure of 2-doubling pairs for ergodic actions is very rigid; the set B must stem from an interval in one-dimensional torus, and the set A is contained in a set S of the form $\tau^{-1}(I_o)$, where I_o is a closed interval in the same one-dimensional torus. Furthermore, S has the same upper Banach density as A (this follows for instance from Corollary A.3 in the appendix). It is straightforward to check that S is a 3-approximate subgroup, and in fact a 2-approximate subgroup if the endpoints of the interval I_o belong to $\tau(G)$. We stress that the converse also holds; if A and B are as in the conclusion of Theorem 1.4, then (A, B) is 2-doubling (modulo null sets).

1.2. **Main dynamical results**

Let us now connect Theorem 1.4 to the main theme of this paper. If A and B are as in this theorem, then it is clear that

$$\nu(AB) \leqslant 2\nu(B) \leqslant d^*(A) + \nu(B) < 1.$$

Theorem 1.5 below tells us that if A is a spread-out subset, then the reverse inequality $\nu(AB) \geqslant \min(1, d^*(A) + \nu(B))$ holds for *all* Borel sets $B \subset Y$, so in the setting at hand, we must have

$$\nu(B) = d^*(A) \quad \text{and} \quad \nu(AB) = d^*(A) + \nu(B) < 1.$$

Theorem 1.4 now follows from the latter part of Theorem 1.8 below.

The general framework for the theorems below reads as follows:
- G - a countable amenable group.
- d^* - the upper Banach density on G.
- (Y, ν) - a standard Borel probability measure space, equipped with an action of G by measure-preserving bijections.

Our first main result (which is proved in Subsection 6.2) asserts that if an action set AB in Y is "small" with respect to $d^*(A)$ and $\nu(B)$, then A must be "close" to a periodic subset.

Theorem 1.5. *Let* $A \subset G$ *be a large set and* $B \subset Y$ *a Borel set with positive measure. Suppose that either,*

    (i) A *is spread-out,* **or**
    (ii) *all finite quotients of* G *are* ABELIAN, *and there is no finite-index subgroup* $G_o < G$ *such that*

$$G_o A \neq G \quad \text{and} \quad d^*(G_o A) < d^*(A) + \frac{1}{[G : G_o]}, \tag{1.2}$$

then $\nu(AB) \geqslant \min\big(1, d^*(A) + \nu(B)\big)$.

Remark 1.6. Concerning Condition (ii): this is of course always satisfied if G is abelian. For a non-abelian example, consider the lampligher group $G = \mathbb{Q} \wr \mathbb{Z}$ (with $\mathbb{Q}$-valued "lamps"). This is the solvable (hence amenable) wreath product of the two abelian groups $(\mathbb{Q}, +)$ and $(\mathbb{Z}, +)$, and it is not hard to show that every homomorphism of G onto a finite group factors through $\mathbb{Z}$, and thus every finite factor group of G is abelian.

Let us now try to understand when the lower bound on $\nu(AB)$ in the previous theorem is attained. We saw in Theorem 1.4 that if G is abelian, then a special role was played by sets of the form $\tau^{-1}(I_o)$, where $(\mathbb{T}, \tau)$ is a torus compactification of G and I_o a closed interval in $\mathbb{T}$. These are examples of so called *Sturmian sets*, which we now define for general groups.

Definition 1.7 (Sturmian set). Let G be a countable group and let M denote either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$, where in the latter case, $\{-1, 1\}$ acts by multiplication on $\mathbb{T}$. We say that a subset $S \subset G$ is *Sturmian* if there exist

- a homomorphism $\tau : G \to M$ with dense image,
- a closed symmetric interval $I_o \subset \mathbb{T}$ and $t_o \in M$,

such that either $S = \tau^{-1}(I_o t_o)$ if $M = \mathbb{T}$ (in which case the assumption that I_o is symmetric can be dropped) or $S = \tau^{-1}((I_o \rtimes \{-1, 1\})t_o)$ if $M = \mathbb{T} \rtimes \{-1, 1\}$. In the latter case, we say that S is *twisted*.

Clearly, abelian groups do not admit twisted Sturmian sets. On the other hand, one can readily check that the infinite dihedral group *only* admits twisted Sturmian sets, and no "un-twisted" ones. Sturmian sets in $\mathbb{Z}^d$ have been extensively studied in complexity theory and tiling theory, see for instance the survey [29], while we seem to be the first to address their twisted analogues.

If (M, τ) is as in Definition 1.7, then we get an ergodic Borel G-space by letting G act on M by multiplication on the left via τ. This action clearly preserves the Haar measure m_M on M and is ergodic (because the image of τ is dense, see for instance Lemma 2.1 below). Let I_o and J_o be symmetric closed intervals of $\mathbb{T}$ with $m_{\mathbb{T}}(I_o + J_o) < 1$, and set

$$A = \tau^{-1}(I_o) \subset G \quad \text{and} \quad B = J_o \subset M,$$

if $M = \mathbb{T}$ (or the twisted versions if $M = \mathbb{T} \rtimes \{-1, 1\}$). Then it is not hard to show that

- A is spread out and $d^*(A) = m_{\mathbb{T}}(I_o)$,
- AB does not contain, modulo m_M-null sets, a Borel set with positive measure which is invariant under a finite-index subgroup of G, and
- $m_M(AB) = d^*(A) + m_M(B) < 1$.

Our second main theorem (which is proved in Subsection 6.2) asserts that upon passing to factors, the setting described above is the only source of examples of ergodic Borel G-spaces for which the lower bound in Theorem 1.5 is attained (this is spelled out precisely for abelian groups below, but the version for general amenable groups will be clear from the proofs).

Theorem 1.8. *Let* $A \subset G$ *be a large set and* $B \subset Y$ *a Borel set with positive measure. Suppose that*

(i) A *is spread-out.*
(ii) AB *does not contain, modulo ν-null sets, a Borel set with positive measure which is invariant under a finite-index subgroup of* G,

and

$$\nu(AB) = d^*(A) + \nu(B) < 1.$$

Then A *is contained in a Sturmian set with the same upper Banach density as* A.

If G *is abelian and* $G \curvearrowright (Y, \nu)$ *is totally ergodic, then Condition (ii) is automatic, and there exist*

(i) *a torus compactification* $(\mathbb{T}, \tau)$ *and closed intervals* $I_o, J_o \subset \mathbb{T}$ *with*

$$d^*(A) = m_{\mathbb{T}}(I_o) \quad \text{and} \quad A \subset \tau^{-1}(I_o) \quad \text{and} \quad \nu(B) = m_{\mathbb{T}}(J_o).$$

(ii) *a G-factor map* $\sigma : (Y, \nu) \to (\mathbb{T}, m_{\mathbb{T}})$ *such that* $B = \sigma^{-1}(J_o)$ *modulo ν-null sets.*

Remark 1.9. The first part of Theorem 1.8 has a curious consequence: If there are sets A and B as above, then G must admit a homomorphism into either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$. In particular, if G is perfect, that is to say, if $G = [G, G]$, then we cannot find a spread-out set $A \subset G$ and $B \subset Y$ as above which satisfy the identity $\nu(AB) = d^*(A) + \nu(B) < 1$.

It is natural to ask how "close" to this case we can get if G is perfect. Corollary 6.7 in Subsection 6.4 shows that for some perfect amenable groups (for instance, the Grigorchuk group), the condition that A is spread-out is so strong that it forces $\nu(AB) = 1$ for every Borel set $B \subset Y$ with positive ν-measure. Similarly, if G is an amenable weakly mixing (minimally almost periodic) group, and thus perfect, then $\nu(AB) = 1$ whenever A is large and $B \subset Y$ has positive ν-measure.

1.3. **Applications to product sets**

In this section we translate our dynamical results into density combinatorial results. While this is rather straightforward to do for *Banach densities*, we need to develop new tools in order to address *asymptotic densities*. This is due to the fact that when dealing with asymptotic densities from a "dynamical" point of view, one needs to use action set theorems for *non-ergodic* Borel G-spaces as well, and our dynamical results above only apply to *ergodic* ones.

Starting from Subsection 6.3.2, we describe how one can transfer inverse theorems for action set with respect to *ergodic* Borel G-spaces to inverse theorems for action sets with respect to *non-ergodic* Borel G-spaces. On the way, we shall need some terminology which we now introduce.

Definition 1.10 (Thick/piecewise periodic/syndetic)**.** We say that a subset $P \subset G$ is
- *thick* if for every finite set $F \subset G$, there exists $g \in G$ such that $Fg \subset P$.
- *periodic* if there exists a finite-index subgroup $G_o < G$ such that $G_o P = P$.
- *piecewise periodic* if there exist a periodic set Q and a thick set T such that $P = Q \cap T$.
- *syndetic* if it intersects non-trivially every thick set in G, or equivalently, if there exists a finite set F in G such that $FP = G$.

1.3.1. *Inverse theorems for Banach densities*

Let us begin our discussions here with a classification of spread-out 2-approximate subgroups of countable amenable groups.

Theorem 1.11 (Warm-up)**.** *Suppose that* $A \subset G$ *is a spread-out* 2-*approximate subgroup such that* A^2 *does not contain a piecewise periodic set. Then* A *is contained in a Sturmian subset of* G *with the same upper Banach density as* A.

If G *does not have any proper finite-index subgroups, then it suffices to assume that* A *is large and* A^2 *is not thick.*

Remark 1.12. We observe, just as we did in Remark 1.9, that this theorem in particular implies that amenable perfect groups do not admit *spread-out* 2-approximate subgroups (whose squares do not contain piecewise periodic sets). In particular, perfect amenable groups without proper finite index subgroups do not admit spread-out 2-approximate subgroups with upper Banach densities strictly between 0 and 1/2.

Theorem 1.11 is a rather direct consequence of Theorem 1.13 and Theorem 1.15 below. Indeed, if A is a 2-approximate subgroup with $d^*(A^2) < 1$, then

$$d^*(A^2) \leqslant 2d^*(A).$$

If A is spread-out, then Theorem 1.13 will show that the reverse inequality also holds, and thus $d^*(A^2) = 2d^*(A)$. If A^2 does not not contain a piecewise periodic set, then Theorem 1.15 now implies that A is contained in a Sturmian set with the same upper Banach density, finishing the proof of Theorem 1.11.

The arguments needed to prove the following two theorems from Theorem 1.5 and Theorem 1.8 are nowadays standard, and are only sketched in the beginning of Subsection 6.3.

Theorem 1.13. *If* $A \subset G$ *is spread-out, then*

$$d^*(AB) \geqslant \min(1, d^*(A) + d^*(B)), \quad \textit{for all large } B \subset G,$$

and

$$d_*(AB) \geqslant \min(1, d^*(A) + d_*(B)), \quad \textit{for all syndetic } B \subset G.$$

If one assumes that every finite quotient of G *is abelian, then, instead of assuming that* A *is spread-out, it suffices to assume that there is no finite-index subgroup* $G_o < G$ *such that (1.2) holds for* A.

Remark 1.14. A version of this theorem for $G = (\mathbb{N}, +)$ was proved by Jin [19] using very different methods. Griesmer [16] proved a version of Theorem 1.13 (as well as of Theorem 1.15 below) for countable abelian groups; his proof was very much inspired by some earlier versions of our Correspondence Principles for product sets (see Proposition 5.1 and Proposition 5.4 below).

Theorem 1.15. *Let* $A \subset G$ *be spread-out and* $B \subset G$ *large and suppose that* AB *does not contain a piecewise periodic set. If* $d^*(AB) = d^*(A) + d^*(B) < 1$, *then* A *is contained in a Sturmian set with the same upper Banach density as* A.

1.3.2. *Inverse theorems for asymptotic densities*

Our inspiration for this subsection comes from a classical result of Kneser [21], generalizing an earlier landmark made by Mann [24], referred to by Khintchine [11] as one of the "three pearls of number theory". It is an inverse result for subsets $A, B \subset \mathbb{N}$ with positive lower asymptotic densities along the Følner sequence $([1, n])$ such that

$$\underline{d}_{[1,n]}(A + B) < \min(1, \underline{d}_{[1,n]}(A) + \underline{d}_{[1,n]}(B)),$$

and roughly asserts that A and B must be contained in periodic subsets which are not "much larger" in size than A and B, and moreover, $A + B$ is a co-finite subset of a periodic set in $\mathbb{N}$. In particular, if A is sufficiently "aperiodic", then

$$\underline{d}_{[1,n]}(A + B) \geqslant \min(1, \underline{d}_{[1,n]}(A) + \underline{d}_{[1,n]}(B))$$

for all $B \subset \mathbb{N}$ with positive lower asymptotic density along $([1, n])$.

Our aim here is to generalize the latter (weaker) formulation of Kneser's Theorem to a general countable amenable group G and to a general Følner sequence (F_n) therein. We shall also prove an inverse theorem in the case when the lower bound is attained. We stress that our results are new already in the case $G = (\mathbb{Z}, +)$ and $F_n = [1, n]$, and in Appendix C we provide several examples showing that our results are optimal in the case $G = (\mathbb{Z}, +)$ and $F_n = [-n, n]$.

In what follows, let G be a countable amenable group. Our first result improves (albeit under stronger conditions on A and B), the lower bound in the (weak) formulation of Kneser's Theorem above, by replacing $\underline{d}_{([1,n])}(A)$ with the (possibly) larger quantity $d^*(A)$. The proof can be found in Subsection 6.3.4.

Theorem 1.16. *Let* (F_n) *be a Følner sequence in* G. *Suppose that*

   (i) $A \subset G$ *is spread-out.*
   (ii) $B \subset G$ *is syndetic.*
   (iii) AB *is not thick.*

Then,

$$\overline{d}_{(F_n)}(AB) \geqslant d^*(A) + \overline{d}_{(F_n)}(B) \quad \textit{and} \quad \underline{d}_{(F_n)}(AB) \geqslant d^*(A) + \underline{d}_{(F_n)}(B). \tag{1.3}$$

If every finite quotient of G *is abelian, then instead of (i), we only need to assume that there is no finite-index subgroup* $G_o < G$ *such that (1.2) holds.*

Remark 1.17. In Proposition C.1 and Proposition C.2 we show that Condition (iii) and Condition (ii) respectively cannot be dispensed with, already in the case $G = (\mathbb{Z}, +)$ and $F_n = [-n, n]$.

Our second theorem addresses the equality case in Theorem 1.16.

Theorem 1.18. *Let* (F_n) *be a Følner sequence in* G. *Suppose that*

   (i) $A \subset G$ *is spread-out.*
   (ii) $B \subset G$ *is syndetic.*
   (iii) AB *does not contain a piecewise periodic subset.*
   (iv) *Either*
$$\overline{d}_{(F_n)}(AB) = d^*(A) + \overline{d}_{(F_n)}(B) < 1 \quad or \quad \underline{d}_{(F_n)}(AB) = d^*(A) + \underline{d}_{(F_n)}(B) < 1.$$

Then A *is contained in a Sturmian set with the same upper Banach density as* A.

Remark 1.19. In Proposition C.3 and Proposition C.4 we show that Condition (ii) and Condition (iii) respectively cannot be dispensed with, already in the case $G = (\mathbb{Z}, +)$ and $F_n = [-n, n]$.

1.4. **Counterexamples**

Some readers might interpret the asymmetry in the roles of the sets A and B in Theorem 1.13 and Theorem 1.15 as a sign of incompleteness; after all, in these theorems, we only make assertions about A, and say nothing about B. Of course, if G is abelian, one can simply swap the order of A and B, and then use the theorems above to deduce things about the set B. However, there is no reason why this trick should work if G is non-abelian.

The aim of the next two results (which are proved in Section 7) is to show that if G is sufficiently non-abelian (certain two-step solvable groups will do), then the roles of A and B are truly asymmetric, and the conclusions about the set A in Theorem 1.13 and Theorem 1.15 do *not* hold for B. Both results are derived from a general counterexample machine for semi-direct products which should be of independent interest.

Theorem 1.20. *There is a countable two-step solvable group* G *and* $A \subset G$ *with* $d^*(A) = 1/2$ *such that for every* $0 < \varepsilon < 1/2$, *there is* $B \subset G$ *with* $d^*(B) = \varepsilon$, *with the property that*
$$d^*(AB) = d^*(A) < d^*(A) + d^*(B) < 1,$$
and for every finite-index subgroup $G_o < G$, *we either have*
$$G_o B = G \quad or \quad d^*(G_o B) > d^*(B) + \frac{1}{[G : G_o]}.$$

Theorem 1.21. *There is a countable two-step solvable group* G *and* $A, B \subset G$ *such that*

   (i) A *and* B *are spread-out,*
   (ii) AB *does not contain a piecewise periodic set,*

and
$$d^*(AB) = d^*(A) + d^*(B) < 1,$$
but B *is* NOT *contained in a Sturmian set with the same upper Banach density.*

1.5. **A few words about the proofs of the dynamical results**

To prove Theorem 1.5 and Theorem 1.8 we argue along the following lines. Let G be a countable amenable group and suppose that G acts by homeomorphisms on a compact metrizable space X. Given a subset $A \subset X$ and $x \in X$, we write $A_x = \{g \in G : gx \in A\}$, which provides us with a subset of G. It is not hard to prove that *every* subset of G can be written on this form for *some*

compact G-space X, *clopen* subset $A \subset X$ and base point $x_o \in X$ (see Subsection 2.1.1). Let us fix such a triple (X, x_o, A) and an ergodic Borel G-space (Y, ν). It follows from Furstenberg's Correspondence Principle (see Section A) that there exists an ergodic G-invariant probability measure μ on X such that $d^*(A_{x_o}) = \mu(A)$.

Towards Theorem 1.5, we suppose that $\mu(A) > 0$ and that $B \subset Y$ is a Borel set with positive ν-measure such that

$$\nu(A_{x_o} B) < \min(1, d^*(A_{x_o}) + \nu(B)). \tag{1.4}$$

If we define $C = (A_{x_o} B)^c$, then $A_{x_o}^{-1} C \subset B^c$ and

$$\nu(A_{x_o}^{-1} C) < d^*(A_{x_o}) + \nu(C).$$

In Section 5 we show that there exists an ergodic joining η of (X, μ) and (Y, ν) such that if we write

$$A' = A \times Y \quad \text{and} \quad C' = X \times C$$

then

$$\eta \otimes \eta(G(A' \times C')) < \eta(A') + \eta(C').$$

In Section 3 and Section 4 we further show that there exist

- a compact group K and a homomorphism $\tau : G \to K$ with dense image.
- a closed subgroup $L < K$ and a G-factor map $\pi : (X \times Y, \eta) \to (K/L, m_{K/L})$, where $m_{K/L}$ denotes the unique K-invariant probability measure on K/L.
- Borel sets $I, J \subset K/L$ with $\nu(A_{x_o}^{-1} C) \geqslant m_K(I^{-1} J)$, where we have identified I and J with their right-L-invariant lifts to K,

such that

$$A_x \subset \tau^{-1}(I\pi(x, y)^{-1}) \quad \text{and} \quad C_y \subset \tau^{-1}(C\pi(x, y)^{-1}),$$

for η-almost every (x, y). In particular,

$$m_K(I^{-1} J) < m_K(I) + m_K(J).$$

Using a classical inverse product set theorems by Kemperman, we conclude that $I^{-1} J$ is invariant under an open normal subgroup. It follows that A_x is contained in some proper periodic subset P of G for μ-almost every x. From this it is not hard to show that there exists a subset $A_o \subset A_{x_o}$ with $d^*(A_o) = d^*(A_{x_o})$ such that A_o is contained in some right translate of P. In particular, A_{x_o} is not spread-out, which finishes the first part of Theorem 1.5.

To prove the second part, we assume that every finite quotient of G is abelian, and we wish to show that not only A_o but the whole of A_{x_o} is contained in some right-translate of P. This is somewhat technical and requires us to use our so called "overshoot inequality" (5.7) together with some classical results of Kneser.

The proof of Theorem 1.8 runs along similar lines, but here we end up with Borel sets $I, J \subset K$ such that

$$m_K(I) = \mu(A) \quad \text{and} \quad m_K(J) = \nu(B) \quad \text{and} \quad m_K(I^{-1} J) = m_K(I) + m_K(J) < 1.$$

A deep fact tells us that since K has a dense *amenable* subgroup, its identity component must be abelian. This allows us to use a recent result by the first author which asserts that K admits either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$ as a factor in such a way that the sets I and J above coincide, modulo null sets, with pull-backs under the factor map of "intervals". This shows that for μ-almost every $x \in X$, the set A_x is contained in a Sturmian set with the same upper Banach density as A_x.

If the G-action on X were minimal, or if $\text{supp}(\mu) = X$, then it would follow from general principles that A_{x_o} is also contained in a Sturmian set with the same upper Banach density. However, these are somewhat degenerate cases. To prove that we can take $x = x_o$ in general requires quite a lot of work (already for abelian G), but the necessary arguments are again based on the "overshoot inequality" (5.7).

Acknowledgments

2. Preliminaries

Throughout this section, let G be a countable group. If Y is a set upon which G acts, then we refer to Y as a *G-space*, and if $B \subset Y$ and $y \in Y$, then we define the "set of returns" of y to B by

$$B_y = \big\{ g \in G \,:\, gy \in B \big\} \subset G. \tag{2.1}$$

We note that for every $B \subset Y$,

$$gB_y = (gB)_y \quad \text{and} \quad B_{gy} = B_y g^{-1} \quad \text{for all } g \in G \text{ and } y \in Y. \tag{2.2}$$

In particular, for every $A \subset G$, we have $AB_y = (AB)_y$ for all $y \in Y$.

2.1. **Dynamical tools and basic notions**

2.1.1. *Hulls*

Let us denote by 2^G the space of all subsets of G, endowed with the sequentially compact Tychonoff topology, with respect to which the set $U = \big\{ A \subset G \,:\, e_G \in A \big\}$ is clopen. We note that the group G acts on 2^G by homeomorphisms via $g.A = Ag^{-1}$, and using the notation in (2.1) above, we have the curious-looking identity $U_A = A$. In particular, every subset of G is the set of returns of itself, viewed as an element in the G-space 2^G, to the set U.

Given $A \subset G$, we shall denote by X_A the closure of the G-orbit of the point $x_o = A$ in 2^G. The pair (X_A, x_o) is a pointed G-space in the sense of Appendix A, and we refer it as the G-*hull* of A. It will often be convenient to abuse notation and denote by A the clopen set $U \cap X_A$ in X_A, so that we can write $A_{x_o} = A$.

2.1.2. *Borel G-spaces and their factors*

Let (Z, η) be a standard Borel probability measure space with Borel σ-algebra $\mathcal{B}_Z$. If it comes equipped with an action of G by bi-measurable η-preserving maps, then we say that (Z, η) is a *Borel G-space*. If the only G-invariant $\mathcal{B}_Z$-measurable subsets of Z are either η-null or η-conull, we say that η is *ergodic*, and that (Z, η) is an *ergodic Borel G-space*. If every finite-index subgroup of G acts ergodically on (Z, η) as well, we say that (Z, η) is a *totally ergodic Borel G-space*.

If (Z, η) and (W, θ) are Borel G-spaces, $Z' \subset Z$ and $W' \subset W$ are conull G-invariant measurable subsets and $\pi : Z' \to W'$ is a measurable and G-equivariant map such that $\eta(\pi^{-1}(C)) = \theta(C)$ for all $C \in \mathcal{B}_W$, then we say that (W, θ) is a *factor Borel G-space* of (Z, η) and $\pi : (Z, \eta) \to (W, \theta)$ is a *G-factor map*. If π in addition is a bijection (which implies that its inverse is measurable as well), we say that (Z, η) and (W, θ) are *isomorphic Borel G-spaces*.

If $\pi : (Z, \eta) \to (W, \theta)$ is a G-factor map, then $\pi^{-1}(\mathcal{B}_W)$ is a (up to η-null sets) a G-invariant sub-σ-algebra of $\mathcal{B}_Z$. Conversely, it is well-known (see for instance Theorem 6.5 in [12]), that to every (essentially) G-invariant sub-σ-algebra $\mathcal{L} \subset \mathcal{B}_Z$ there correspond

   (i) a factor Borel G-space (W, θ), and
   (ii) a G-factor map $\pi : (Z, \eta) \to (W, \theta)$,

such that $\mathcal{L} = \pi^{-1}(\mathcal{B}_W)$ modulo η-null sets. We shall refer to any Borel G-space (W, θ) with this property as a *factor Borel G-space associated to $\mathcal{L}$*; they are all isomorphic as Borel G-spaces.

Any Borel G-space (Z, η) gives rise to a strongly continuous unitary representation of G on the Hilbert space $L^2(Z, \eta)$ via $g \cdot f = f \circ g^{-1}$, which we refer to as the *regular representation* of (Z, η); the term *Koopman representation* is often also used in the literature. It is easy to prove that η is ergodic if and only if $L^2(Z, \eta)^G \cong \mathbb{C}$. We say that (Z, η) has *discrete spectrum* if $L^2(Z, \eta)$ decomposes into a direct sum of FINITE-DIMENSIONAL irreducible representations.

2.1.3. *Discrete spectrum and isometric G-spaces*

Let us now introduce a class of Borel G-spaces that will play a key role in this paper. Let (K, τ) be a metrizable compactification of G (see Appendix B for the necessary terminology), and let $L < K$ be a closed subgroup. Then there is an action of G by continuous maps on the quotient space K/L given by $g.tL = \tau(g)tL$. If $\mathfrak{m}_{K/L}$ denotes the unique K-invariant probability measure on K/L, then $(K/L, \mathfrak{m}_{K/L})$ is obviously a Borel G-space (Borel standardness follows from metrizability). We shall refer to (K, L, τ) as an *isometric G-space*; this strange choice of terminology is standard and comes from the fact that there in this setting always exists a K-invariant, and thus G-invariant, metric on K/L.

Lemma 2.1. *Let (K, L, τ) be an isometric G-space. Then,*

   (i) $\mathfrak{m}_{K/L}$ is the unique G-invariant Borel probability measure on K/L.
   (ii) $G \curvearrowright (K/L, \mathfrak{m}_{K/L})$ is ergodic and has discrete spectrum.

Proof. (i) The stabilizer of any probability measure on K/L is a *closed* subgroup of K. Since $\tau(G)$ is dense in K, any G-invariant probability measure on K/L must be K-invariant.

(ii) By continuity of the regular representation of K on $L^2(K/L)$, any G-invariant element must be K-invariant, and thus constant. By Peter-Weyl's Theorem, the left-regular representation of K on $L^2(K)$ decomposes into finite-dimensional irreducible representations, whence the regular representation on $L^2(K/L)$ as well. Each of these finite-dimensional representations is irreducible under the G-action as well (by denseness of $\tau(G)$ in K). $\square$

Let us now state a strong converse to Lemma 2.1 (ii) due to Mackey (Theorem 1, [23]). For *abelian* G, this result is more often referred to as a special instance of the classical Halmos-von Neumann Theorem.

Proposition 2.2. *Every ergodic Borel* G-*space with discrete spectrum is isomorphic as a Borel* G-*space to* $(K/L, m_{K/L})$ *for some isometric* G-*space* (K, L, τ).

Let (Z, η) be an ergodic Borel G-space, and denote by $\mathcal{K}$ the smallest G-invariant sub-σ-algebra of $\mathcal{B}_Z$ with respect to which all elements in $L^2(Z, \eta)$ with finite-dimensional cyclic sub-spaces under the G-action are measurable. We note that $\mathcal{K} = \mathcal{B}_Z$ if and only if (Z, η) has discrete spectrum, as every such finite-dimensional cyclic sub-space decomposes into a direct sum of irreducibles. If (W, θ) is a factor Borel G-space associated to $\mathcal{K}$, then $G \curvearrowright (W, \theta)$ clearly has discrete spectrum, so by Proposition 2.2 it is isomorphic to $(K/L, m_{K/L})$ for some isometric G-space (K, L, τ). In particular, it follows from the discussions in the last subsection that there is a G-factor map $\pi : (Z, \eta) \to (K/L, m_{K/L})$ such that $\pi^{-1}(\mathcal{B}_{K/L}) = \mathcal{K}$ modulo η-null sets. We shall refer to $(K/L, m_{K/L})$ as the *Kronecker-Mackey triple* associated to (Z, η), and to both $\mathcal{K}$ and $(K/L, m_{K/L})$ as the *Kronecker-Mackey factor* of (Z, η). The G-factor map π will be referred to as the *Kronecker-Mackey* G-*factor map*. It is clear from the definition of $\mathcal{K}$ that any other G-invariant sub-σ-algebra of $\mathcal{B}_Z$ whose associated factor Borel G-space has discrete spectrum is contained in $\mathcal{K}$. The following lemma will be useful in the next section.

Lemma 2.3. *If* $\mathcal{E}_G$ *denotes the sub-σ-algebra of* $\mathcal{B}_Z \otimes \mathcal{B}_Z$ *consisting of* G-*invariant subsets of* $Z \times Z$, *then* $\mathcal{E}_G \subset \mathcal{K} \otimes \mathcal{K}$.

Proof. We first note that any G-invariant function $f \in L^2(Z \times Z, \eta \otimes \eta)$ decomposes as $f_1 + if_2$, where f_1 and f_2 are G-invariant, and $\overline{f_j(z, z')} = f_j(z', z)$, for $j = 1, 2$. It thus suffices to show that any G-invariant f with $f(z, z') = \overline{f(z', z)}$ is $\mathcal{K} \otimes \mathcal{K}$-measurable. To prove this, fix such a (non-zero) G-invariant function f, and consider the (non-zero) operator $T_f : L^2(Z, \eta) \to L^2(Z, \eta)$ given by

$$(T_f \phi)(z) = \int_Z f(z, z') \phi(z') \, d\eta(z').$$

It is a well-known classical fact that T_f is self-adjoint and Hilbert-Schmidt. Hence, by the Spectral Theorem of such operators, there is an orthonormal basis of eigenfunctions (ψ_j) for T_f with eigenvalues (λ_j) such that $T_f \phi = \sum_j \lambda_j \langle \phi, \psi_j \rangle \psi_j$ for all $\phi \in L^2(Z, \eta)$; in particular, unwrapping this identity yields

$$f = \sum_j \lambda_j \psi_j \otimes \overline{\psi_j}, \tag{2.3}$$

where convergence holds in the L^2-sense. Since η is G-invariant, T_f is G-equivariant, and thus every eigenspace of T_f is G-invariant. Since T_f is a G-invariant operator, each eigenspace is G-invariant. By compactness of T_f, each eigenspace corresponding to a non-zero eigenvalue (such eigenvalues exist since T_f is non-zero) is finite-dimensional. We conclude that the G-cyclic sub-spaces for the corresponding ψ_j's are finite-dimensional, whence ψ_j is $\mathcal{K}$-measurable. By (2.3), we can conclude that f is $\mathcal{K} \otimes \mathcal{K}$-measurable. $\square$

2.1.4. *Shadows*

Let (Z, η) be a Borel G-space. Given a η-complete sub-σ-algebra $\mathcal{F}$ of $\mathcal{B}_Z$ and a $\mathcal{B}_Z$-measurable subset $A \subset Z$, we can consider the conditional expectation $\mathbb{E}[\chi_A \mid \mathcal{F}]$, pick a η-almost everywhere defined pointwise realization of this element in $L^2(Z, \eta)$, and denote by $A_{\mathcal{F}}$ the positivity set of this pointwise realization. We shall refer to any $A_{\mathcal{F}}$ constructed in this manner as a $\mathcal{F}$-*shadow of*

the set A. It is clear that all possible choices of $A_{\mathcal{F}}$ only differ by η-null sets, and that $A \subset A_{\mathcal{F}}$ modulo η-null sets for all such choices. Moreover, $A_{\mathcal{F}}$ is $\mathcal{F}$-measurable by construction. The following lemma will be very useful in the next section. Recall that $\mathcal{E}_G$ denotes the sub-σ-algebra of $\mathcal{B}_Z \otimes \mathcal{B}_Z$ consisting of G-invariant subsets of $Z \times Z$.

Lemma 2.4. *Let* $\mathcal{F} \subset \mathcal{B}_Z$ *be a* G-*invariant sub-σ-algebra, and suppose that* $\mathcal{E}_G \subset \mathcal{F} \otimes \mathcal{F}$. *Then, for any* $\mathcal{B}_Z$-*measurable sets* $A, B \subset Z$, *we have, modulo* $\eta \otimes \eta$-*null sets,*

   (i) $(A \times B)_{\mathcal{F} \otimes \mathcal{F}} = A_{\mathcal{F}} \times B_{\mathcal{F}}$.
   (ii) $G(A \times B) = G(A_{\mathcal{F}} \times B_{\mathcal{F}})$.

Proof. (i) Obvious; true for any sub-σ-algebra $\mathcal{F}$.

   (ii) Since $A \subset A_{\mathcal{F}}$ and $B \subset B_{\mathcal{F}}$ modulo η-null sets, it suffices to prove that the G-invariant set $E = G(A_{\mathcal{F}} \times B_{\mathcal{F}}) \setminus G(A \times B)$ is a $\eta \otimes \eta$-null set. By our assumption on $\mathcal{F}$, the set E is $\mathcal{F} \otimes \mathcal{F}$-measurable, whence

$$0 = \eta \otimes \eta(E \cap (A \times B)) = \int_E \mathbb{E}[\chi_{A \times B} \mid \mathcal{F} \otimes \mathcal{F}] \, d\eta \otimes \eta = \int_E \mathbb{E}[\chi_A \mid \mathcal{F}] \, \mathbb{E}[\chi_B \mid \mathcal{F}] \, d\eta \otimes \eta,$$

and since the integrand on the right is strictly positive on the direct product $A_{\mathcal{F}} \times B_{\mathcal{F}}$, we conclude that $\eta \otimes \eta(E \cap (A_{\mathcal{F}} \times B_{\mathcal{F}})) = 0$. Since E is G-invariant and G is countable, this implies that $0 = \eta \otimes \eta(E \cap G(A_{\mathcal{F}} \times B_{\mathcal{F}})) = \eta \otimes \eta(E)$, which finishes the proof. □

   Combined with Lemma 2.3, and the discussion proceeding it, this observation yields the following corollary.

Corollary 2.5. *Let* (Z, η) *be an ergodic Borel* G-*space, and let* (K, L, τ) *denote its Kronecker-Mackey triple, and* $\pi : (Z, \eta) \to (K/L, \mathfrak{m}_{K/L})$ *the associated* G-*factor map. Then, for all measurable* $A, B \subset Z$, *there are Borel sets* $I, J \subset K/L$ *such that*

$$A \subset \pi^{-1}(I) \quad \text{and} \quad B \subset \pi^{-1}(J)$$

modulo η-*null sets, and* $\eta \otimes \eta(G(A \times B)) = \mathfrak{m}_{K/L} \otimes \mathfrak{m}_{K/L}(G(I \times J))$.

Proof. Note that $A \subset A_{\mathcal{K}}$ and $B \subset B_{\mathcal{K}}$ modulo η-null sets, and by Lemma 2.4 (ii) we know that, $\eta \otimes \eta(G(A \times B)) = \eta \otimes \eta(G(A_{\mathcal{K}} \times B_{\mathcal{K}}))$. Since $A_{\mathcal{K}}$ and $B_{\mathcal{K}}$ are $\mathcal{K}$-measurable, there are Borel sets $I, J \subset K/L$ such that $A_{\mathcal{K}} = \pi^{-1}(I)$ and $B_{\mathcal{K}} = \pi^{-1}(J)$, which finishes the proof. □

3. THE JOINING TRICK

We define joinings of Borel G-*spaces. For compact pointed* G-*spaces, we discuss how one can use joinings to transfer inclusions of return times at generic points to inclusion of return times at base points.*

   Let (X, μ) and (W, θ) be Borel G-spaces. A G-invariant Borel probability measure ξ on $X \times W$ such that $\mu(A) = \xi(A \times W)$ and $\theta(I) = \xi(X \times I)$ for all Borel sets $A \subset Z$ and $I \subset W$ is called a *joining* of (Z, η) and (W, θ). Note that $\eta \times \theta$ is always a joining. We denote the set of all joinings of (Z, η) and (W, θ) by $\mathcal{J}_G(\eta, \theta)$. The following result is standard (see for instance Theorem 6.2 in [15]).

Proposition 3.1. *If* (X, μ) *and* (W, θ) *are ergodic, then there are ergodic measures in* $\mathcal{J}_G(\mu, \theta)$.

   Let G be a countable amenable group, and let X and W be compact metrizable spaces, equipped with actions of G by homeomorphisms, and suppose that there exists a point $x_o \in X$ with a dense G-orbit. Let μ be an ergodic G-invariant measures on X.

Lemma 3.2. *With the notation and assumptions above,*

    (i) there exists a μ-conull subset $X' \subset X$ such that $\mathrm{supp}(\mu) = \overline{Gx}$ for all $x \in X'$.

    (ii) for every closed G-invariant subset $Z \subset X \times W$ whose projection to X contains $\mathrm{supp}(\mu)$, there exists an ergodic $\xi \in \mathcal{P}_G(Z)$ which projects to μ.

In particular, if $\mathcal{P}_G(W) = \{\theta\}$, then the measure ξ in (ii) is an ergodic joining of (X, μ) and (W, θ).

Proof. (i) Since X is compact and metrizable, so is $\mathrm{supp}(\mu)$, and thus there is a countable basis (U_n) for the restricted topology on $\mathrm{supp}(\mu)$. By ergodicity of μ, we have $\mu(GU_n) = 1$ for all n, so $X' = \bigcap_n GU_n$ is μ-conull. For all $x \in X'$, the G-orbit of x meets every U_n, and is thus dense in $\mathrm{supp}(\mu)$.

(ii) Fix a closed G-invariant set $Z \subset X \times W$, write $p : Z \to X$ for the projection, and assume that $\mathrm{supp}(\mu) \subset p(Z)$. By a standard use of Hahn-Banach's Theorem, the set of probability measures on Z which project to μ is non-empty, and it is clearly weak*-compact and convex in $\mathcal{P}(Z)$. Since G is amenable, there is a G-fixed point ξ' in this set, and since μ is ergodic, every ergodic component ξ of ξ' will project to μ as well. $\qquad\square$

Let us also record the following corollary of this lemma; see Appendix A for terminology concerning pointed G-spaces.

Corollary 3.3. *Let G be a countable amenable group, (X, x_o) a pointed G-space, μ an ergodic G-invariant probability measure on X and (K, L, τ) an isometric G-space. Then, for every $t \in K/L$ and μ-almost every $x \in X$, there exist*

    (i) an ergodic joining ξ of (X, μ) and $(K/L, m_{K/L})$ supported on $Z = \overline{G(x, t)}$.

    (ii) a point $t_o \in K/L$ such that $Z \subset Z_o := \overline{G(x_o, t_o)}$.

Proof. (i) Immediate from Lemma 3.2 applied to $(W, \theta) = (K/L, m_{K/L})$, using the fact that $m_{K/L}$ is the unique G-invariant probability measure on K/L (Lemma 2.1).

(ii) Pick $x \in \mathrm{supp}(\mu)$ and $t \in K/L$, and find a sequence (g_n) in G such that $g_n x_o \to x$. Choose a sub-sequence (g_{n_k}) such that $\tau(g_{n_k})$ converges in K to some k, and set $t_o = k^{-1}t$. One readily checks that $g_{n_k} t_o \to t$, and thus $(x, t) \in \overline{G(x_o, t_o)}$. $\qquad\square$

We now arrive at the punchline of this subsection. In what follows, let G be a countable amenable group, and

    • (X, x_o) pointed G-space and μ an ergodic G-invariant probability measure on X.

    • $A \subset X$ an open μ-Jordan measurable subset, i.e. $\mu(\overline{A}) = \mu(A)$.

    • (K, L, τ) an isometric G-space, and $I \subset K/L$ a closed $m_{K/L}$-measurable subset, which we shall identify with its right-L-invariant lift to K.

Our aim will be to show the following lemma, which roughly asserts that if $A_x \subset I_t^o$ for some x and t, where I^o denotes the interior of I, then there is a "big" subset $A_o \subset A_{x_o}$ such that $A_o \subset I_{t_o}$ for some $t_o \in K/L$. This will be the first step in a "bootstrap argument" used in the proofs of Theorem 1.5 and 1.8.

Lemma 3.4. *Suppose that $A_x \subset I_t^o$ for some $x \in X$ such that $\mathrm{supp}(\mu) = \overline{Gx}$ and $t \in K/L$. Then there exist $t_o \in K/L$ and an extreme invariant mean λ on G such that if we set $A_o = (A \times I^o)_{(x_o, t_o)}$, then*

    (i) $\lambda(A_o) = \mu(A)$ and $A_o \subset \tau^{-1}(It_o^{-1})$.

    (ii) for every Borel G-space (Y, ν) and Borel set $C \subset Y$, we have $\nu(A_o^{-1}C) \geqslant \nu(A_x^{-1}C)$.

Proof. (i) Let $Z = \overline{G(x, t)}$, and use Corollary 3.3 to produce an ergodic joining ξ of (X, μ) and $(K/L, m_{K/L})$ supported on Z, and a point $t_o \in K/L$ such that $Z \subset \overline{G(x_o, t_o)}$. It follows from the

inclusion $A_x \subset I_t$ that the *open* set $D = A \times I^c$ satisfies $D \cap Z = \emptyset$, whence $\xi(D) = 0$ since ξ is supported on Z, and thus $\mu(A) = \xi(A \times I)$. Since ξ is an ergodic G-invariant measure on $Z_o = \overline{G(x_o, t_o)}$ we can apply Proposition A.4 to the pointed G-space (Z_o, z_o) where $z_o = (x_o, t_o)$, and find an extreme invariant mean λ such that $\xi = S^*_{z_o}\lambda$. Since $A \times I$ is ξ-Jordan measurable, Lemma A.2 (i) now shows that if we write $A_o = (A \times I^o)_{z_o}$, then $\lambda(A_o) = \mu(A)$. The inclusion $A_o \subset \tau^{-1}(It_o^{-1})$ is immediate.

(ii) Since $A_o \supset U_{z_o}$, where $U = A \times I^o$ is open, the lower bound for every (Y, ν) and $C \subset Y$ follows from the proof of Lemma 5.2, by using the fact that $U_z = A_x$ (which is equivalent to the inclusion $A_x \subset I_t^o$). $\hfill\square$

3.0.1. *Removing Jordan measurability*

In most of our arguments, assuming that the set $A \subset X$ is μ-Jordan measurable is rather harmless; in fact, in most of our applications, A will be clopen. However, at one subtle point in the proof of our main density results, it will be useful to instead refer to the following weak cousin to Lemma 3.4.

Lemma 3.5. *Suppose that $U \subset X$ is open, and there is a Borel set $Q \subset U$ with positive μ-measure which is invariant under a finite-index normal subgroup $G_o < G$. Then there is a thick set $T \subset G$ and a non-empty G_o-invariant set $Q_o \subset G$ such that $Q_o \cap T \subset U_{x_o}$.*

Proof. One readily checks that the map $\sigma : X \to 2^{G/G_o}$ given by $x \mapsto Q_x$ is Borel and G-equivariant, and $Q = \sigma^{-1}(V)$, where $V = \{D \subset G/G_o : G_o \in D\}$ is clopen. In particular, we have $V_{\sigma(x)} \subset U_x$ for all $x \in X$, and thus $\xi(U^c \times V) = 0$ for the graph joining $\xi = (id \times \sigma)_*\mu$. Just as in Corollary 3.3, we can utilize the ergodicity of ξ to find $t_o \in K/L$ such that $\overline{G(x_o, t_o)} \supset \text{supp}(\xi)$. We then use Proposition A.4 to find an invariant mean λ such that $S^*_{(x_o, t_o)}\lambda = \xi$, where $S^*_{(x_o, t_o)}$ is defined in Subsection A.2. Since $U^c \times V$ is closed, Lemma A.2 tells us that $\lambda(U^c_{x_o} \cap V_{t_o}) \leqslant \eta(U^c \times V) = 0$, and thus $T = (U^c_{x_o} \cap V_{t_o})^c$ is thick by Lemma A.7. The set $Q_o = V_{t_o}$ is clearly G_o-invariant, and one readily checks that $Q_o \cap T \subset U_{x_o}$. $\hfill\square$

3.1. **Extra features of the joining trick if the action is minimal (optional)**

If one were to impose on the set A in Theorem 1.5 or Theorem 1.8 the additional (somewhat unnatural) assumption that its G-hull is a minimal G-space (or at least that there exists a G-invariant measure on the hull of full support), then many arguments in the coming sections would become significantly shorter and less technical, in view of the following stronger version of Lemma 3.4. We retain the notation introduced in the previous subsection.

Lemma 3.6. *Suppose that $G \curvearrowright X$ is minimal, $A \subset X$ is open and $I \subset K/L$ is closed. If $A_x \subset I_t$ for some $(x, t) \in X \times K/L$, then there exists $t_o \in K/L$ such that $A_{x_o} \subset I_{t_o}$.*

Proof. Let $Z = \overline{G(x, t)} \subset X \times K/L$, and choose a closed G-invariant minimal subset $Z' \subset Z$. Since X is minimal, we conclude that Z' projects onto X, and thus there is at least one $t_o \in K/L$ such that $(x_o, t_o) \in Z'$. We further note that since $A \times I^c$ is open, and $(A \times I^c)_{(x,t)} = \emptyset$, we have $(A \times I^c) \cap Z' = \emptyset$, and thus in particular $A_{x_o} \cap I^c_{t_o} = \emptyset$, whence $A_{x_o} \subset I_{t_o}$. $\hfill\square$

4. How to take differences in Borel G-spaces

We discuss a way to associate to pairs of Borel sets in an ergodic Borel G-space their "difference set" in the associated Kronecker-Mackey factor.

Let G be a countable group, and (Z, η) an ergodic Borel G-space. Let $A, B \subset Z$ be Borel sets. If (K, L, τ) denotes the Kronecker-Mackey triple associated to (Z, η) (see previous section for definitions), and $\pi : (Z, \eta) \to (K/L, m_{K/L})$ the corresponding G-factor map, then Corollary 2.5 provides us with Borel sets $I, J \subset K/L$ such that $A \subset \pi^{-1}(I)$ and $B \subset \pi^{-1}(J)$ and

$$\eta \otimes \eta(G(A \times B)) = m_{K/L} \otimes m_{K/L}(G(I \times J)).$$

We may identify I and J with their right-L-invariant lifts to K under the canonical map $K \mapsto K/L$, after which we can write this identity as $\eta \otimes \eta(G(A \times B)) = m_K \otimes m_K(G(I \times J))$. It is now tempting to argue as follows: Since $\tau(G)$ is dense in K, we "should be able to" replace $G(I \times J)$ with $K(I \times J)$ without increasing the $\eta \otimes \eta$-measure; the latter set is the pull-back of the set $I^{-1}J$ under the multiplication map $(x, y) \mapsto x^{-1}y$, and since $m_K \otimes m_K$ is mapped to m_K under this multiplication map, we have $\eta \otimes \eta(G(A \times B)) = m_K(I^{-1}J)$. Unfortunately, already the first line of the argument above fails; replacing I and J with $I \cup N_I$ and $J \cup N_J$ where N_I and N_J are m_K-null sets such that the difference set $N_I^{-1}N_J$ has positive m_K-measure show that additional arguments are required. Fortunately for us, upon passing to conull subsets of I and J in the first step (which will not affect the measure of $G(I \times J)$), the rest of the argument runs as before. The exact correction can be stated as follows.

Proposition 4.1. *If* $I, J \subset K$ *are Borel sets, then there exist conull subsets* $I' \subset I$ *and* $J' \subset J$ *such that*

$$m_K \otimes m_K(G(I' \times J')) = m_K \otimes m_K(K(I' \times J')).$$

Furthermore, if I *and* J *are right-invariant under a subgroup* $L < K$, *then so are* I' *and* J'.

4.1. **Difference arithmetics for shadows**

Combining Proposition 4.1 with the rest of the argument above, as well as with Corollary 2.5, yields the following corollary which will play a key role in this paper. We stress that this result seems new already for actions of $G = (\mathbb{Z}, +)$.

Corollary 4.2. *Let* (Z, η) *be an ergodic Borel* G-space, and let (K, L, τ) *denote its Kronecker-Mackey triple, and* $\pi : (Z, \eta) \to (K/L, m_{K/L})$ *the associated* G-factor map. Then, for all measurable $A, B \subset Z$, there are Borel sets $I, J \subset K/L$ *such that*

$$A \subset \pi^{-1}(I) \quad and \quad B \subset \pi^{-1}(J)$$

modulo η-null sets, and $\eta \otimes \eta(G(A \times B)) = m_K(I^{-1}J)$, *where we have identified* I *and* J *with their right-L-invariant lifts to* K *under the canonical map* $K \mapsto K/L$.

The rest of this section will be devoted to the proof of Proposition 4.1. It will be useful to adopt a slightly more general perspective on things. In what follows, let H be a compact metrizable group, $M < H$ a closed subgroup and $\Gamma < M$ a dense countable subgroup. Soon enough, we shall apply our results below to the setting:

$$H = K \times K \quad and \quad M = \big\{(k, k) : k \in K\big\} \quad and \quad \Gamma = \big\{(\tau(g), \tau(g)) : g \in G\big\}. \quad (4.1)$$

We say that a decreasing sequence (B_j) of closed sets in H with positive m_H-measures is *Dirac* if their intersection equals $\{e_H\}$, and we say that a Borel set $D \subset H$ is *balanced with respect to* (B_j) if

$$\lim_j \frac{m_H(D \cap sB_j)}{m_H(B_j)} = 1, \quad \text{for all } s \in D.$$

It is easy to see that every compact and metrizable group admits a Dirac sequence (B_j), and given any such sequence, we can form $\rho_j = \frac{\chi_{B_j}}{m_H(B_j)}$ in $L^1(H)$. It is quite standard to prove that for every Borel set $D \subset H$, we have $\rho_j * \chi_D \to \chi_D$ in the L^2-norm, and thus, upon passing to a sub-sequence,

m_H-almost everywhere. Clearly, if D is *right*-invariant under some subgroup Q of H, then the set on which this sub-sequence converges is again right-invariant under Q. Unwrapping this, we conclude:

Lemma 4.3. *If* (B_j) *is Dirac and* $D \subset H$ *is Borel with positive* m_H*-measure, then there exists a conull subset* $D' \subset D$ *and a sub-sequence* (B_{j_k}) *such that* D' *is balanced with respect to* (B_{j_k})*. If* Q *is a subgroup of* H*, and* D *is right-*Q*-invariant, then so is* D'*.*

Let us now assume that $D \subset H$ is a Borel set with positive m_H-measure which is balanced with respect to some Dirac sequence (B_j). We claim that $m_H(\Gamma D) = m_H(MD)$. To prove this, we argue by contradiction, and define $C = MD \setminus \Gamma D$, and assume that C has positive m_H-measure. Then for every j, the function $f_j(s) = m_H(C \cap sB_j)$ is continuous and left Γ-invariant, whence left M-invariant as well. Furthermore, fix $0 < \varepsilon < 1/2$, and use the lemma above to produce $C' \subset C$ with $m_H(C') = m_H(C)$ which is balanced with respect to some sub-sequence (B_{j_k}). Fix $s \in C'$ and write $s = md$ for some $m \in M$ and $d \in D$. Then, since both C and D are balanced with respect to (B_{j_k}), we have for large k,

$$f_{j_k}(d) = m_H(C \cap dB_{j_k}) = f_{j_k}(md) = m_H(C \cap sB_{j_k}) \geqslant (1 - \varepsilon)m_H(B_{j_k}),$$

and

$$m_H(D \cap dB_{j_k}) \geqslant (1 - \varepsilon)m_H(B_{j_k}),$$

and thus $m_H(C \cap D \cap dB_{j_k}) \geqslant (1 - 2\varepsilon)m_H(B_{j_k}) > 0$. In particular, $C \cap D \neq \emptyset$, which is a contradiction, and thus $m_H(C) = 0$.

Let us now apply all of this to prove Proposition 4.1, using H, M and Γ from (4.1). Let $I, J \subset K$ be Borel sets, and fix a Dirac sequence (B_j). By using Lemma 4.3 twice, we can produce a sub-sequence (B_{j_k}) and $I' \subset I$ and $J' \subset J$ with $m_K(I') = m_K(I)$ and $m_K(J') = m_K(J)$ which are both balanced with respect to (B_{j_k}). Note that they can both be chosen to be right-L-invariant. We conclude that $D = I' \times J'$ is balanced with respect to $B_{j_k} \times B_{j_k}$ in $H = K \times K$, so the argument above tells us that

$$m_K \otimes m_K(K(I' \times J')) = m_H(MD) = m_H(\Gamma D) = m_K \otimes m_K(G(I' \times J')).$$

5. ACTION SETS VERSUS PRODUCT SETS IN COMPACT GROUPS

We show that action sets for an ergodic action of a countable group G *can be "nicely shadowed" by product sets in a group compactification of* G*. In certain situations, when a priori upper bounds are imposed on the action sets, topological regularity for the involved sets in the group compactification can be deduced.*

This is a long, and somewhat technical, section, which we partition it into three main sub-sections. The same notation is kept throughout the section, but with each new sub-section, additional assumptions on the basic objects will be imposed.

5.1. **A correspondence principle for action sets**

Let G be a countable, not necessarily amenable, group. Throughout this section, our key players will be:

- a pointed G-space (X, x_o) and an ergodic $\mu \in \mathcal{P}_G(X)$
- a non-empty open set $A \subset X$.
- an ergodic Borel G-space (Y, ν) and a Borel set $C \subset Y$ with positive ν-measure.

Our first goal will be to establish the following theorem, which is one of the principal building blocks in the proofs of our main results.

Theorem 5.1. *For every ergodic joining* $\eta \in \mathcal{J}_G(\mu, \nu)$, *there exist*

  (i) a metrizable compactification (K, τ) *of* G,
  (ii) a closed subgroup $L < K$ *and a* G-*factor map* $\pi : (X \times Y, \eta) \to (K/L, \mathfrak{m}_{K/L})$,
  (iii) Borel sets $I, J \subset K/L$,

such that

$$A \times Y \subset \pi^{-1}(I) \quad and \quad X \times C \subset \pi^{-1}(J), \quad modulo\ \eta\text{-null sets}, \tag{5.1}$$

and

$$\nu(A_{x_o}^{-1}C) \geqslant \eta \otimes \eta(G(A \times C)) = \mathfrak{m}_K(I^{-1}J), \tag{5.2}$$

where we have identified I *and* J *with their right* L-*invariant lifts to* K *under the canonical quotient map* $K \to K/L$. *In particular, we have* $\mathfrak{m}_K(I) \geqslant \mu(A)$ *and* $\mathfrak{m}_K(J) \geqslant \nu(C)$.

This theorem admits many immediate, yet interesting corollaries; however, we feel that this is not the proper place in the paper to state them, and prefer to save them to Section 6 where the main results are proved.

Let us start the proof of Theorem 5.1 by picking an ergodic $\eta \in \mathcal{J}_G(\mu, \nu)$ once and for all; this is possible by Proposition 3.1. Moreover, in order to shorten some lines and formulas, we shall throughout the rest of the section write

$$A' = A \times Y \quad and \quad C' = X \times B,$$

whenever this is convenient. The following lemma symmetrizes the roles of A and C so that the results of Section 4 can be applied. It is the only place in the proof of Theorem 5.1 where the assumption that A is open is used.

Lemma 5.2. *With the notation above,*

  (i) $\nu(A_{x_o}^{-1}C) \geqslant \eta \otimes \eta(G(A' \times C'))$.
  (ii) there is a μ-*conull subset* $X' \subset X$ *such that* $\nu(A_x^{-1}C) = \eta \otimes \eta(G(A' \times C'))$ *for all* $x \in X'$.

Proof. First note that the continuous map $\sigma : (X \times Y)^2 \to X \times Y$ given by $((x_1, y_1), (x_2, y_2)) \mapsto (x_1, y_2)$ is G-equivariant, and satisfies

$$\sigma_*(\eta \otimes \eta) = \mu \otimes \nu \quad and \quad \sigma^{-1}(A \times C) = A' \times C'.$$

Hence, $\eta \otimes \eta(G(A' \times C')) = \mu \otimes \nu(G(A \times C))$. Secondly,

$$\mu \otimes \nu(G(A \times C)) = \int_X \left(\int_Y \chi_{G(A \times C)}(x, y)\, d\nu(y) \right) d\mu(x) = \int_X \nu(A_x^{-1}C)\, d\mu(x).$$

Since the measurable function $x \mapsto \nu(A_x^{-1}C)$ is G-invariant, there exists by ergodicity of μ a μ-conull subset $X' \subset X$ on which this function equals its μ-integral, which in this case is $\eta \otimes \eta(G(A' \times C'))$, showing (ii).

To prove (i), it suffices to establish the lower bound $\nu(A_{x_o}^{-1}C) \geqslant \nu(A_x^{-1}C)$ for all $x \in X$, as (i) then follows upon integration against μ. To show this lower bound, let us fix $\varepsilon > 0$ and $x \in X$, and assume, without loss of generality, that A_x is non-empty. By σ-additivity of the measure ν, we can find a finite subset $F \subset A_x$ such that $\nu(F^{-1}C) \geqslant \nu(A_x^{-1}C) - \varepsilon$.

Since the set A is assumed to be open, the non-empty set $A_F = \{z \in X : F \subset A_z\}$ is open as well. Since x_o has a dense G-orbit in X, there exists at least one $g \in G$ such that $g.x_o \in A_F$, whence $Fg \subset A_{x_o}$. Finally, since ν is G-invariant, we conclude that $\nu(A_{x_o}^{-1}C) \geqslant \nu(F^{-1}C) \geqslant \nu(A_x^{-1}C) - \varepsilon$. Since $\varepsilon > 0$ and $x \in X$ were arbitrary, we are done. $\qquad \square$

5.1.1. *Proof of Theorem 5.1*

Let (K, L, τ) be the Kronecker-Mackey factor of $(X \times Y, \eta)$ and denote by π the corresponding G-factor map $(X \times Y, \eta) \to (K/L, m_{K/L})$. By Corollary 4.2 applied to $(X \times Y, \eta)$, we can find $I, J \subset K/L$ such that

$$A' \subset \pi^{-1}(I) \quad \text{and} \quad C' \subset \pi^{-1}(J), \quad \text{modulo } \eta\text{-null sets,}$$

and, $\eta \otimes \eta(G(A' \times C')) = m_K(I^{-1}J)$, so in combination with the lemma above,

$$\nu(A_{x_o}^{-1}C) \geqslant \eta \otimes \eta(G(A' \times C')) = m_K(I^{-1}J),$$

where I and J have been identified with their right L-invariant lifts to K under the canonical quotient map $K \to K/L$.

5.2. **Forcing regularity from small doubling**

The guiding question in this subsection is:

What can be said about the Borel sets $I, J \subset K$ *that we end up with in Theorem 5.1?*

In general, the answer is "very little". However, as we will see below, if one *assumes* certain *a priori* upper bounds on $\nu(A_{x_o}^{-1}C)$ in terms of $d^*(A_{x_o})$ and $\nu(C)$, then powerful tools from the research field of product set theory in groups become available, and will force I and J to be "nicely contained" in pretty regular sets. This regularity will allow us in the next subsection to utilize the arguments from Section 3 ("The joining trick") in order to establish relations between the sets A_{x_o} and I_{t_o} for a certain point $t_o \in K/L$. However, before we state our second main result of this section, we will need some technical definitions.

Definition 5.3. Let K be a compact metrizable group with Haar probability measure m_K, and let $I, J \subset K$ be Borel sets. We say that

(i) (I, J) *reduces* to a pair (I_o, J_o) of Borel sets in a factor group M of K if

$$I \subset p^{-1}(I_o) \quad \text{and} \quad J \subset p^{-1}(J_o) \quad \text{and} \quad m_K(I^{-1}J) = m_M(I_o^{-1}J_o), \tag{5.3}$$

where $p : K \to M$ denotes the factor map. We denote this by $(I, J) \preceq_p (I_o, J_o)$.

(ii) (I, J) is *left-balanced* if the inclusion $s^{-1}J \subset I^{-1}J$ implies that $s \in I$.

(iii) $I \subset K$ is *periodic* if it is invariant under an open normal subgroup.

(iv) $I \subset K$ is *sub-periodic* if there exists a conull Borel subset $I' \subset I$ and a normal open subgroup U of K such that $I'U \neq K$.

(v) $I \subset K$ is *Sturmian* if either
   (a) $K = \mathbb{T}$ and I is a closed interval.
   (b) $K = \mathbb{T} \rtimes \{-1, 1\}$ and $I = (I' \rtimes \{-1, 1\})k$ for some symmetric closed interval $I' \subset \mathbb{T}$, and $k \in K$.

(vi) (I, J) is a *Sturmian pair* if both I and J are Sturmian in the sense of (a) or (b) simultaneously.

Throughout the rest of the section we identify I and J from Theorem 5.1 with their right L-invariant lifts to K under the canonical quotient map from $K \to K/L$. The important role of the theorem below will hopefully become clear in the next sub-section.

Theorem 5.4. *With the notation and assumptions above, we have*

(i) $\nu(A_{x_o}^{-1}C) < 1 \implies \bar{I} \neq K.$

(ii) $\nu(A_{x_o}^{-1}C) < \min(1, \mu(A) + \nu(C)) \implies (I, J)$ *reduces to* (I_o, J_o) *in a* finite *quotient group* M *of* K. *In particular,* I *is contained in a proper periodic subset of* K. *Furthermore, if* M *is abelian, then we can choose* (I_o, J_o) *to be left-balanced and satisfy*

$$m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) - m_M(\{e_M\}). \tag{5.4}$$

(iii) G *is amenable and* $\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1 \implies$ *Either* I *or* J *is a sub-periodic set, or the pair* (I, J) *reduces to a Sturmian pair. In the latter case, we also have* $m_K(I) = \mu(A)$ *and* $m_K(J) = \nu(C)$.

We see that in each of the sub-cases in the theorem above, an a priori bound on $\nu(A_{x_o}^{-1}C)$ forces some regularity on I and J. Theorem 5.1 tells us that such bounds immediately imply analogous bounds on $m_K(I^{-1}J)$. By using the following series of results of Kemperman [20], Kneser [22] and the first author [3], the proof of Theorem 5.4 will be swift.

Theorem 5.5. *Let* K *be a compact metrizable group with identity component* K^o, *and let* $I, J \subset K$ *be Borel sets with positive measures.*

(i) $m_K(I^{-1}J) < 1 \implies \bar{I} \neq K$.

(ii) $m_K(I^{-1}J) < \min(1, m_K(I) + m_K(J)) \implies (I, J)$ *reduces to* (I_o, J_o) *in a* finite *quotient group* M *of* K. *In particular,* I *is contained in a proper periodic subset of* K. *Furthermore, If* M *is abelian, then we can choose* (I_o, J_o) *to be left-balanced and satisfy*

$$m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) - m_M(\{e_M\}).$$

If we assume that every finite quotient of G *is* abelian, *then* M *is abelian.*

(iii) K^o *is abelian, and* $m_K(I^{-1}J) = m_K(I) + m_K(J) < 1 \implies$ *Either* I *or* J *is a sub-periodic set, or the pair* (I, J) *reduces to a Sturmian pair.*

Remark 5.6. We encourage the reader to verify that if (I, J) is Sturmian then neither I nor J is sub-periodic and $m_K(I^{-1}J) = \min(1, m_K(I) + m_K(J))$.

Concerning the exact credits in Theorem 5.5: The first assertion in (ii) is due to Kemperman (Theorem 1, [20]), while the second assertion in (ii) is due to Kneser (Satz 1, [22]), modulo the comment about left-balance; this follows from the simple observation:

Lemma 5.7. *Let* M *be a compact group, and let* $I_o, J_o \subset M$ *be closed sets. Then there exists a closed set* $I_1 \subset M$ *such that*

$$I_o \subset I_1 \quad and \quad I_o^{-1}J_o = I_1^{-1}J_o \quad and \quad (I_1, J_o) \text{ is left-balanced.}$$

Proof. Define $I_1^{-1} := \bigcap_{y \in J_o} I_o^{-1}J_o y^{-1}$, and verify the conditions above. $\square$

Going back to the credits in Theorem 5.5: In the case when K is connected, and thus $K^o = K$, (iii) is due to Kneser (Satz 4, [22]). Note that in this case, sub-periodic subsets do not exist. The general case of (iii) is due to the first author (Theorem 1.8, [3]).

Finally, (i) should be attributed to Weil [28] (based on an earlier observation by Steinhaus), although this exact form is not stated there. However, it is not hard to deduce "our" version: Note that if I is dense in K, but $D = (I^{-1}J)^c$ has positive Haar measure, then the dense set I^{-1} would intersect the product set JD^{-1} trivially; however, Weil shows that JD^{-1} always has non-empty interior, which leads to a contradiction. Hence $m_K(I^{-1}J) = m_K(D^c) = 1$.

5.2.1. *Proof of Theorem 5.4*

Before we begin, recall from Theorem 5.1 that

$$\mathfrak{m}_K(I) \geqslant \mu(A) \quad \text{and} \quad \mathfrak{m}_K(J) \geqslant \nu(C) \quad \text{and} \quad \nu(A_{x_o}^{-1}C) \geqslant \mathfrak{m}_K(I^{-1}J),$$

and (K, τ) is a metrizable compactification of G.

Hence, if $\nu(A_{x_o}^{-1}C) < 1$, then $\mathfrak{m}_K(I^{-1}J) < 1$ as well, which by Theorem 5.5 settles (i). If $\nu(A_{x_o}^{-1}C) < \min(1, \mu(A) + \nu(C))$, then $\mathfrak{m}_K(I^{-1}J) < \min(1, \mathfrak{m}_K(I) + \mathfrak{m}_K(J))$ as well, which by Theorem 5.5 settles (ii). Concerning (iii), we note that if $\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1$, then

$$\mathfrak{m}_K(I^{-1}J) \leqslant \nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) \leqslant \min(1, \mathfrak{m}_K(I) + \mathfrak{m}_K(J)).$$

If any of these inequalities is strict, then Theorem 5.5 (ii) implies that I is contained in a proper periodic subset of K, which in particular means that I is sub-periodic. Hence, if assume that neither I nor J is sub-periodic, then we must have $\mathfrak{m}_K(I^{-1}J) = \mathfrak{m}_K(I) + \mathfrak{m}_K(J) < 1$. Since G is amenable and (K, τ) is a group compactification of G, Lemma B.1 (i) guarantees that K^o is abelian, so Theorem 5.5 (iii) now tells us that (I, J) reduces to a Sturmian pair.

5.3. **Proving containment using the joining trick**

Let us now take a closer look at what the inclusions (5.1) and Theorem 5.4 together imply for the set $A_{x_o} \subset G$. To get interesting results, it is necessary to assume from now on that

A is not only open, but also μ-Jordan measurable.

From the inclusions (5.1), we know that $A \times Y \subset \pi^{-1}(I)$ modulo η-null sets. We conclude that for η-almost every $(x, y) \in X \times Y$, we have

$$A_x = (A \times Y)_{(x,y)} \subset \pi^{-1}(I)_{(x,y)} = \tau^{-1}(I\pi(x, y)^{-1}), \tag{5.5}$$

where we have identified I with its right-L-invariant lift to K. Of course, we also have

$$C_y = (X \times C)_{(x,y)} \subset \tau^{-1}(J\pi(x, y)^{-1}), \quad \text{for } \eta\text{-almost every } (x, y). \tag{5.6}$$

These expressions are well-defined since π maps into K/L and I and J are both right-L-invariant. However, it will be convenient in what follows to sometimes pass to conull subsets of I and J, which may or may not be right-L-invariant. To make the expressions that we end up with well-defined, we will adopt the following convention. Fix a Borel section $\sigma : K/L \to K$ (such sections always exist, see for instance Corollary A.8 in [30]), and replace π with $\sigma \circ \pi$; this will have no affect on the expressions above, but we can now think of π as a (not necessarily G-equivariant map) from $X \times Y$ to K.

As a warm-up for the things that will come, let us first assume that I is sub-periodic in K, that is to say, there is a conull subset $I' \subset I$ and an open subgroup $U < K$ such that $UI' \neq K$. By passing to finite-index subgroups, we may without loss of generality assume that U is normal in K. Let p_o denote the canonical quotient map from K to the finite group K/U, and let I_o denote the image of I' under p_o. Then $I_o \neq K/U$, and $A_x \subset \tau_o^{-1}(I_o\pi(x, y)^{-1})$ for η-almost every (x, y), where $\tau_o = p_o \circ \tau$. Since K/U is finite, I_o is definitely closed and $\mathfrak{m}_{K/U}$-Jordan measurable.

Lemma 3.4 now shows that there exists a subset $A_o \subset A_{x_o}$ and an extreme invariant mean λ on G such that $\lambda(A_o) = \lambda(A_{x_o})$ and $A_o \subset P = \tau_o^{-1}(I_o t_o^{-1})$, where P is a proper subset, invariant under the finite-index normal subgroup $G_o = \tau^{-1}(U)$. Furthermore, if we in addition assume that the G-action on X is minimal (or if μ has full support), then Lemma 3.6 tells us that we in fact have the stronger inclusion $A_{x_o} \subset P$.

If J is sub-periodic, then we can argue along the same lines as above; if $J' \subset J$ is conull and $U < K$ is an open subgroup such that $UJ' \neq K$, then $G_o C_y \subset \tau^{-1}(UJ'\pi(x,y)^{-1}) \neq G$, with $G_o = \tau^{-1}(U)$, for ν-generic $y \in Y$, and thus $G_o C$ cannot be ν-conull, as we would then have $G_o C_y = G$ for ν-almost every y.

We summarize these observations in the following proposition.

Proposition 5.8. *With the notation above:*

    (i) If I is sub-periodic, then there exist a finite-index subgroup $G_o < G$, an extreme invariant mean λ on G, and a subset $A_o \subset A_{x_o}$ with $\lambda(A_o) = \mu(A)$ such that $G_o A_o \neq G$. If we in addition assume that $G \curvearrowright X$ is minimal, then $G_o A_{x_o} \neq G$. In particular, if A_{x_o} is spread-out, then I cannot be sub-periodic.

    (ii) If J is sub-periodic, then there is a finite-index subgroup $G_o < G$ such that $\nu(G_o C) < 1$. In particular, if $G \curvearrowright (Y, \nu)$ is totally ergodic, then J cannot be sub-periodic.

5.3.1. *The "overshoot" bound*

Let us now assume that the pair (I, J) of Borel sets in K from Theorem 5.1 reduces to a pair (I_o, J_o) in a quotient group M of K under the quotient map $p : K \to M$, and let us further assume that I_o and J_o are both closed and m_M-Jordan measurable. Then, since $I \subset p^{-1}(I_o)$, the inclusions in (5.1) imply that

$$A_x = (A \times X)_{(x,y)} \subset \pi^{-1}(p^{-1}(I_o)\pi(x,y)^{-1}) = \tau_o^{-1}(I_o \pi_p(x,y)^{-1}),$$

for η-almost every (x,y), where $\tau_o = p \circ \tau$ and $\pi_p = p \circ \pi$, adopting the convention above about the map π when being multiplied with sets which are not L-invariant (see the discussion after equation (5.6)).

Upon passing to a further η-conull subset we can even ensure that $A_x \subset \tau_o^{-1}(I_o^\circ t^{-1})$ for η-almost every (x,y), where $t = \pi_p(x,y)$. In particular, the conditions of Lemma 3.4 are satisfied, so we conclude that there exist $t_o \in M$, an extreme invariant mean λ on G and $A_o \subset A_{x_o}$ such that

$$\lambda(A_o) = \mu(A) \quad \text{and} \quad A_o \subset \tau_o^{-1}(I_o t_o^{-1}).$$

and $\nu(A_o^{-1}C) \geqslant \nu(A_x^{-1}C)$. We can further choose x so that it belongs to the set X' in Lemma 5.2, ensuring that

$$\nu(A_x^{-1}C) = \eta \otimes \eta(G(A \times C)) = m_K(I^{-1}J) = m_M(I_o^{-1}J_o).$$

In particular, if M is finite, then A_o is contained in a proper periodic subset. If $G \curvearrowright X$ is minimal, Lemma 3.6 tells us that we can choose $t_o \in M$ such that $A_{x_o} \subset \tau_o^{-1}(I_o t_o^{-1})$. Without the assumption of minimality, this inclusion might not hold, and we have to take a different route. Crucial to this alternative is an "overshoot"-inequality which we now formulate. Pick $s \in A_{x_o} \setminus A_o$, and note that, since η is a joining of (X, μ) and (Y, ν), and $X \times C \subset \pi_p^{-1}(J_o)$ modulo η-null sets,

$$\begin{aligned}
\nu(A_{x_o}^{-1}C) &\geqslant \nu((A_o \cup \{s\})^{-1}C) = \nu(A_o^{-1}C) + \nu(C) - \nu(A_o^{-1}C \cap s^{-1}C) \\
&\geqslant m_M(I_o^{-1}J_o) + \nu(C) - \eta(A_o^{-1}(X \times C) \cap s^{-1}(X \times C)) \\
&\geqslant m_M(I_o^{-1}J_o) + \nu(C) - m_M(\tau_o(A_o)^{-1}J_o \cap \tau_o(s)^{-1}J_o) \\
&\geqslant m_M(I_o^{-1}J_o) + \nu(C) - m_M(t_o I_o^{-1}J_o \cap \tau_o(s)^{-1}J_o).
\end{aligned}$$

We shall write the resulting lower bound more suggestively as

$$\nu(A_{x_o}^{-1}C) - \nu(C) \geqslant m_M(I_o^{-1}J_o) - m_M(J_o) + m_M(\tau_o(s)^{-1}J_o \setminus t_o I_o^{-1}J_o), \tag{5.7}$$

for all $s \in A_{x_o} \setminus A_o$, and refer to it as the "overshoot" bound. In relation with this bound, it is important to also keep in mind the lower bounds $m_M(I_o) \geqslant \mu(A)$ and $m_M(J_o) \geqslant \nu(C)$.

5.3.2. *Consequences of the "overshoot" bound*

Inequality (5.7) turns out to be a powerful tool; let us consider the following two applications of it.

Case I: First assume that M is finite,

$$\nu(A_{x_o}^{-1}C) < \min(1, \mu(A) + \nu(C))$$

and

$$m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) - m_M(\{e_M\}).$$

By Theorem 5.4 (ii) this is for instance the case if every finite quotient of G is abelian. Then (5.7) simplifies to

$$m_M(I_o) > \mu(A) \geqslant m_M(I_o) - m_M(\{e_M\}) + m_M(\tau(s)^{-1}J_o \setminus t_o I_o^{-1}J_o),$$

whence the rightmost term is strictly less than $m_M(\{e_M\})$ and thus zero. This readily implies the inclusion $\tau(s)^{-1}J_o \subset t_o I_o^{-1}J_o$, so if we assume that (I_o, J_o) is left-balanced, then we conclude that $\tau_o(s) \in I_o t_o^{-1}$ for all $s \in A_{x_o} \setminus A_o$, and thus $A_{x_o} \subset \tau_o^{-1}(I_o t_o^{-1})$.

Furthermore, if we denote by G_o the stabilizer of $\tau^{-1}(I_o t_o^{-1})$, then this subgroup must have finite index in G, and thus $m_M(\{e\}) \geqslant \frac{1}{[G:G_o]}$. Since $G_o A_{x_o} \subset \tau_o^{-1}(I_o t_o^{-1})$, we have

$$m_M(I_o) \geqslant d^*(G_o A_{x_o}),$$

and thus

$$\mu(A) \geqslant m_M(I_o) - m_M(\{e_M\}) \geqslant d^*(G_o A_{x_o}) - \frac{1}{[G:G_o]}.$$

Case II: Let us now assume that J_o is equal to the closure of its interior in M, (I_o, J_o) is left-balanced, and

$$\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1 \quad \text{and} \quad m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o). \tag{5.8}$$

Then (5.7) implies that

$$m_M(I_o) = \mu(A) \quad \text{and} \quad m_M(J_o) = \nu(C) \quad \text{and} \quad m_M(\tau(s)^{-1}J_o \setminus t_o I_o^{-1}J_o) = 0,$$

whence $\tau_o(s)^{-1}J_o^{\circ} \subset t_o I_o^{-1}J_o$. By our assumption on J_o, we conclude that $\tau_o(s)^{-1}J_o \subset t_o I_o^{-1}J_o$, and since the pair is left-balanced, this implies that $\tau_o(s) \in I_o t_o^{-1}$, for all s as above, whence

$$m_M(I_o) = \mu(A) \quad \text{and} \quad m_M(J_o) = \nu(C) \quad \text{and} \quad A_{x_o} \subset \tau_o^{-1}(I_o t_o^{-1}). \tag{5.9}$$

We are now lead to the question:

When can we ensure that the conditions in Case II are satisfied?

Let us recall from Proposition 5.8 that if I and J are not sub-periodic, then

  (i) there is no finite-index subgroup $G_o < G$, extreme invariant mean λ on G and subset $A_o \subset A_{x_o}$ with $\lambda(A_o) = \mu(A)$ such that $G_o A_o \neq G$.
  (ii) there is no finite-index subgroup $G_o < G$ such that $\nu(G_o C) < 1$.

Let us assume that (i) and (ii) are satisfied for A and C. By Theorem 5.4 (iii), we then know that (I, J) reduces to a Sturmian pair (I_o, J_o) in either $M = \mathbb{T}$ or $M = \mathbb{T} \rtimes \{-1, 1\}$. Such pairs are clearly m_M-Jordan measurable, left-balanced and J_o equals the closure of its interior, so the conditions of Case II are satisfied and we get the conclusions in (5.9).

5.3.3. *Our findings*

We shall now summarize, and slightly expand upon, our findings above in two propositions. In both of these, G is assumed to be amenable, and our key players are:

  (i) a pointed G-space (X, x_o) and an ergodic $\mu \in \mathcal{P}_G(X)$.
  (ii) a non-empty open μ-Jordan measurable set $A \subset X$.
  (iii) an ergodic Borel G-space (Y, ν) and a Borel set $C \subset Y$ with positive ν-measure.

The proof of the following proposition follows from Proposition 5.8 and Case I above.

Proposition 5.9. *Suppose that* $\nu(A_{x_o}^{-1}C) < \min(1, \mu(A) + \nu(C))$. *Then there is a subset* $A_o \subset A_{x_o}$, *a finite-index subgroup* $G_o < G$ *such that* $G_o A_o \neq G$ *and* $\lambda(A_o) = \mu(A)$ *for some extreme invariant mean* λ *on* G.

If every finite quotient of G *is* ABELIAN, *then there is a finite-index subgroup* $G_o < G$ *such that*

$$d^*(G_o A_{x_o}) < \mu(A) + \frac{1}{[G : G_o]}.$$

The proof of our next observation is contained under Case II above, modulo the last part, which will be proved below.

Proposition 5.10. *Suppose that* $\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1$, *and*
  (i) *there is no subset* $A_o \subset A_{x_o}$ *and finite-index subgroup* $G_o < G$ *such that* $G_o A_o \neq G$ *and* $\lambda(A_o) = \mu(A)$ *for some extreme invariant mean* λ *on* G.
  (ii) *there is no finite-index subgroup* $G_o < G$ *such that* $\nu(G_o C) < 1$.

Then A_{x_o} *is contained in a Sturmian set* S *with* $d^*(S) = d_*(S) = \mu(A)$.

Furthermore, if G *is abelian,* $G \curvearrowright (Y, \nu)$ *is totally ergodic and* $(\mathbb{T}, \tau)$ *denotes the torus compactification of* G *from which the Sturmian set* S *comes, then there is a G-factor map* $\sigma : (Y, \nu) \to (\mathbb{T}, m_\mathbb{T})$ *and a closed interval* $J_o \subset \mathbb{T}$ *such that* $C = \sigma^{-1}(J_o)$ *modulo* ν-null sets.

Remark 5.11. It follows from Corollary A.3 that all invariant means assign the same value to any given Sturmian set S in G, so in particular, the identity $d^*(S) = d_*(S)$ is automatic. Concerning the last assertion in Proposition 5.10: The assumption that G is abelian is strictly speaking not necessary, but it simplifies the proofs significantly, why we choose to use it here. The general statement (without assuming that G is abelian) would also be somewhat technical to write down. Furthermore, total ergodicity is not necessary to assume, condition (ii) in Proposition 5.10 suffices.

5.3.4. *The proof of the last part of Proposition 5.10*

Let us assume that G is abelian, $A_{x_o} \subset G$ satisfies (i) in Proposition 5.10 and $G \curvearrowright (Y, \nu)$ is totally ergodic, so that (ii) in Proposition 5.10 is automatically satisfied. Since G is abelian, the group compactification (K, τ) is abelian as well, and thus all subgroups of K are normal, so we may without loss of generality assume that L is trivial, and thus, there is a G-factor map $\pi : (X \times Y, \eta) \to (K, m_K)$. By Proposition 5.8, our assumptions on A_{x_o} and on the action $G \curvearrowright (Y, \nu)$ force I and J to not be sub-periodic in K, and thus we are in the setting of the conclusion (5.9) with $M = \mathbb{T}$. In particular, if we denote by $p : K \to \mathbb{T}$ the quotient map and $\pi_p = p \circ \pi$, then

$$X \times C = \pi_p^{-1}(J_o), \quad \text{modulo } \eta\text{-null sets,} \tag{5.10}$$

where $J_o \subset \mathbb{T}$ is a closed (proper) interval.

We wish to construct a G-factor map $\sigma : (Y, \nu) \to (\mathbb{T}, m_\mathbb{T})$ such that $C = \sigma^{-1}(J_o)$ modulo ν-null sets; but the way to proceed is perhaps not that clear; since η is typically very far from a product measure, there is absolutely no reason to expect the G-factor map π_p to "split" naturally into components which only depend on x and y respectively. For our construction, we shall instead use the "tightness" of (5.10), combined with the fact that closed proper intervals J_o in $\mathbb{T}$ have trivial stabilizers. It will be convenient to take a bird's eye view on the matters first.

Let (W, θ) be a Borel G-space, and let $D \subset W$ be a Borel set. We define the map $\sigma_D : W \to 2^G$ by $w \mapsto D_w = \{g \in G : gw \in D\}$. It is not hard to check that this map is Borel and G-equivariant (recall that our action on 2^G is defined by $g.B = Bg^{-1}$). Let $U \subset 2^G$ be the clopen set $\{B : e_G \in B\}$, and note that $\sigma_D^{-1}(U) = D$. The sought-after G-factor map $\sigma : (Y, \nu) \to (\mathbb{T}, m_\mathbb{T})$ above will be constructed from π_p and σ's from different Borel G-spaces and Borel sets therein.

Before we get into the construction, let us consider a special case first. If (K, τ) is a group compactification of G and $J \subset K$ is a Borel set with trivial stabilizer in K, that is to say, $J \neq Jk$ for all $k \neq e_K$, then it readily follows that the Borel map $\sigma_J : K \to 2^G$ is injective, and thus have a Borel measurable inverse σ_J^{-1} defined on the image of σ_J, which is also Borel (see Theorem A.4 in [30]). In particular, this applies to the torus compactification $(\mathbb{T}, \tau_o)$ above and the closed proper interval $J_o \subset \mathbb{T}$.

Let us now turn to the construction of $\sigma : (Y, \nu) \to (\mathbb{T}, m_\mathbb{T})$. From $(X \times Y, \eta)$ we have two natural G-equivariant Borel maps, namely π_p into $\mathbb{T}$ and $\sigma_{X \times C}$ into 2^G. We note that

$$\sigma_{X \times C}(x, y) = (X \times C)_{(x,y)} = C_y = \sigma_C(y).$$

From (5.10), we have $\sigma_{(X \times C)}(x, y) = \pi_p^{-1}(J_o)_{(x,y)} = \sigma_{J_o}(\pi_p(x, y))$, so since σ_{J_o} has a Borel measurable inverse, we can define $\sigma : (Y, \nu) \to (\mathbb{T}, m_\mathbb{T})$ by

$$\sigma(y) := (\sigma_{J_o}^{-1} \circ \sigma_C)(y) = \pi_p(x, y), \quad \text{for } \eta\text{-almost every } (x, y), \tag{5.11}$$

and since $\sigma_C^{-1}(U) = C$, we note that

$$\sigma^{-1}(J_o) = \sigma_C^{-1}(\sigma_{J_o}(J_o)) \subseteq \sigma_C^{-1}(U \cap \operatorname{Im} \sigma_{J_o}) \subset C.$$

Since $\sigma : Y \to \mathbb{T}$ is Borel and G-equivariant, and the G-action on $\mathbb{T}$ is uniquely ergodic, the push-forward of ν must equal $m_\mathbb{T}$. Since $m_\mathbb{T}(J_o) = \nu(C)$, we conclude that $\sigma^{-1}(J_o) = C$ modulo ν-null sets.

Of course, since we also know that

$$A \times Y = \pi_p^{-1}(I_o), \quad \text{modulo } \eta\text{-null sets,}$$

we could have done the same thing for the set $A \subset X$, and produced a G-factor map

$$\sigma' : (X, \mu) \to (\mathbb{T}, m_\mathbb{T}),$$

which again would have to coincide with $\pi_p(x, y)$ for η-almost every (x, y), and $\sigma'^{-1}(I_o) = A$ modulo μ-null sets. In particular, $\sigma = \sigma'$ η-almost everywhere.

The question of which η on $X \times Y$ that gives rise to such curious-looking identities as (5.11) naturally arises. The reader is invited to check that a relatively independent joining (as well as any ergodic component thereof) over a common factor Borel G-space, which has a further factor Borel G-space of the form $(\mathbb{T}, m_\mathbb{T})$ will do the job (see Section 6 in [15] for more details).

We can summarize these observations as follows.

Proposition 5.12. *Suppose that* G *is abelian,* $G \curvearrowright (Y, \nu)$ *is totally ergodic,*

$$\nu(A_{x_o}^{-1} C) = \mu(A) + \nu(C) < 1,$$

and condition (i) in Proposition 5.10 is satisfied. Then, for every ergodic joining η *of* (X, μ) *and* (Y, ν)*, there are*

   (i) *a torus compactification* $(\mathbb{T}, \tau_o)$ *and closed intervals* $I_o, J_o \subset \mathbb{T}$.
   (ii) G-*factor maps* $\alpha : (X, \mu) \to (\mathbb{T}, \mathfrak{m}_{\mathbb{T}})$ *and* $\beta : (Y, \nu) \to (\mathbb{T}, \mathfrak{m}_{\mathbb{T}})$,

such that $\eta\big(\{(x, y) : \alpha(x) = \beta(y)\}\big) = 1$, *and* $A = \alpha^{-1}(I_o)$ *and* $C = \beta^{-1}(J_o)$ *modulo null sets.*

6. Proofs of the main theorems

Our main dynamical results are now rather straightforward consequences of Proposition 5.9 and Proposition 5.10. However, in order to prove our main density results, we must deal with non-ergodic measures, which makes it necessary to investigate how well our ergodic-theoretical conclusions behave when passing to ergodic components.

6.1. General framework

In what follows, let

   - G be a countable amenable group,
   - ξ an extreme invariant mean on G, and
   - A is a subset of G with $\xi(A) > 0$.

As described in Subsection 2.1.1, we can associate to the set A a pointed G-space (X, x_o) and (abusing notation) a *clopen* subset $A \subset X$ such that $A = A_{x_o}$. From now on, we shall only work with the triple (X, x_o, A), and write A_{x_o} for the set in G.

By Proposition A.4, the G-invariant probability measure $\mu := S_{x_o}^* \xi$ on X is ergodic, and since $A \subset X$ is clopen, we have $\xi(A_{x_o}) = \mu(A)$ by Lemma A.2.

The notation X, x_o, A, ξ *and* μ *will be fixed throughout the rest of the section.*

6.2. Proofs of the dynamical results

To formulate our dynamical results, we need in addition to the objects introduced above, an *ergodic* Borel G-space (Y, ν) and a Borel set $B \subset Y$ with positive ν-measure. The following theorem generalizes Theorem 1.5.

Theorem 6.1. *Suppose that*

$$\nu(A_{x_o} B) < \min\big(1, \mu(A) + \nu(B)\big).$$

Then there exist a finite-index subgroup $G_o < G$, *an extreme invariant mean* λ *on* G *and* $A_o \subset A_{x_o}$ *with* $\lambda(A_o) = \mu(A)$ *such that* $G_o A_o \neq G$.

If one assumes that all finite quotients of G *are abelian, then there exists a proper finite index subgroup* $G_o < G$ *such that* $G_o A \neq G$, *and*

$$d^*(G_o A) < \mu(A) + \frac{1}{[G : G_o]}.$$

6.2.1. *Proof of Theorem 1.5 assuming Theorem 6.1*

Let us assume that
$$\nu(A_{x_o}B) < \min\left(1, \mu(A) + \nu(B)\right),$$
and choose ξ such that $\xi(A_{x_o}) = d^*(A_{x_o})$; this can be done by Proposition A.6. Theorem 6.1 now entails that there is a finite-index subgroup G_o, an extreme invariant mean λ on G, and a subset $A_o \subset A_{x_o}$ with $\lambda(A_o) = \mu(A)$ such that $G_o A_o \neq G$. Since
$$d^*(A_{x_o}) \geqslant d^*(A_o) \geqslant \lambda(A_o) = \mu(A) = \xi(A_{x_o}) = d^*(A_{x_o}),$$
and thus $d^*(A_o) = d^*(A_{x_o})$, we see that A_{x_o} is *not* spread-out, contradicting Assumption (i) in Theorem 1.5. Assuming that all finite quotients of G are abelian, Assumption (ii) in Theorem 1.5 is violated along the same lines using Theorem 6.1.

6.2.2. *Proof of Theorem 6.1*

Set $C = (A_{x_o}B)^c$, and note that $A_{x_o}^{-1}C \subset B^c$. Since $1 - \nu(C) < \mu(A) + \nu(B)$ and $\nu(B) > 0$, we get
$$\nu(A_{x_o}^{-1}C) \leqslant 1 - \nu(B) < \min(1, \mu(A) + \nu(C)),$$
which places us in the setting of Proposition 5.9, and all properties of $A_{x_o} \subset G$ in Theorem 6.1 readily follow from this.

6.2.3. *Proof of Theorem 1.8*

The following theorem generalizes Theorem 1.8 in the same way that Theorem 6.1 generalizes Theorem 1.5, so we omit the proof of the latter.

Theorem 6.2. *Suppose that*
$$\nu(A_{x_o}B) = \mu(A) + \nu(B) < 1.$$
If A_{x_o} is spread-out, and $A_{x_o}B$ does not contain, modulo ν-null sets, a Borel set with positive measure which is invariant under a finite-index subgroup $G_o < G$, then A_{x_o} is contained in a Sturmian set S with $\xi(S) = \xi(A_{x_o}) = \mu(A)$.

Furthermore, if one in addition assumes that G is abelian and $G \curvearrowright (Y, \nu)$ is totally ergodic, then, for every ergodic joining η of (X, μ) and (Y, ν), there exist

(i) a torus compactification $(\mathbb{T}, \tau_o)$ and closed intervals $I_o, J_o \subset \mathbb{T}$.
(ii) G-factor maps $\alpha : (X, \mu) \rightarrow (\mathbb{T}, m_{\mathbb{T}})$ and $\beta : (Y, \nu) \rightarrow (\mathbb{T}, m_{\mathbb{T}})$,

such that

- $\eta(\{(x, y) : \alpha(x) = \beta(y)\}) = 1$,
- $A = \alpha^{-1}(I_o)$ *and* $B = \beta^{-1}(J_o)$ *modulo null sets.*

6.2.4. *Proof of Theorem 6.2*

Set $C = (A_{x_o}B)^c$, and note that $A_{x_o}^{-1}C \subset B^c$. Since $1 - \nu(C) = \mu(A) + \nu(B) < 1$ and $\nu(B) > 0$, we get $\nu(A_{x_o}^{-1}C) \leqslant 1 - \nu(B) = \mu(A) + \nu(C) < 1$. Since A_{x_o} is assumed to be spread-out, we see by Proposition 5.9 that the first inequality cannot be strict, whence
$$\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1.$$
which places us in the setting of Proposition 5.10 and Proposition 5.12.

Assume that there exists a finite-index subgroup $G_o < G$ such that $\nu(G_o C) < 1$. Then, by taking complements, we see that the Borel set $D := \bigcap_{g \in G_o} gA_{x_o}B \subset X$ has positive ν-measure

and is G_o-invariant, contradicting our second assumption in Theorem 6.2. Hence the conditions of Proposition 5.10 are satisfied, and the conclusions about the set A_{x_o} follow.

Let us now assume that G is abelian and $G \curvearrowright (Y, \nu)$ is totally ergodic. Let us also fix an ergodic joining η of (X, μ) and (Y, ν). Since $\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1$, Proposition 5.12 ensures the existence of a torus compactification $(\mathbb{T}, \tau_o)$ and G-factor maps

$$\alpha : (X, \mu) \to (\mathbb{T}, m_{\mathbb{T}}) \quad \text{and} \quad \beta : (Y, \nu) \to (\mathbb{T}, m_{\mathbb{T}})$$

such that $\eta(\{(x, y) : \alpha(x) = \beta(y)\}) = 1$ holds, and closed intervals $I_o, H_o \subset \mathbb{T}$ such that

$$A = \alpha^{-1}(I_o) \quad \text{and} \quad C = \beta^{-1}(H_o),$$

modulo null sets. Furthermore, upon going into the arguments of Proposition 5.10, we see that there is an element $t_o \in \mathbb{T}$ such that $A_{x_o} \subset \tau_o^{-1}(I_o t_o^{-1})$. It follows from chain of identities above that

$$A_{x_o}^{-1}C = B^c \subset \tau_o^{-1}(t_o I^{-1})\beta^{-1}(H_o),$$

modulo null sets. It is also not hard to see that since both I_o and H_o are intervals and β is G-equivariant, we have

$$\tau_o^{-1}(t_o I^{-1})\beta^{-1}(H_o) = \beta^{-1}(t_o I_o^{-1} H_o)$$

modulo null sets. Since $J_o := (t_o I_o^{-1} H_o)^c$ is again an interval in $\mathbb{T}$ with

$$m_{\mathbb{T}}(J_o) = 1 - m_{\mathbb{T}}(I_o^{-1} H_o) = 1 - m_{\mathbb{T}}(I_o) - m_{\mathbb{T}}(H_o) = \nu(B),$$

we conclude that $B = \beta^{-1}(J_o)$ modulo null sets.

Remark 6.3. Note that in both proofs above, we went from action sets of the form $A_{x_o}B$ to actions sets of the form $A_{x_o}^{-1}C$. This was of course done so that the results in the previous section could be applied, but it is natural to ask whether this conversion is necessary - surely this is not the case for *abelian* G. The necessity for this twist can be backtracked to Lemma 5.2; in the proof of this lemma, we heavily use that the map $x \mapsto \nu(A_x^{-1}C)$ is G-invariant. This is not the case for the map $x \mapsto \nu(A_x B)$, unless of course G is abelian.

6.3. **Proofs of the density results**

We retain the notation for X, x_o, A, ξ and μ introduced in the beginning of the section. We shall further fix a subset $B \subset G$, and add conditions on it as we go along. We associate to B a pointed G-space (Y, y_o) so that there is a clopen set $B \subset Y$ (abuse of notation) such that $B = B_{y_o}$, where the set B on the left is in G, and the set B on the right is the clopen set in B. To avoid this abuse of notation, we shall from now on only refer to the set in G as B_{y_o}.

6.3.1. *Proofs of Theorem 1.13 and Theorem 1.15*

By Proposition A.6, we can find *extreme* invariant means λ_+ and λ_- on G such that

$$\lambda_+(B_{y_o}) = d^*(B_{y_o}) \quad \text{and} \quad \lambda_+(A_{x_o}B_{y_o}) \leqslant d^*(A_{x_o}B_{y_o}) \tag{6.1}$$

and

$$\lambda_-(B_{y_o}) \geqslant d_*(B_{y_o}) \quad \text{and} \quad \lambda_-(A_{x_o}B_{y_o}) = d_*(A_{x_o}B_{y_o}). \tag{6.2}$$

By Proposition A.4, the G-invariant probability measures $\nu_+ = S_{y_o}^* \lambda_+$ and $\nu_- = S_{y_o}^* \lambda_-$ are *ergodic*, and by Lemma A.2 we have

$$\nu_+(B) = d^*(B_{y_o}) \quad \text{and} \quad \nu_+(A_{x_o}B) \leqslant d^*(A_{x_o}B_{y_o}) \tag{6.3}$$

and
$$\nu_-(B) \geqslant d_*(B_{y_o}) \quad \text{and} \quad \nu_-(A_{x_o}B_{y_o}) \leqslant d_*(A_{x_o}B_{y_o}). \tag{6.4}$$

If we enforce the assumptions in Theorem 1.13 on the set A_{x_o}, then Theorem 1.5 readily implies Theorem 1.13.

Towards the proof of Theorem 1.15, suppose that A_{x_o} is spread-out, B_{y_o} large and $A_{x_o}B_{y_o}$ does not contain a piecewise periodic set. If $d^*(A_{x_o}B_{y_o}) = d^*(A_{x_o}) + d^*(B_{y_o}) < 1$, then, with the notation above,
$$\nu_+(A_{x_o}B) \leqslant d^*(A_{x_o}) + \nu_+(B)$$

Since A_{x_o} is spread-out and ν_+ is ergodic, Theorem 1.5 shows that the inequality cannot be strict and thus $\nu_+(A_{x_o}) = d^*(A_{x_o}) + \nu_+(B) < 1$. At this point, Theorem 1.8 tells us A_{x_o} is contained in a Sturmian set with the same upper Banach density as A_{x_o}, *provided* that $A_{x_o}B$ does not contain a Borel set Z which is invariant under a finite-index subgroup G_o. To prove that this is not the case, we argue by contradiction, and apply Lemma 3.5 to the ergodic G-space (Y, ν) with $Q = Z$ and $U = A_{x_o}B \subset Y$. We conclude that there is a non-empty G_o-invariant set $Q_o \subset G$ and a thick set $T \subset G$ such that
$$U_{y_o} = A_{x_o}B_{y_o} \supset Q_o \cap T,$$
contradicting our assumption that $A_{x_o}B_{y_o}$ does not contain a piecewise periodic set.

6.3.2. *How to deal with asymptotic densities*

As we have just seen, Theorem 1.13 and Theorem 1.15 are rather direct consequences of Theorem 1.5 and Theorem 1.8. The main reason for this is that the measures ν_+ and ν_- on Y that we end up with are *ergodic* - or, equivalently, the maximixing/minimizing invariant means λ_+ and λ_- are extreme points in $\mathcal{L}_G$. This will no longer be the case when we study asymptotic densities.

Towards the proofs of Theorem 1.16 and Theorem 1.18, we shall begin by fixing a Følner sequence (F_n) in G once and for all. We can then use (A.4) and (A.5) to produce (not necessarily extreme) invariant means $\overline{\lambda}$ and $\underline{\lambda}$ such that
$$\overline{\lambda}(B) = \overline{d}_{(F_n)}(B_{y_o}) \quad \text{and} \quad \overline{\lambda}(A_{x_o}B_{y_o}) \leqslant \overline{d}_{(F_n)}(A_{x_o}B_{y_o})$$
and
$$\underline{\lambda}(B) \geqslant \underline{d}_{(F_n)}(B_{y_o}) \quad \text{and} \quad \underline{\lambda}(A_{x_o}B_{y_o}) = \underline{d}_{(F_n)}(A_{x_o}B_{y_o}).$$

If we write $\overline{\nu} = S_{y_o}^*\overline{\lambda}$ and $\underline{\nu} = S_{y_o}^*\underline{\lambda}$, then by Lemma A.2,
$$\overline{\nu}(B) = \overline{d}_{(F_n)}(B_{y_o}) \quad \text{and} \quad \overline{\nu}(A_{x_o}B) \leqslant \overline{d}_{(F_n)}(A_{x_o}B_{y_o}), \tag{6.5}$$
and
$$\underline{\nu}(B) \geqslant \underline{d}_{(F_n)}(B_{y_o}) \quad \text{and} \quad \underline{\nu}(A_{x_o}B) \leqslant \underline{d}_{(F_n)}(A_{x_o}B_{y_o}). \tag{6.6}$$

It will be useful in the subsequent arguments to recast our assumptions on B_{y_o} and AB_{y_o} in Theorem 1.16 and Theorem 1.18 as properties of sets in Y. The following lemma does so.

Lemma 6.4. *With the notation and assumptions above,*

    (i) B_{y_o} *is syndetic* $\iff \nu(B) > 0$ *for all* $\nu \in \mathcal{P}_G(Y)$.
    (ii) $A_{x_o}B_{y_o}$ *is not thick* $\implies \nu(A_{x_o}B) < 1$ *for all* $\nu \in \mathcal{P}_G(Y)$.
    (iii) $A_{x_o}B_{y_o}$ *does not contain a piecewise periodic set* $\implies$ *For every* $\nu \in \mathcal{P}_G^{erg}(Y)$, $A_{x_o}B$ *does not contain a Borel set with positive ν-measure which is invariant under a finite-index subgroup.*

Proof. (i) and (ii) are immediate consequences of Lemma A.2 and Lemma A.7.

(iii) Assume that there exists $\nu \in \mathcal{P}_G^{erg}(Y)$ and a Borel set $Z \subset A_{x_o}B$ with positive ν-measure, which is invariant under a finite-index subgroup. Apply Lemma 3.5 to the ergodic G-space (Y, ν) with $Q = Z$ and $U = A_{x_o}B$. We conclude that there is a G_o-invariant set $Q_o \subset G$ and a thick set $T \subset G$ such that

$$U_{y_o} = A_{x_o}B_{y_o} \supset Q_o \cap T,$$

showing in particular that $A_{x_o}B_{y_o}$ contains a piecewise periodic set. $\square$

6.3.3. *Ergodic decompositions*

In what follows, let ν be a G-invariant (not necessarily ergodic) Borel probability measure on Y. In the applications that will follow, we will consider the cases $\nu = \overline{\nu}$ and $\nu = \underline{\nu}$. It is a standard fact in ergodic theory (see for instance Theorem 4.8 in [12]), that one can decompose ν into *ergodic components*, that is to say, there exists a probability measure κ on $\mathcal{P}_G(Y)$, which is concentrated on the set of ergodic measures, such that

$$\nu(D) = \int_{\mathcal{P}^{erg}(Y)} \nu'(D)\, d\kappa(\nu'), \quad \text{for all Borel sets } D \subset Y. \tag{6.7}$$

6.3.4. *Proof of Theorem 1.16*

Recall that ν is fixed. Let us assume that

 (i) A_{x_o} is spread-out (or, if every finite quotient of G is abelian, that (1.2) does not hold for any finite-index subgroup G_o).
 (ii) B_{y_o} is syndetic; Lemma 6.4 implies that $\nu'(B) > 0$ for all $\nu' \in \text{supp}(\kappa)$.
 (iii) $A_{x_o}B_{y_o}$ is not thick; Lemma 6.4 implies that $\nu'(A_{x_o}B) < 1$ for all $\nu' \in \text{supp}(\kappa)$.

By Theorem 1.5, applied to each ergodic component ν' of ν, we can now conclude

$$1 > \nu'(A_{x_o}B) \geqslant \mu(A) + \nu'(B),$$

and thus, by (6.7),

$$\begin{aligned} \nu(A_{x_o}B) &= \int_{\mathcal{P}^{erg}(Y)} \nu'(A_{x_o}B)\, d\kappa(\nu') \\ &\geqslant \int_{\mathcal{P}^{erg}(Y)} (\mu(A) + \nu'(B))\, d\kappa(\nu') \\ &= \mu(A) + \nu(B). \end{aligned}$$

If we apply these inequalities to the measures $\overline{\nu}$ and $\underline{\nu}$, together with (6.5) and (6.6), Theorem 1.16 readily follows (by choosing μ, or equivalently, ξ, so that $\mu(A) = \xi(A_{x_o}) = d^*(A_{x_o})$).

6.3.5. *Proof of Theorem 1.18*

Let us now assume that

 (i) A_{x_o} is spread-out.
 (ii) B_{y_o} is syndetic; Lemma 6.4 implies that $\nu'(B) > 0$ for all $\nu' \in \text{supp}(\kappa)$.
 (iii) $A_{x_o}B_{y_o}$ does not contain a piecewise periodic subset; Lemma 6.4 implies that for every $\nu' \in \text{supp}(\kappa)$, the action set $A_{x_o}B$ does not contain a Borel set with positive ν'-measure which is invariant under a finite-index subgroup.
 (iv) $\nu(A_{x_o}B) = \mu(A) + \nu(B) < 1$.

We claim that
$$\nu'(A_{x_o}B) = \mu(A) + \nu'(B) < 1, \quad \text{for } \kappa\text{-a.e. } \nu'.$$
Indeed, by Theorem 1.5 applied to each ν' (using the assumption that A_{x_o} is spread-out), we know that $\nu'(A_{x_o}B) \geqslant \mu(A) + \nu'(B)$ for κ-a.e. ν'. However, by (6.7) and our assumption (iv) above, we also have
$$\int_{\mathcal{P}(Y)} \nu'(A_{x_o}B) \, d\kappa(\nu') = \nu(A_{x_o}B) = \mu(A) + \nu(B) = \int_{\mathcal{P}(Y)} \big(\mu(A) + \nu'(B)\big),$$
so the inequality $\nu'(A_{x_o}B) \geqslant \mu(A) + \nu'(B)$ cannot be strict on a set of positive κ-measure.

We now see that all conditions of Theorem 1.8 are satisfied for every ergodic component ν', and thus we conclude that A_{x_o} is contained in a Sturmian set S with upper Banach density equal to $\mu(A)$. When μ (or equivalently, ξ) is chosen so that $\xi(A_{x_o}) = d^*(A_{x_o})$, Theorem 1.18 follows.

6.4. **Some auxiliary consequences of our arguments (optional)**

Let us now go back all the way to Theorem 5.1 and, as promised, harvest some additional results from the "Correspondence Principle" proved there.

Theorem 6.5. *Let* G *be a finitely generated amenable group, and suppose that* $G \curvearrowright (Y,\nu)$ *is an ergodic Borel* G-*space. Let* $A \subset G$ *be a large set and* $B \subset Y$ *a Borel set with positive* ν-*measure.*

(i) If G *is simple, then* $\nu(AB) = 1$.
(ii) If G *is a torsion group and* $A \subset G$ *is spread-out, then* $\nu(AB) = 1$.

Proof. Let (X, x_o) be the hull associated to A, abuse notation (as many times before) and denote by A the clopen set in X so that $A = A_{x_o}$. Fix a Borel set $B \subset Y$ with positive ν-measure, and set $C = (A_{x_o}B)^c$. If we assume that $\nu(A_{x_o}B) < 1$, then
$$\nu(C) > 0 \quad \text{and} \quad \nu(A_{x_o}^{-1}C) < \nu(B^c) < 1.$$
Let I and J denote the sets in Theorem 5.1.

(i) If G is simple, then Lemma B.1 (iv) shows that any group compactification (K, τ) of G is trivial, and thus $1 = m_K(I^{-1}J) \leqslant \nu(A_{x_o}^{-1}C)$ by Theorem 5.1.

(ii) If G is a torsion group, then Lemma B.1 (iii) shows that any group compactification (K, τ) of G is totally disconnected. If A_{x_o} is spread-out, then it follows from Proposition 5.8 that the set $I \subset K$ cannot be sub-periodic. In particular, for every open subgroup $U < K$, we have $UI = K$. Since the open subgroups form a neighborhood basis of the identity, this implies in turn that I must be dense in K, and thus $m_K(I^{-1}J) = 1$ by Theorem 5.5 (i), which, via Theorem 5.1, shows that $\nu(A_{x_o}^{-1}C) = 1$ $\qquad\qquad\square$

Remark 6.6. In fact, the proof of (ii) gives a bit more: If G is a finitely generated amenable torsion group and $G \curvearrowright (Y, \nu)$ is a (non-trivial) totally ergodic Borel G-space, then it is weakly mixing. Indeed, if $G \curvearrowright (Y, \nu)$ is *not* weakly mixing, then by Lemma 2.2, there exist

(i) a metrizable group compactification (K, τ) of G, and a closed *proper* subgroup $L < K$.
(ii) a G-factor map $\pi \colon (Y, \nu) \to (K/L, m_{K/L})$.

By assumption, every finite-index subgroup $G_o < G$ acts ergodically on (Y, ν) and thus also on $(K/L, m_{K/L})$. It is not hard to see that this implies that $UL = K$ for every *open* subgroup $U < K$. By Lemma B.1 (iii), K is totally disconnected, so we can find a decreasing chain (U_n) of open subgroups such that $\bigcap_n U_n = \{e_K\}$. Since L is proper, there exists $t \in K$ such that $tL \cap L = \emptyset$, and since $\bigcap_n tL \cap U_n = \emptyset$, we can by compactness find n such that $tL \cap U_n = \emptyset$, and thus $t \notin U_nL$, which is a contradiction. We conclude that $L = K$, and thus K/L is trivial, whence $G \curvearrowright (Y, \nu)$ is weakly mixing.

The following corollary is immediate.

Corollary 6.7. *Let G be a finitely generated amenable group, and let $A \subset G$ be a large set. Then $d^*(AB) = 1$ for every large set $B \subset G$ if either G is simple, or if G is a torsion group and A is spread-out.*

Remark 6.8. The first assertion (when G is simple) was essentially proved by Bergelson and Furstenberg in [1] - they prove the same result under the assumption that G is minimally almost periodic, that is to say, G admits no non-trivial group compactification (this is the only property that we use as well).

7. COUNTEREXAMPLE MACHINE FOR SEMI-DIRECT PRODUCTS

We develop a "machine" which supplies counterexamples to certain conjectural "symmetrized" versions of our main results concerning upper Banach densities of product sets in groups which are far from being abelian.

7.1. **General setting**

Throughout this section, let G be a countable group which is a product of two distinguished subgroups N and L, where N is *abelian* and *normal* in G, and $N \cap L = \{e_G\}$. We shall assume that there is a proper *finitely generated* subgroup Λ of N with the property that for *every* finite subset $F \subset N$, there is an element $l \in L$ such that $lFl^{-1} \subset \Lambda$. The reader can verify that these assumptions imply that

- G is amenable $\iff$ L is amenable.
- N is *not* finitely generated.

The two main examples to keep in mind are

$$G = \mathbb{Z}[1/p] \rtimes \langle p \rangle \quad \text{and} \quad L \cong \langle p \rangle \quad \text{and} \quad N \cong \mathbb{Z}[1/p] \quad \text{and} \quad \Lambda \cong \mathbb{Z}, \tag{7.1}$$

for some prime number p, which acts by multiplication on $\mathbb{Z}[1/p]$, and

$$G = \mathbb{Q} \rtimes \mathbb{Q}^* \quad \text{and} \quad L \cong \mathbb{Q}^* \quad \text{and} \quad N \cong \mathbb{Q} \quad \text{and} \quad \Lambda \cong \mathbb{Z}. \tag{7.2}$$

In both of these examples, the group G is two-step solvable, and hence amenable.

We shall from now on assume that L, and hence G, is amenable. Also, to avoid confusion, we denote by d_G^* and d_L^* the upper Banach densities on G and L respectively. Towards the proofs of Theorem 1.20 and Theorem 1.21 we record in the next proposition some peculiar behaviors of the sets

$$S = L\Lambda \quad \text{and} \quad T = (S^{-1}S)^c \tag{7.3}$$

with respect to the upper Banach density on G.

Proposition 7.1. *With the notation and assumptions above, we have*

(i) $d_G^*(S) = d_G^*(T) = 1$.

(ii) For any $A_o, B_o \subset L$, we have $d_G^*(AB) \leqslant d_L^*(A_o B_o)$, where

$$A = NA_o \cap S \quad \text{and} \quad B = (NB_o \cap T) \sqcup \{e_G\}.$$

Property (i) will be proved below. Towards the proof of (ii), we note that

$$\begin{aligned} AB &= (NA_o \cap S)(NB_o \cap T) \cup (NA_o \cap S) \\ &\subseteq (NA_o B_o \cap ST) \sqcup (NA_o \cap S). \end{aligned}$$

Furthermore, since S is left L-invariant and every element $g \in G$ can be written on the form nl for some $n \in N$ and $l \in L$, we see that for every invariant mean λ on G and $g \in G$,

$$\lambda(NA_oB_o \cap gST) = \lambda(NA_oB_o \cap ST) \quad \text{and} \quad \lambda(NA_o \cap gS) = \lambda(NA_o \cap S).$$

In particular, if we assume that λ is extreme in $\mathcal{L}_G$, then Proposition A.1 implies that

$$\lambda(NA_oB_o \cap ST) = \lambda(NA_oB_o)\lambda(ST) \quad \text{and} \quad \lambda(NA_o \cap S) = \lambda(NA_o)\lambda(S).$$

We note that $\lambda_o(\cdot) = \lambda(N\cdot)$ defines an invariant mean on L (this is simply the push-forward of λ under the quotient map $G \to G/N$), and thus $\lambda(NA_oB_o) \leqslant d_L^*(A_oB_o)$ by Proposition A.6. Hence, for every extreme invariant mean λ on G,

$$
\begin{aligned}
\lambda(AB) \;&\leqslant\; \lambda(NA_oB_o \cap ST) + \lambda(NA_o \cap S) \\
&\leqslant\; \lambda(NA_oB_o)\,\lambda(ST \sqcup S) \leqslant \lambda(NA_oB_o) \\
&\leqslant\; d_L^*(A_oB_o),
\end{aligned}
$$

where the second inequality follows from monotonicity of λ and the fact that $ST \cap S = \emptyset$. Finally, Proposition A.6 allows us to pick an extreme λ in $\mathcal{L}_G$ such that $\lambda(AB) = d_G^*(AB)$, which finishes the proof of (ii).

Let us now turn to the proof of (i). To prove that $d_G^*(S) = 1$ it suffices by Lemma A.7 (i) to show that for every finite subset $F \subset G$, there exists $g \in G$ such that $Fg \subset S$. In our setting, we may without loss of generality assume that F is of the form $F_L F_N$ where $F_L \subset L$ and $F_N \subset N$ are finite sets. By our assumptions on L, N and Λ, we can now find $l \in L$ such that $lF_N l^{-1} \subset \Lambda$, and thus

$$Fl^{-1} = (F_L l^{-1})(lF_N l^{-1}) \subset L\Lambda = S,$$

which finishes the proof that $d_G^*(S) = 1$. Towards the proof of the second claim, we note that in order to show that $d^*(T) = 1$, or equivalently, $d_*(S^{-1}S) = 0$, it suffices by Lemma A.7 (ii) to show that there is *no* finite subset $F \subset G$ such that $FS^{-1}S = F\Lambda L\Lambda = G$; in particular, it would be enough to show that there is no finite set of the form $F_N F_L$, where $F_N \subset N$ and $F_L \subset L$ are finite subsets, such that

$$F_N F_L \Lambda L\Lambda \cap N = N.$$

To reach a contradiction, let us assume that such sets F_N and F_L exist, and note that since the intersection $N \cap L$ is trivial, this implies that

$$F_N F_L \Lambda L\Lambda \cap N = F_N \Big(\bigcup_{l \in F_L} l\Lambda l^{-1} \Big)\Lambda = N. \tag{7.4}$$

By assumption, Λ is generated by some *finite* set Q_o, so we conclude from (7.4) that N must be generated by the finite set $F_N \cup \bigcup_{l \in F_L} lQ_o l^{-1} \cup Q_o$. However, we observed already in the beginning of the section that under our assumptions, N cannot be finitely generated. This finishes the proof of (i).

7.2. **Constructing counterexamples**

The counterexamples in Theorem 1.20 and Theorem 1.21 are constructed along similar lines, so let us begin by describing their basic structures. Let (K, τ_o) be a metrizable group compactification of L; since N is normal, we may extend it to a group compactification (K, τ) of G by setting $\tau(nl) = \tau_o(l)$ for $n \in N$ and $l \in L$. Let $I, J \subset K$ be closed m_K-Jordan measurable subsets which are equal to the closures of their interiors, and such that IJ is m_K-Jordan measurable as well. Set

$$A = \tau^{-1}(I) \cap S \quad \text{and} \quad B = (\tau^{-1}(J) \cap T) \sqcup \{e_G\}. \tag{7.5}$$

We note that A and B are constructed from the sets $A_o = \tau_o^{-1}(I)$ and $B_o = \tau_o^{-1}(J)$ exactly as in Proposition 7.1, whence

$$d_G^*(AB) \leqslant d_L^*(A_o B_o) \leqslant d_L^*(\tau_o^{-1}(IJ)) = m_K(IJ), \qquad (7.6)$$

where we in the last equality used Corollary A.3 and our assumption IJ is m_K-Jordan measurable. The lemma below records some further important properties of these sets.

Lemma 7.2. *With the notation and assumptions above, we have:*
  (i) $d_G^*(A) = m_K(I)$ *and* $d_G^*(B) = m_K(J)$
  (ii) Let $G_o < G$ *be a finite-index subgroup:*
     (a) If $\tau(G_o)J = K$, *then* $G_o B = G$.
     (b) If J *has trivial stabilizer and* $\tau(G_o) \cap J = \emptyset$, *then*

$$d_G^*(G_o B) > d_G^*(B) + \frac{1}{[G : G_o]}.$$

  (iii) If K *is connected, then* A *and* B *are spread-out in* G, *and, if in addition,* $IJ \cup I \neq K$, *then* AB *does not contain a piecewise periodic set.*
  (iv) If $e_K \notin J$, *then* B *is not contained in a Sturmian set in* G *with the same upper Banach density.*

Proof. (i) Since $d_G^*(S) = d_G^*(T) = 1$, we can by Proposition A.6 find invariant means λ_1 and λ_2 on G such that $\lambda_1(S) = \lambda_2(T) = 1$, and thus, in combination with Corollary A.3,

$$\lambda_1(A) = \lambda_1(\tau^{-1}(I)) = m_K(I) \quad \text{and} \quad \lambda_2(A) = \lambda_2(\tau^{-1}(J)) = m_K(J).$$

Clearly, $A \subset \tau^{-1}(I)$ and $B \subset \tau^{-1}(J) \cup \{e_G\}$, whence $d_G^*(A) \leqslant m_K(I)$ and $d_G^*(B) \leqslant m_K(J)$, which finishes the proof in view of Proposition A.6.

(ii) Since T is thick, Lemma B.2 (ii) tells us that $G_o(\tau^{-1}(J) \cap T) = \tau^{-1}(\tau(G_o)J)$, and thus $G_o B = \tau^{-1}(\tau(G_o)J) \cup G_o$. This finishes (a). If the conditions in (b) hold, then we claim that the union is disjoint; if not, $J \cap \tau(G_o) \neq \emptyset$, contradicting our assumption. Hence, for any invariant mean λ on G,

$$\lambda(G_o B) \geqslant \lambda(\tau^{-1}(\tau(G_o)J)) + \lambda(G_o) > \lambda(\tau^{-1}(J)) + \frac{1}{[G : G_o]} = m_K(J) + \frac{1}{[G : G_o]},$$

where we in the last equality used Corollary A.3, and in the second inequality the fact that m_K-Jordan measurability and trivial stabilizer of J implies that $J' = \tau(G_o)J \setminus J$ contains a non-empty open set in K, and thus $\lambda(\tau^{-1}(J'))$ is strictly positive by Lemma A.2 (iii). The fact that $\lambda(G_o) = 1/[G : G_o]$ is left to the reader.

(iii) First note that $AB \subset \tau^{-1}(IJ \cup I)$, so if AB contains $Q \cap U$ for some right G_o-invariant set Q and thick set U, then $\overline{\tau(Q \cap U)} \subset IJ \cup I \neq K$. However, this contradicts Lemma B.2 (iii). It thus remains to show that A and B are spread-out. To do this, first note that if $A' \subset A$ has the same upper Banach density, then we can write $A' = \tau^{-1}(I) \cap U_I$ for some thick set $U_I \subset G$. We wish to prove that $G_o A' = G$ for any finite-index subgroup G_o of G. By Lemma B.2 (ii), $G_o A' = \tau^{-1}(\tau(G_o)I) = G$, where the last identity follows since K is connected and thus the image of G_o under τ is dense in K. Similarly, any subset $B' \subset B$ with the same upper Banach density can be written of the form $\tau^{-1}(J) \cap U_J$ for some thick set U_J, possibly adding e_G depending on whether it belongs to B' or not. Again, by Lemma B.2 (ii) and the fact that K is connected, $G_o B' = G$, so B is spread-out.

(iv) Suppose that (M, θ) is a group compactification of G and $J' \subset M$ a closed m_M-Jordan measurable subset equal to the closure of its interior such that

$$m_M(J') = d_G^*(\theta^{-1}(J')) = d_G^*(B) = m_K(J) \quad \text{and} \quad B = (\tau^{-1}(J) \cap T) \sqcup \{e_G\} \subset \theta^{-1}(J').$$

We wish to show that such M, τ and J' cannot exist, proving in particular that B cannot be contained in a Sturmian set with the same upper Banach density. To disprove existence, assume that these things exist, and consider the homomorphism $\xi : G \to K \times M$ defined by $\xi(g) = (\tau(g), \theta(g))$, and denote by E the closure of $\xi(G)$ in $K \times M$. Then (E, ξ) is a group compactification of G and E maps onto both K and M. We set

$$C = (J \times M) \cap E \quad \text{and} \quad D = (K \times J') \cap E,$$

and leave it to the reader to show that C and D are closed, m_E-Jordan measurable and equal to the closures of their interiors in E. Furthermore, $m_E(C) = m_K(J) = m_M(J') = m_E(D)$. Since $\xi^{-1}(C) = \tau^{-1}(J)$ and $\xi^{-1}(D) = \theta^{-1}(J')$, we see that $\xi^{-1}(C) \sqcup \{e_G\} \subset \xi^{-1}(D)$, whence

$$C = \overline{C^o} \subset \overline{C \cap \xi(G)} \subset D.$$

In particular, since C and D are m_E-Jordan measurable and $m_E(C) = m_E(D)$, the open set $D^o \setminus C$ is null, and thus empty, whence $D^o \subset C$. Since D equals the closure of its interior, we conclude that $C = D$. However, going back a few lines, we see that this implies that $e_E \in C$, and thus $e_K \in J$, which contradicts our assumption. $\qquad\square$

7.2.1. *Proof of Theorem 1.20*

Suppose that L admits a connected group compactification (K_1, τ_1) and an index two subgroup L_2, for instance, we could take G, L, N and Λ as in (7.1) with $L_2 = \langle p^2 \rangle$. Let $K = L/L_2 \times K_1$ and define $\tau_o : L \to K$ by $\tau_o(l) = (lL_2, \tau_1(l))$. Since K_1 is connected, (K, τ_o) is a group compactification of L.

Fix $0 < \varepsilon < 1/2$ and choose a closed m_{K_1}-Jordan measurable subset $J_1 \subset K_1$ with $m_{K_1}(J_1) = 2\varepsilon$, equal to the closure of its interior. Pick an element δ in L such that $\delta L_2 \cap L_2 = \emptyset$, and set

$$I = L_2 \times K_1 \quad \text{and} \quad J = \delta L_2 \times J_1.$$

One checks that I and J are closed m_K-Jordan measurable sets, equal to the closures of their interiors, and $m_K(I) = 1/2$ and $m_K(J) = \varepsilon$. Moreover, J has trivial stabilizer in K, and $IJ = \delta L_2 \times K_1$, which is again m_K-Jordan measurable, and $m_K(IJ) = 1/2$. Let A and B be as in (7.5). Then, by Lemma 7.2 (i) and (7.6),

$$m_K(I) = d_G^*(A) \leqslant d_G^*(AB) \leqslant m_K(IJ) = \frac{1}{2} < d_G^*(A) + d_G^*(B) < 1,$$

which finishes the first part of Theorem 1.20. For the second part, note that if $G_o < G$ is any finite-index subgroup of G, then, since K_1 is connected, we have either $\overline{\tau(G_o)} = L_2 \times K_1$ or $\overline{\tau(G_o)} = K$. Indeed, if the first case does not occur, then the closed subgroup $\overline{\tau(G_o)} < K$ projects onto both L/L_2 and K_1. If this subgroup is not equal to K, then Goursat's Lemma tells us that K_1 admits a non-trivial homomorphism onto L/L_2 which is absurd.

In the case when $\overline{\tau(G_o)} = L_2 \times K_1$, then $\tau(G_o) \cap J = \emptyset$, so by Lemma 7.2 (ii,b),

$$d_G^*(G_o B) > d_G^*(B) + \frac{1}{[G : G_o]}.$$

In the case when $\overline{\tau(G_o)} = K$, then $\tau(G_o)J = K$, so by Lemma 7.2 (ii,a), we have $G_o B = G$, which finishes the proof of Theorem 1.20.

7.2.2. *Proof of Theorem 1.21*

Suppose that L admits a homomorphism $\tau_o : L \to K \cong \mathbb{T}$ with dense image; either Example (7.1) or Example (7.2) would do. Pick a closed interval $I \subset \mathbb{T}$ with $m_{\mathbb{T}}(I) < 1/3$, such that $(I+I) \cap I = \emptyset$; in particular, $0 \notin I$. Let A and B be as in (7.5) with $I = J$. By Lemma 7.2 (iii) and (iv), A and B are both spread-out, but B is not contained in a Sturmian set with the same upper Banach density as B. Furthermore, by Lemma 7.2 (i) and (7.6),

$$d_G^*(AB) \leqslant m_K(I+I) = 2m_K(I) = d_G^*(A) + d_G^*(B) < 1.$$

Since A is spread-out, Theorem 1.13 tells us that the first inequality cannot be strict, so (A, B) provides the counterexample in Theorem 1.21.

APPENDIX A. INVARIANT MEANS AND FURSTENBERG'S CORRESPONDENCE PRINCIPLE

We define amenable groups, invariant means and asymptotic densities along Følner sequences. We also state Furstenberg's well-known Correspondence Principle in a slightly unorthodox form, and list some of its useful applications.

A.1. **Amenable groups and invariant means**

Throughout this section, let G be a countable group. We denote by $\ell^\infty(G)$ the Banach space of real-valued bounded functions on G, endowed with the uniform norm. If λ belongs to the dual of $\ell^\infty(G)$, then we shall twice abuse notation when we refer to this element. Firstly, we shall identify λ with the bounded finitely additive measure λ on G, defined by $\lambda(A) = \lambda(\chi_A)$ for $A \subset G$. Secondly, if $f \in \ell^\infty(G)$, it will sometimes be convenient to write

$$\lambda(f) = \int_G f(g) \, d\lambda(g),$$

although the right hand side is not an integral in the Lebesgue sense. We denote by $\mathcal{M}_G$ the weak*-closed and convex set of positive and unital functionals on $\ell^\infty(G)$, so called *means* on G. We say that $\lambda \in \mathcal{M}_G$ is *invariant* if $\lambda(gA) = \lambda(A)$ for all $g \in G$. We denote by $\mathcal{L}_G$ the (possibly empty) set of invariant means on G. We say that G is *amenable* if $\mathcal{L}_G$ is non-empty, and we refer to [26] for more information about this class of groups. It suffices for now to say that every solvable group is amenable, as is every locally finite group and every group of sub-exponential growth. On the other hand, any group which contains a free subgroup on more than two generators is *not* amenable.

If G is amenable, then $\mathcal{L}_G$ must contain extreme points by Krein-Milman's Lemma. We denote the set of such extreme elements by $\mathcal{L}_G^{\text{ext}}$. The following result, which is quite standard (see for instance [3] for a proof), points out an important "ergodicity" property of such means.

Proposition A.1 (Weak Ergodic Theorem). *If $\lambda \in \mathcal{L}_G^{ext}$, then*

$$\int_G \lambda(gA \cap B) \, d\eta(g) = \lambda(A) \, \lambda(B),$$

for all $A, B \subset G$ and $\eta \in \mathcal{L}_G$.

A.2. **Pointed G-spaces**

Let G be a countable amenable group and suppose that it acts by homeomorphisms on a compact second countable space X. We shall assume that there exists $x_o \in X$ with a dense G-orbit, and we refer to the pair (X, x_o) as a *pointed* G-*space*. Let C(X) denote the Banach space of real-valued continuous functions on G endowed with the uniform norm, and $\mathcal{P}(X) \subset C(X)^*$ the weak*-closed and convex set of regular Borel probability measures on X. We say that $\mu \in \mathcal{P}(X)$ is *invariant* if $\mu(gB) = \mu(B)$ for all $g \in G$ and every Borel set $B \subset X$, and write $\mathcal{P}_G(X)$ for the set of invariant probability measures. We see that there is a positive and unital linear map $S_{x_o} : C(X) \to \ell^\infty(G)$ defined by $(S_{x_o} f)(g) = f(gx_o)$, which intertwines the left-regular representations of G on C(X) and $\ell^\infty(G)$. It readily follows that $S_{x_o}^*(\mathcal{L}_G) \subset \mathcal{P}_G(X)$. In particular, since G is amenable, $\mathcal{P}_G(X)$ is always non-empty. By Krein-Milman's Theorem, the set $\mathcal{P}_G^{\text{erg}}(X)$ of extreme points is then non-empty, and it turns out that it coincides with the set of *ergodic* measures in $\mathcal{P}_G(X)$ (see for instance Theorem 4.4. in [12]).

If $\lambda \in \mathcal{L}_G$ and $\mu = S_{x_o}^* \lambda$, then for every *clopen* set $U \subset X$, the indicator function χ_U is continuous on X, and thus we have $\lambda(U_{x_o}) = \lambda(S_{x_o} \chi_U) = \mu(U)$. However, this observation is only useful for *disconnected* spaces; as many of the G-spaces that we will work with are connected, it will be useful to know if this type of identity holds for more general classes of sets (that some regularity on U has to be assumed is clear already from easy examples). The following lemma provides some useful answers in this direction. Recall that if μ is a regular Borel probability measure on X, then a set $U \subset X$ is μ-*Jordan measurable* if $\mu(\overline{U}) = \mu(U^o)$.

Lemma A.2. *Let $\lambda \in \mathcal{L}_G$ and $\mu = S_{x_o}^* \lambda$.*

  (i) If $U \subset X$ is μ-Jordan measurable, then $\lambda(U_{x_o}) = \mu(U)$.
  (ii) If $U, V \subset X$ are μ-Jordan measurable, and $\nu(U \setminus V) = 0$, then $\lambda(U_{x_o} \cap V_{x_o}) = \mu(U)$.
  (iii) If $U \subset X$ is open, then $\lambda(U_{x_o}) \geqslant \mu(U)$.

In particular, if $A \subset G$ is any subset, and $U \subset$ is clopen, then

$$\lambda(AU_{x_o}) \geqslant \mu(AU) \quad and \quad \lambda(U_{x_o}) = \mu(U).$$

Proof. (i) If U is μ-Jordan measurable, then by Proposition 2.3.3 in [29], there exist, for every $\varepsilon > 0$, continuous functions f_- and f_+ on X such that

$$f_- \leqslant \chi_U \leqslant f_+ \quad \text{and} \quad \mu(f_+ - f_-) \leqslant \varepsilon,$$

and thus

$$\mu(f_-) = \lambda(S_{x_o} f_-) \leqslant \lambda(U_{x_o}) = \lambda(S_{x_o} \chi_U) \leqslant \lambda(S_{x_o} f_+) = \mu(f_+),$$

whence $0 \leqslant \mu(f_+) - \lambda(U_{x_o}) \leqslant \varepsilon$. By letting $\varepsilon \searrow 0$, we conclude that $\mu(U) = \lambda(U_{x_o})$.
  (ii) One checks that $U \cap V$ is μ-Jordan measurable, and thus by (i),

$$\lambda(U_{x_o} \cap V_{x_o}) = \lambda((U \cap V)_{x_o}) = \mu(U \cap V) = \mu(U).$$

(iii) See Lemma 2.1 in [2].

$\square$

In the case when $\mathcal{P}_G(X) = \{\mu\}$, then μ is of course extremal in $\mathcal{P}_G(X)$, and thus ergodic, and for every μ-Jordan measurable subset $U \subset X$ and $\lambda \in \mathcal{L}_G$, we have $\lambda(U_{x_o}) = \nu(U)$, no matter if λ is extremal in $\mathcal{L}_G$ or not. In particular, let (K, L, τ) be an isometric G-action. Then, by Lemma 2.1, the unique K-invariant Borel probability measure $m_{K/L}$ is also the unique G-invariant probability measure on $X = K/L$. We conclude:

Corollary A.3. *If* $I \subset K/L$ *is a* $m_{K/L}$*-Jordan measurable set, then* $\lambda(I_t) = m_{K/L}(I)$ *for every* $t \in K/L$ *and* $\lambda \in \mathcal{L}_G$. *In particular, if* L *is trivial, then* $\lambda(\tau^{-1}(I)) = m_K(I)$ *for every* $\lambda \in \mathcal{L}_G$.

A.3. **Furstenberg's Correspondence Principle**

We now state Furstenberg's Correspondence Principle in terms of the transpose map $S^*_{x_o}$ above. This formulation is perhaps somewhat unorthodox, but can be readily proved along the same lines as in Furstenberg's seminal paper [14] where this principle first appeared. A detailed proof of a slightly more general statement in the language below can be found in [2].

Proposition A.4 (Furstenberg's Correspondence Principle). *The map* $S^*_{x_o} : \mathcal{L}_G \to \mathcal{P}_G(X)$

(i) is affine, weak*-continuous and onto.
(ii) maps $\mathcal{L}_G^{ext}$ *onto* $\mathcal{P}_G^{erg}(X)$.

We stress that it is not at all automatic for weak*-continuous affine maps between weak*-closed and convex sets to map extreme points to extreme points; indeed, consider the unit square $[0,1]^2$ in $\mathbb{R}^2$, and the linear map which projects it onto one of its diagonals. The two corners which are not touched by the diagonal are certainly extreme points of the square but will be mapped to midpoint of the diagonal, which is not extreme anymore.

A.4. **Følner sequences and densities**

The notions and results in this subsection are well-known, and we only include a brief discussion for completeness, and to make referencing easier.

Definition A.5. A sequence (F_n) of finite subsets of G is *Følner* if

$$\overline{\lim_n} \frac{|F_n \triangle gF_n|}{|F_n|} = 0, \quad \text{for all } g \in G. \tag{A.1}$$

If (F_n) is a Følner sequence in G, then we define the *upper* and *lower asymptotic density* of a subset $A \subset G$ along (F_n) by

$$\overline{d}_{(F_n)}(A) = \overline{\lim_n} \frac{|A \cap F_n|}{|F_n|} \quad \text{and} \quad \underline{d}_{(F_n)}(A) = \underline{\lim_n} \frac{|A \cap F_n|}{|F_n|} \tag{A.2}$$

respectively, and the *upper* and *lower Banach densities* by

$$d^*(A) = \sup\left\{\overline{d}_{(F_n)}(A) : (F_n) \text{ Følner}\right\} \quad \text{and} \quad d_*(A) = \inf\left\{\underline{d}_{(F_n)}(A) : (F_n) \text{ Følner}\right\} \tag{A.3}$$

respectively.

We note that every Følner sequence (F_n) naturally gives rise to invariant means. Indeed, consider the sequence (λ_n) in $\mathcal{M}_G$ defined by

$$\lambda_n(f) = \frac{1}{|F_n|} \sum_{g \in F_n} f(g), \quad \text{for } f \in \ell^\infty(G).$$

It readily follows from the Følner condition (A.1) that any weak*-accumulation point is invariant. In particular, for any $A \subset G$, there are $\overline{\lambda}, \underline{\lambda}$ such that

$$\overline{d}_{(F_n)}(A) = \overline{\lambda}(A) \quad \text{and} \quad \underline{d}_{(F_n)}(A) = \underline{\lambda}(A). \tag{A.4}$$

Of course, for a subset $B \subset G$ different from A, we can only be sure of the inequalities

$$\overline{\lambda}(B) \leqslant \overline{d}_{(F_n)}(B) \quad \text{and} \quad \underline{\lambda}(B) \geqslant \underline{d}_{(F_n)}(B). \tag{A.5}$$

The following proposition is well-known to experts, but hard to find a good reference for, why we supply a proof here.

Proposition A.6. *For every* $A \subset G$,

$$d^*(A) = \sup\left\{\lambda(A) : \lambda \in \mathcal{L}_G\right\} \quad and \quad d_*(A) = \inf\left\{\lambda(A) : \lambda \in \mathcal{L}_G\right\},$$

and there are extreme λ_+, λ_- *in* $\mathcal{L}_G$ *such that* $d^*(A) = \lambda_+(A)$ *and* $d_*(A) = \lambda_-(A)$.

Proof. Assuming the identities for d^* and d_*, the second assertion is immediate from the fact that the map $\lambda \mapsto \lambda(A)$ is weak*-continuous and affine on $\mathcal{L}_G$, and such maps always attain their minima and maxima at extreme points.

Concerning the identities, we first note that (A.4) implies that

$$d^*(A) \leqslant \sup\left\{\lambda(A) : \lambda \in \mathcal{L}_G\right\} \quad and \quad d_*(A) \geqslant \inf\left\{\lambda(A) : \lambda \in \mathcal{L}_G\right\}.$$

Let us prove that the first inequality is in fact an identity; the second inequality can be treated completely analogously. Pick an extreme λ at which the supremum above is realized, and denote by (X, x_o) the G-hull associated to the set A, as in Subsection 2.1.1. Abusing notation, we can find a *clopen* subset $A \subset X$ such that our set in G can be represented as A_{x_o}. Let $\mu = S^*_{x_o}\lambda$; by Proposition A.4, μ is ergodic and $\lambda(A_{x_o}) = \mu(A)$.

By the strong mean Ergodic Theorem (see e.g. [4]), the averages

$$\lim_n \frac{1}{|F_n|} \sum_{g \in F_n} \chi_A(gx) = \lim_n \frac{|A_x \cap F_n|}{|F_n|}, \quad \text{for } x \in X,$$

converge in $L^2(X, \mu)$ to the constant function $\mu(A)$, whence, upon passing to a sub-sequence (n_k), μ-almost surely to $\mu(A)$. Pick $x \in X$ for which this sub-sequence converges. Since A is clopen, we can find (g_{n_k}) such that $A_{g_{n_k}.x_o} \cap F_{n_k} = A_x \cap F_{n_k}$ for every k, and thus

$$|A_{x_o} \cap F_{n_k} g_{n_k}| = |A_{g_{n_k}.x_o} \cap F_{n_k}| = |A_x \cap F_{n_k}|,$$

which shows that

$$d^*(A_{x_o}) \geqslant \lim_k \frac{|A_{x_o} \cap F_{n_k} g_{n_k}|}{|F_{n_k}|} = \mu(A),$$

where the inequality follows from the fact that $(F_{n_k} g_{n_k})$ is a Følner sequence in G. $\square$

A.5. **Thickness and syndeticity**

Recall that a subset $A \subset G$ is *thick* if for every finite subset $F \subset G$ there is $g \in G$ such that $Fg \subset A$, and *syndetic* if there exists a finite set $F \subset G$ such that $FA = G$. For a proof of the following well-known density characterizations of thick and syndetic sets, see for instance Subsections 2.5 and 2.6 in [5].

Lemma A.7. *Let* G *be a countable amenable group, and let* $A \subset G$. *Then,*

    (i) A *is thick* $\iff$ $d^*(A) = 1$.
    (ii) A *is syndetic* $\iff$ $d_*(A) > 0$.

APPENDIX B. GENERALITIES ON GROUP COMPACTIFICATIONS

We collect here some basic facts about group compactifications of countable groups that will be used in some of our proofs.

Let G be a countable group. We say that (K, τ) is a *group compactification* of G if K is a compact Hausdorff group and $\tau : G \to K$ is a homomorphism with dense image. We stress that we do *not* assume that τ is injective. We shall always denote the (unique) Haar probability measure on K by m_K and the identity element in K by e_K.

As it turns out, many group-theoretical properties of G can be transported to topological properties of K. We record here some instances of this phenomenon.

Lemma B.1. *If* (K, τ) *is a group compactification of* G, *then*

    (i) G *amenable* $\implies$ K^o *is abelian.*
    (ii) G *has no non-trivial finite index subgroups* $\implies$ K *is connected.*
    (iii) G *is a finitely generated torsion group* $\implies$ K *is totally disconnected.*
    (iv) G *is a finitely generated simple group* $\implies$ K *is trivial.*

Proof. (i) See the Appendix in [3].

(ii) If K is not connected, then there is a non-trivial proper open subgroup U of K. Since K is compact, U must have finite index in K, and thus $G_o = \tau^{-1}(U)$ has finite index in G.

(iii) By Corollary 2.36 in [18], we can find a net (N_α) of closed normal subgroups of K and integers (n_α) such that

$$\bigcap_\alpha N_\alpha = \{e_K\} \quad \text{and} \quad K_\alpha := K/N_\alpha \overset{\iota_\alpha}{\hookrightarrow} U(n_\alpha),$$

where $U(n)$ denotes the unitary group in dimension n, and ι_α is injective for every α. Note that for every α, the subgroup $\Gamma_\alpha = \iota_\alpha \circ \tau(G)$ of the linear group $U(n_\alpha)$ is finitely generated and torsion. By Jordan-Schur's Theorem, these properties imply that Γ_α is finite, whence K_α is finite, and thus N_α must be open in K for every α. Since the intersections of all N_α is trivial, K is totally disconnected.

(iv) First note that (ii) implies that K must be connected, and thus Peter-Weyl's Theorem shows that if K is non-trivial, then it admits a non-trivial compact and connected Lie group K' as a quotient group. Since G is simple, the composition of τ with this quotient map is still injective (otherwise the kernel would be a non-trivial normal subgroup of G). In particular, G can be viewed as a finitely generated subgroup of K'. However, Malcev's Theorem now says that any such subgroup must be residually finite, and thus far from simple, which leads us to conclude that K is trivial. $\quad\square$

The next lemma contains some auxiliary observations about pull-backs of sets in a group compactification (K, τ) of a countable group G.

Lemma B.2. *Let* (K, τ) *be a compactification of* G, *and fix a thick subset* $T \subset G$, *a non-empty open subset* $U \subset K$ *and a finite-index subgroup* $G_o < G$. *Then,*

    (i) for every $s \in G$, *the set* $\tau^{-1}(U) \cap G_o s$ *is non-empty iff it is syndetic.*
    (ii) $G_o(\tau^{-1}(U) \cap T) = \tau^{-1}(\tau(G_o)U)$.
    (iii) if K *is connected, then* $\overline{\tau(Q \cap T)} = K$ *for every non-empty right* G_o-*invariant set* $Q \subset G$.

Proof. (i) Note that the closed subgroup $H = \overline{\tau(G_o)} < K$ has finite index, hence open. We note that $\tau^{-1}(U) \cap G_o s = \tau^{-1}(U\tau(s)^{-1} \cap H)s$, and thus, if this set is non-empty, then $V = U\tau(s)^{-1} \cap H$ is a non-empty open subset of K. Since $\tau(G)$ is dense in K, there is a finite set $F \subset G$ such that $\tau(F)V = K$, whence $F(\tau^{-1}(U) \cap G_o s) = \tau^{-1}(FV)s = G$, which shows that $\tau^{-1}(U) \cap G_o s$ is syndetic in G.

(ii) Set $D = \tau^{-1}(U)$, and define $D_+ = \{s \in G_o \backslash G : D \cap G_o s \neq \emptyset\}$. Note that

$$G_o(D \cap T) = \bigsqcup_{s \in D_+} G_o((D \cap G_o s) \cap T).$$

By (i), if $s \in D_+$, then $D \cap G_o s$ is in fact syndetic in G, and thus intersects the thick set T non-trivially, whence $G_o((D \cap G_o s) \cap T) = G_o s$ for all $s \in D_+$, which finishes the proof.

(iii) Fix an open identity neighborhood V in K and an exhaustion (F_n) of finite subsets of G. Since T is thick, we can find a sequence (g_n) such that $F_n g_n \subset T$ for all n. Since Q is right G_o-invariant and K compact, we may pass to further sub-sequence (or sub-net, if K is not sequentially compact), so that for some $g \in G$ and $t \in K$ we have

$$Qg_n^{-1} = Qg^{-1} \quad \text{and} \quad \tau(g_n)^{-1} \in Vt$$

for all n, and thus

$$\tau(Q \cap T)V \supset \tau(Q \cap F_n g_n)\tau(g_n)^{-1}t^{-1} = \tau(Qg^{-1} \cap F_n)t^{-1}$$

for all n, whence $\tau(Q \cap T)V \supset \tau(Qg^{-1})t^{-1}$. Since K is connected, $\tau(G_o)$ is dense in K, and thus $\tau(Qg^{-1})$ is dense as well. Since V is arbitrary, we conclude that $\tau(Q \cap T)$ is dense. $\qquad\square$

APPENDIX C. PECULIAR SUMSETS IN $\mathbb{Z}$ RELATIVE TO THE FØLNER SEQUENCE $\{[-n, n]\}$

We show that different attempts to weaken the assumptions in Theorem 1.16 and Theorem 1.18 fail, already for $G = (\mathbb{Z}, +)$ and the Følner sequence $F_n = [-n, n]$.

To keep things simple, let us in this appendix only focus on the group $G = (\mathbb{Z}, +)$, the Følner sequence $F_n = [-n, n]$ in G and its associated *lower* asymptotic density, which we here denote by

$$\underline{d}(A) = \varliminf_{n \to \infty} \frac{|A \cap [-n, n]|}{2n + 1}, \quad \text{for } A \subset \mathbb{Z}.$$

Our two first examples concern attempts to weaken Theorem 1.16, while our third and fourth example deal with failed conjectural strengthenings of Theorem 1.18. In each example, the weakened assumption is marked in CAPITAL letters.

Proposition C.1. *There exist* $A, B \subset \mathbb{Z}$ *with* $\underline{d}(A + B) < d^*(A) + \underline{d}(B) < 1$ *such that*

(i) A *is not contained in a proper periodic set,*
(ii) B *is syndetic,*
(iii) $A + B$ *is* THICK.

Proposition C.2. *There exist* $A, B \subset \mathbb{Z}$ *with* $\underline{d}(A + B) < d^*(A) + \underline{d}(B) < 1$ *such that*

(i) A *is not contained in a proper periodic set,*
(ii) B *is* NOT SYNDETIC, *but* $\underline{d}(B) > 0$,
(iii) $A + B$ *is not thick.*

Proposition C.3. *There exist* $A, B \subset \mathbb{Z}$ *with* $\underline{d}(A + B) = d^*(A) + \underline{d}(B) < 1$ *such that*

(i) A *is spread-out,*
(ii) B *is* NOT SYNDETIC, *but* $\underline{d}(B) > 0$,
(iii) $A + B$ *does not contain a piecewise periodic set,*

but A *is* not *contained in a Sturmian set with the same upper Banach density as* A.

Proposition C.4. *There exist* $A, B \subset \mathbb{Z}$ *with* $\underline{d}(A + B) = d^*(A) + \underline{d}(B) < 1$ *such that*

(i) A *is spread-out,*
(ii) B *is is syndetic,*
(iii) $A + B$ *is* THICK,

but A *is* not *contained in a Sturmian set with the same upper Banach density as* A.

The examples above are constructed by similar procedures, so we will discuss them in parallel. We start by fixing an irrational $\alpha \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given a proper closed interval $I \subset \mathbb{T}$, we shall write

$$C_I = \{n \in \mathbb{Z} : n\alpha \in I\}.$$

In Proposition C.1, we pick $I \subset \mathbb{T}$ with $m_{\mathbb{T}}(I) < 1/3$, and set $A = C_I \cap \mathbb{N}$ and $B = C_I \cup \mathbb{N}$. In Proposition C.2, we pick $I \subset \mathbb{T}$ with $m_{\mathbb{T}}(I) < 1/2$, and set $A = B = C_I \cap \mathbb{N}$. In Proposition C.3, we pick $I \subset \mathbb{T}$ such that $(I + I) \cup (I + n\alpha) = \emptyset$ for some integer n, and set $A = (C_I \cap \mathbb{N}) \cup \{n\}$ and $B = C_I \cap \mathbb{N}$. The matter of verifying that these choices indeed lead to the examples in the propositions is entirely routine, and left to the reader. Proposition C.4 is more involved. To construct our example here, we first need to produce a thick set $T \subset \mathbb{N}$ such that

$$\underline{d}_{([1,n])}(T) = \frac{1}{10} \quad \text{and} \quad \underline{d}_{([1,n])}(T + T) = \frac{2}{10},$$

with the property that the sequence $F_n = [1, n] \setminus (T + T)$ is Følner. This is tedious, but still a matter of routine. We now choose $I \subset \mathbb{T}$ with $m_{\mathbb{T}}(I) = 4/9$ such that for some $m \in \mathbb{Z}$,

$$m_{\mathbb{T}}((I + I) \cup (I + m\alpha)) = \left(2 + \frac{1}{24}\right)m_{\mathbb{T}}(I).$$

The exact numbers here are not so important; the construction has some wiggle room. Once T, I and m have been produced, we set $A = (C_I \cap T) \cup \{m\}$ and $B = C_I \cup T$, and note that B is syndetic *and* thick, so $A + B$ is thick as well. To check the remaining properties in Proposition C.4 is again a matter of routine.

References

1. Vitaly Bergelson and Hillel Furstenberg. WM groups and ramsey theory. *Topology and its Applications*, 156(16):2572–2580, 2009.
2. Michael Björklund. Product set phenomena for measured groups. *Ergodic Theory and Dynamical Systems*, (1):1–29, 2017.
3. Michael Björklund. Small product sets in compact groups. *Fund. Math.*, 238(1):1–27, 2017.
4. Michael Björklund and Alexander Fish. Ergodic theorems for coset spaces. *J. d'Anal. Math.*
5. Michael Björklund and Alexander Fish. Product set phenomena for countable groups. *Adv. Math.*, 275:47–113, 2015.
6. Michael Björklund and Tobias Hartnick. Analytic properties of approximate lattices. *Preprint, https://arxiv.org/abs/1709.09942*.
7. Michael Björklund and Tobias Hartnick. Approximate lattices. *To appear in Duke Math. J., https://arxiv.org/abs/1612.09246*.
8. Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. Aperiodic order and spherical diffraction, i: Auto-correlation of model sets. *Accepted in Proc. London Math. Soc.*
9. Jean Bourgain and Alexander Gamburd. Uniform expansion bounds for cayley graphs of $2(\mathbb{F}_l)$. *Ann. of Math.*
10. Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.*, 116:115–221, 2012.
11. A. J. Chintschin. *Drei Perlen der Zahlentheorie*. Akademie-Verlag, Berlin, 1984. Reprint of the 1951 translation from the Russian, With a foreword by Helmut Koch.
12. Manfred Einsiedler and Thomas Ward. Homogeneous dynamics: a study guide. In *Introduction to modern mathematics*, volume 33 of *Adv. Lect. Math. (ALM)*, pages 171–201. Int. Press, Somerville, MA, 2015.
13. G. A. Freiman. *Foundations of a structural theory of set addition*. American Mathematical Society, Providence, R. I., 1973. Translated from the Russian, Translations of Mathematical Monographs, Vol 37.
14. Harry Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Analyse Math.*, 31:204–256, 1977.
15. Eli Glasner. *Ergodic theory via joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
16. John T. Griesmer. Small-sum pairs for upper Banach density in countable abelian groups. *Adv. Math.*, 246:220–264, 2013.
17. H. A. Helfgott. Growth and generation in $SL_2(\mathbb{Z}/p\mathbb{Z})$. *Ann. of Math. (2)*, 167(2):601–623, 2008.

18. Karl H. Hofmann and Sidney A. Morris. *The structure of compact groups*, volume 25 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2013. A primer for the student—a handbook for the expert, Third edition, revised and augmented.

19. Renling Jin. Characterizing the structure of A + B when A + B has small upper Banach density. *J. Number Theory*, 130(8):1785–1800, 2010.

20. J. H. B. Kemperman. On products of sets in a locally compact group. *Fund. Math.*, 56:51–68, 1964.

21. Martin Kneser. Abschätzung der asymptotischen Dichte von Summenmengen. *Math. Z.*, 58:459–484, 1953.

22. Martin Kneser. Summenmengen in lokalkompakten abelschen Gruppen. *Math. Z.*, 66:88–110, 1956.

23. George W. Mackey. Ergodic transformation groups with a pure point spectrum. *Illinois J. Math.*, 8:593–600, 1964.

24. Henry B. Mann. A proof of the fundamental theorem on the density of sums of sets of positive integers. *Ann. of Math. (2)*, 43:523–527, 1942.

25. Yves Meyer. *Algebraic numbers and harmonic analysis*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972. North-Holland Mathematical Library, Vol. 2.

26. Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

27. Terence Tao. Product set estimates for non-commutative groups. *Combinatorica*, 28(5):547–594, 2008.

28. André Weil. *L'intégration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940. [This book has been republished by the author at Princeton, N. J., 1941.].

29. Reinhard Winkler. Hartman sets, functions and sequences—a survey. In *Probability and number theory—Kanazawa 2005*, volume 49 of *Adv. Stud. Pure Math.*, pages 517–543. Math. Soc. Japan, Tokyo, 2007.

30. Robert J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

DEPARTMENT OF MATHEMATICS, CHALMERS, GOTHENBURG, SWEDEN
E-mail address: micbjo@chalmers.se

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, AUSTRALIA
E-mail address: alexander.fish@sydney.edu.au