KNESER-TYPE THEOREMS FOR COUNTABLE AMENABLE GROUPS

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ABSTRACT. We develop in this paper a general ergodic-theoretical technique which reduces to a large extent the study of lower bounds on ASYMPTOTIC densities (with respect to some fixed Følner sequence) of product sets in a countable amenable group G, to the study of lower bounds of products of Borel sets in a metrizable compactification of G, for which a plethora of classical (as well as more recent) results are known. We apply this technique to extend Kneser's classical sum set bound in the additive group $(\mathbb{Z}, +)$ (a density analogue of Mann's celebrated theorem) to asymptotic densities on general countable amenable groups, as well as to answer some old questions concerning the sharpness of this bound.

Roughly speaking, our main combinatorial results assert that if the product of two "large" sets in G is "small", then the "left" addend is very well controlled by either a proper periodic set or a Sturmian set in G. As an application of our results, we provide a classification of "spread-out" approximative subgroups of spread two in a general countable amenable group.

We also develop a "Counterexample Machine" which produces in certain two-step solvable groups, pairs of large subsets with "small" product sets whose "right" (but not left) addend fails to be "nicely" controlled by well-structured Bohr sets.

1. INTRODUCTION

1.1. Kneser's density theorem

Let G be a countable discrete group. Given two subsets $A, B \subset G$, we define their **product set** AB by

$$AB = \{ab \ : \ a \in A, \ \ b \in B\}.$$

It is a fundamental problem in the research field of arithmetic combinatorics to understand the structure of "large" sets $A, B \subset G$ for which the product set AB is "small", where the intentionally vague notions "large" and "small" are allowed to vary.

The problem has so far mostly been studied in the case when $G = (\mathbb{Z}, +)$; in this setting, we (temporarily) define a set $A \subset \mathbb{Z}$ to be *large* if its lower asymptotic density $\underline{d}(A)$, given by

$$\underline{d}(A) = \underline{\lim}_{n} \frac{|A \cap [1, n]|}{n}$$

is positive. Extending a celebrated result by Mann, known as one of Chintschine's "Three pearls of number theory" [5], Kneser proved in [15] that if $A \subset \mathbb{Z}$ is large and "sufficiently spread-out", meaning that $A \cap \mathbb{N}$ is not contained in a proper periodic set $P \subset \mathbb{Z}$ such that

$$\underline{d}(P) < \underline{d}(A) + \frac{1}{m}, \tag{1.1}$$

where m is the smallest positive integer such that $P + m\mathbb{Z} = P$ (known as the **period** of P), then

$$\underline{d}(A + B) \geq \min\left(1, \underline{d}(A) + \underline{d}(B)\right), \quad \text{for all } B \subset \mathbb{Z}.$$

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In other words, if the product (sum) set $A+B$ is "small" in the sense that

$$\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B),$$

then neither A nor B can be spread out in the sense described above, and must thus be "controlled" by proper periodic sets P and Q, which are not much larger than A and B (in the sense that (1.1) hold for (A,P) and (B,Q)).

1.2. **Main combinatorial results**

The (combinatorial) aim of this paper is two-fold. Firstly, we wish to extend (as much as possible) of Kneser's result to more general (not necessarily abelian) groups. Secondly, we wish to address the "spread-out" cases when Kneser's lower bound is attained. In order to formulate our results, we first need to introduce some notation.

A sequence (F_n) of *finite* subsets of G is called **Følner** if

$$\overline{\lim_n} \frac{|F_n \Delta g F_n|}{|F_n|} = 0, \quad \text{for all } g \in G,$$

and we say that G is **amenable** if it admits a Følner sequence. We stress that we do not assume that (F_n) exhausts G. For instance, $G = (\mathbb{Z}, +)$ is amenable, and

$$F_n = [1,n] \quad \text{and} \quad F_n = [-n,n] \quad \text{and} \quad F_n = [n!, n! + n^n]$$

are all examples of Følner sequences in G. More generally, every solvable (in particular, abelian) group is amenable, as is every locally finite group and every group of intermediate growth. However, free groups of rank at least two, are non-amenable, as is every (countable) group which contains such a free group.

Let us from now on assume that G is amenable, and let (F_n) be a Følner sequence in G. Given a subset $A \subset G$, we define its **upper** and **lower asymptotic density** by

$$\overline{d}_{(F_n)}(A) = \overline{\lim_n} \frac{|A \cap F_n|}{|F_n|} \quad \text{and} \quad \underline{d}_{(F_n)}(A) = \underline{\lim_n} \frac{|A \cap F_n|}{|F_n|}.$$

Note that if $G = (\mathbb{Z}, +)$, then $\underline{d} = \underline{d}_{([1,n])}$. Since there is no canonical choice of a Følner sequence in G, it is also natural to consider the following "uniformizations" of the asymptotic densities. We define the **upper** and **lower Banach density** of $A \subset G$ by

$$d^*(A) = \sup\{\overline{d}_{(F_n)}(A) \,:\, (F_n) \text{ is a Følner sequence in } G\}$$

and

$$d_*(A) = \inf\{\underline{d}_{(F_n)}(A) \,:\, (F_n) \text{ is a Følner sequence in } G\}.$$

We say that a set $D \subset G$ is

- **large** if $d^*(D) > 0$.
- **syndetic** if there exists a finite set $F \subset G$ such that $FD = G$.
- **thick** if there is, for every finite set $F \subset G$, an element $g \in G$ such that $Fg \subset D$.
- **periodic** if its *stabilizer group* $\mathrm{Stab}_G(D)$, defined by

  $$\mathrm{Stab}_G(D) = \{g \in G \,:\, gD = D\}$$

  has finite index in G.
- **piecewise periodic** if there is a periodic set P and a thick set T such that $D = P \cap T$.

- **spread-out** if there is no subset $D_o \subset D$ with $d^*(D_o) = d^*(D)$ which is contained in a proper periodic set. In particular, D is large. We note that if G lacks proper finite-index subgroups, then every large set is automatically spread-out.

Our extension of Kneser's Theorem now takes the following form.

Theorem 1.1. *If $A \subset G$ is spread-out, $B \subset G$ is syndetic and AB is* NOT *thick, then*

$$\overline{d}_{(F_n)}(AB) \geq d^*(A) + \overline{d}_{(F_n)}(B),$$

and

$$\underline{d}_{(F_n)}(AB) \geq d^*(A) + \underline{d}_{(F_n)}(B).$$

If all finite quotients of G are ABELIAN*, then instead of assuming that A is spread-out, it suffices to assume that A is large and* NOT *contained in a proper periodic subset $P \subset G$ with*

$$d^*(P) < d^*(A) + \frac{1}{[G : \mathrm{Stab}_G(P)]}. \tag{1.2}$$

Remark 1.2. We stress that this result is completely new even for $G = (\mathbb{Z}, +)$ and the Følner sequence $([-n, n])$. Furthermore, in the appendix to this paper, we show that in this particular case, all of the assumptions above are necessary. An analogous result for the *upper* asymptotic density for

$$G = (\mathbb{Z}, +) \quad \text{and} \quad F_n = [1, n] \quad \text{and} \quad A = B$$

was recently established by Jin [12].

The proof of Theorem 1.1 also yields the following analogous result for the upper Banach density (a similar statement can also be deduced for the lower Banach density).

Scholium 1.3. *If $A \subset G$ is spread-out and $B \subset G$ is large, then*

$$d^*(AB) \geq \min\left(1, d^*(A) + d^*(B)\right).$$

If all finite quotients of G are ABELIAN*, then it suffices to assume that A is large and* NOT *contained in a proper periodic subset $P \subset G$ such that* (1.2) *holds.*

Remark 1.4. This result was first established in the (semigroup) case

$$G = (\mathbb{N}, +) \quad \text{and} \quad F_n = [1, n]$$

by Jin [13], and extended to countable *abelian* groups by Griesmer [10] (his proof is very much inspired by earlier versions of our Correspondence Principles).

Let us now give an example of a countable two-step (hence non-abelian) solvable (hence amenable) group to which the last assertion in Theorem 1.1 (and Scholium 1.3) applies.

Example 1 (Generalized lamplighter group)**.** Let $G = \mathbb{Q} \wr \mathbb{Z}$, i.e. the wreath product of the two abelian groups $(\mathbb{Q}, +)$ and $(\mathbb{Z}, +)$ (also known as a generalized lamplighter group). It is not hard to show that every homomorphism of G onto a finite group factors through $\mathbb{Z}$, and thus every finite factor group of G is abelian. Indeed, since $\mathbb{Q}$ does not have any proper finite-index subgroups, the restriction to the subgroup $\bigoplus_{\mathbb{Z}} \mathbb{Q}$ of G, of any homomorphism from G into a finite group, must be trivial.

Let us address the question when the lower bound in Theorem 1.1 is attained. As it turns out, there are quite few examples of pairs of sets which hit the lower bound, and they are essentially all encompassed by the following construction. Let $\mathbb{T}$ denote the one-dimensional torus, which we think of as $(\mathbb{R}/\mathbb{Z}, +)$, and let M denote either $\mathbb{T}$ or the "twisted" torus $\mathbb{T} \rtimes \{-1, 1\}$, where the group $(\{-1, 1\}, \cdot)$ acts on $\mathbb{T}$ by multiplication. We say that a set $A \subset G$ is **Sturmian**

if there exists a homomorphism $\tau : G \to M$, a closed and symmetric interval $I' \subset \mathbb{T}$ and $a \in M$ such that either

$$A = \tau^{-1}(I' + a) \quad \text{or} \quad A = \tau^{-1}((I' \rtimes \{-1, 1\})a),$$

depending on whether $M = \mathbb{T}$ or $M = \mathbb{T} \rtimes \{-1, 1\}$. The "untwisted" Sturmian sets ($M = \mathbb{T}$) have been extensively studied in complexity theory and tiling theory, see e.g. the survey [24]. On the other hand, it seems that our paper is the first one to deal with their "twisted" analogues. Of course, for *abelian* G, only "untwisted" Sturmian sets exist. Let us now consider an example for which *only* TWISTED Sturmian sets exist.

Example 2 (Dihedral group)**.** Let $G = \mathbb{Z} \rtimes \{-1, 1\}$ denote the infinite dihedral group, and note that the commutator group $[G, G]$ equals $2\mathbb{Z} \rtimes \{1\}$. We conclude that if (K, τ_K) is any *abelian* compactification of G, then K must be finite. In particular, G does not admit any homomorphism into $\mathbb{T}$ with a dense image. On the other hand, there are plenty of homomorphisms into $M = \mathbb{T} \rtimes \{-1, 1\}$ with dense images; for instance, let $a \in \mathbb{T}$ be irrational, and define the homomorphism $\tau_M : G \to \mathbb{T} \rtimes \{-1, 1\}$ by

$$\tau_M(m, \varepsilon) = (ma, \varepsilon), \quad \text{for } (m, \varepsilon) \in G.$$

One readily checks the image is dense in M.

The relevance of Sturmian sets to our problem at hand is explained by the following theorem, which is completely new already for $G = (\mathbb{Z}, +)$.

Theorem 1.5. *If $A \subset G$ is spread-out, $B \subset G$ is syndetic, AB does* NOT *contain a piecewise periodic set and if either*

$$\overline{d}_{(F_n)}(AB) = d^*(A) + \overline{d}_{(F_n)}(B) < 1,$$

or

$$\underline{d}_{(F_n)}(AB) = d^*(A) + \underline{d}_{(F_n)}(B) < 1,$$

then A is contained in a Sturmian subset with the same upper Banach density as A.

Remark 1.6. We show in the appendix of this paper that the assumptions in the theorem are necessary already in the case $G = (\mathbb{Z}, +)$ and $F_n = [-n, n]$.

Just as with Theorem 1.1 and Scholium 1.3, the proof of Theorem 1.5 can be readily adapted to yield the following result.

Scholium 1.7. *If $A \subset G$ is spread-out, $B \subset G$ is large, AB does not contain a piecewise periodic set and*

$$d^*(AB) = d^*(A) + d^*(B) < 1,$$

then A is contained in a Sturmian subset with the same upper Banach density as A.

Remark 1.8. Inspired by personal communication, as well as by earlier versions of this paper, Griesmer outlined proofs of Scholium 1.3 and Scholium 1.7 for countable ABELIAN groups in [10]. His version of Scholium 1.7 does not assume that A is spread-out, and he gives a rather technical characterization of the cases for which the lower bound is attained. We stress that by digging deeper into our proofs, we can provide a similar list in our generality. As this would probably expand the paper's length with at least 10 pages, we have refrained from doing so.

1.3. **An application: Spread-out approximate subgroups of spread 2**

Let us briefly mention an application of Scholium 1.3 and Scholium 1.7. We say that a symmetric subset $A \subset G$ is an **approximate subgroup** of G if $e_G \in A$ and there is a finite set $F \subset G$ such that $A^2 \subset FA$. The smallest possible cardinality $|F|$ is called the **spread** of A. Of course, an approximate subgroup of spread one is nothing but a subgroup of G. The problem of understanding FINITE approximate subgroups of various groups have attracted a lot of attention in recent years, and the paper [4] by Breuillard, Green and Tao presents a general structure theory of such approximate subgroups.

Let us here focus on *spread-out* (and thus large - in particular, INFINITE) approximate subgroups of a countable AMENABLE group G of spread TWO. If A is such an approximate subgroup, then it is readily checked that $d^*(A^2) \leq 2d^*(A)$. Since A is spread-out, Scholium 1.3 shows that this inequality cannot be strict, and thus either

$$d^*(A^2) = 1 \quad \text{or} \quad d^*(A^2) = 2d^*(A) < 1.$$

In the first case, Lemma 2.12 below implies that A^2 is thick. In the second case, Scholium 1.7 applies, and shows that either A^2 contains a piecewise periodic set, or A is contained in a Sturmian set with the same upper Banach density as A. We can summarize this discussion in the following corollary.

Corollary 1.9. *Let G be a countable amenable group and suppose that $A \subset G$ is spread-out, A^2 does not contain a piecewise periodic set and there exist $x, y \in G$ such that*

$$A^2 \subset xA \cup yA. \tag{1.3}$$

Then A is contained in a Sturmian subset of G with the same upper Banach density as A. If G does not have any proper finite-index subgroups, then it suffices to assume that A is large, A^2 is not thick and (1.3) *holds.*

1.4. **Main ergodic-theoretical results**

Ever since the very influential work of Furstenberg [7], it has been a common practice in ergodic Ramsey theory to prove results in arithmetic combinatorics by first converting them into essentially equivalent "dynamical" versions. By the general "structure theory" of ergodic systems, we can then often reduce the proofs of these dynamical versions to simpler classes of systems for which a larger set of technical tools is available. In our setting, this strategy works as follows.

Let Y be a set equipped with an action of G. Given a subset $A \subset G$ and a set $B \subset Y$, we define their **action set** AB by

$$AB = \bigcup_{a \in A} aB = \{a \cdot b \ : \ a \in A, \ \ b \in B\}.$$

Note that if $Y = G$, on which G acts by left multiplication, then action sets are the same as product sets. If Y is a Borel space, the G-action on Y is Borel measurable and $B \subset Y$ is Borel measurable, then AB is again Borel measurable (since G is countable).

In what follows, let (Y, ν) be a standard Borel probability measure space, equipped with an action of a countable amenable group G by bi-measurable maps which preserve ν. We refer to (Y, ν) as a **Borel G-space**. Throughout this section, we shall always assume that our Borel G-spaces are **ergodic**, i.e. any G-invariant Borel set of Y must be either ν-null or ν-conull. In Section 3, we show how the following "dynamical" analogue of Theorem 1.1 in fact implies it. We stress that this result is completely new already for $G = (\mathbb{Z}, +)$.

Theorem 1.10. *If $A \subset G$ is spread-out, then for every Borel set $B \subset Y$ with positive measure, we have*

$$\nu(AB) \geq \min\left(1, d^*(A) + \nu(B)\right).$$

If all finite quotients of G are ABELIAN, *then it suffices to assume that A is large and* NOT *contained in a proper periodic subset $P \subset G$ with*

$$d^*(P) < d^*(A) + \frac{1}{[G : \mathrm{Stab}_G(P)]}. \tag{1.4}$$

The proof of Theorem 1.10 is outlined in Section 5 and heavily utilizes our "Correspondence Principles for action sets", which constitute the main technical core of this paper. The aim of these Correspondence Principles is to reduce the study of action sets in general ergodic systems, to the study of products of Borel sets in compact metrizable groups, which contain (some factor of) G as a dense subgroup. Since G is amenable, this puts serious constraints on the identity components of these compact groups. In fact, by an application of Tits' alternative, they must be abelian (see Proposition 2.9 below).

Luckily, there already exist in the literature a plethora of results which deal with products of "large" Borel sets in the latter kind of groups. Some of these results are strong enough to be plugged into our "Correspondence Machinery", and return Theorem 1.10.

Example 3 (Grigorchuk group)**.** Before we state our dynamical version of Theorem 1.5, let us briefly comment on the role of the assumption that the set A in Theorem 1.10 must be spread-out. While this assumption is harmless if G lacks proper finite-index subgroups, it is quite a strong assumption (when action sets are concerned) if G is a FINITELY GENERATED AMENABLE TORSION GROUP, e.g. the Grigorchuk group. In the appendix of Section 5, we show that if G is such a group, and $A \subset G$ is spread-out, then $\nu(AB) = 1$ for every ergodic Borel G-space and Borel measurable set $B \subset Y$ with positive measure.

In Section 3, we show how the following result, which addresses the sharpness of Theorem 1.10, can be used to deduce Theorem 1.5.

Theorem 1.11. *If $A \subset G$ is spread-out, $B \subset Y$ is a Borel set with positive measure such that AB does not contain, modulo null sets, a Borel subset with positive measure which is invariant under a finite-index subgroup of G, and*

$$\nu(AB) = d^*(A) + \nu(B) < 1,$$

then A is contained in a Sturmian subset of G with the same upper Banach density as A. In particular, if every finite-index subgroup of G acts ergodically on (Y, ν), then it suffices to assume that A is spread-out and B has positive measure.

1.5. **Counterexamples**

Our two last theorems in this paper address the asymmetry in the roles of A and B in Scholium 1.3 and Scholium 1.7. We note that the first of these results says that if $A, B \subset G$ are large and

$$d^*(AB) < 1 \quad \text{and} \quad d^*(AB) < d^*(A) + d^*(B),$$

then A cannot be spread-out. If G is abelian (or only admits abelian finite factors), then A is in fact contained in a proper periodic set for which (1.2) holds. However, in general, we make no such assertion about the set B. The same asymmetry appears in Scholium 1.7. We stress that by looking deeper into the proofs of Scholium 1.3 and Scholium 1.7 some very rough information CAN be deduced about B, but this information is certainly not on the same level as the one we deduce about A.

The aim of the following two theorems is to show that if G is "sufficiently non-abelian", then the situation is truly asymmetric, and the corresponding containment statements for the set B do simply not hold. Both examples will be produced by our "Counterexample Machine", which we construct in Section 11.

Theorem 1.12. *There exist a countable two-step solvable group G with the following property: There is a large set $A \subset G$ with $d^*(A) = 1/2$ such that for every $0 < \varepsilon < 1/2$, there is a set $B \subset G$ with $d^*(B) = \varepsilon$ such that*

$$d^*(AB) = d^*(A) < d^*(A) + \varepsilon < 1,$$

and whenever P is a proper periodic subset of G with $B \subset P$, then

$$d^*(P) > d^*(B) + \frac{1}{[G : \mathrm{Stab}_G(P)]}.$$

Theorem 1.13. *There exist a countable two-step solvable group G and spread-out subsets $A, B \subset G$ such that AB does not contain a piecewise periodic set, $B \subset G$ is not contained in Sturmian subset of G with the same upper Banach density as B, and*

$$d^*(AB) = d^*(A) + d^*(B) < 1.$$

1.6. **Organization of the proofs**

In Section 3, we reduce Theorem 1.10 and Theorem 1.11 to Theorem 3.1 and Theorem 3.2 respectively. In Section 4, we show how one can deduce Theorem 1.1 and Theorem 1.5 from Theorem 1.10 and Theorem 1.11. In Section 5, we formulate our main Correspondence Principles, and show how one can establish Theorem 3.1 and Theorem 3.2 from them. Finally, in Section 6 and Section 10 we prove the Correspondence Principles mentioned in the introduction. Theorem 1.12 and Theorem 1.13 are proved in Section 11.

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2. Preliminaries

Throughout this section, let G be a countable group.

2.1. **General notation**

Let Y be a set on which G acts. If $A \subset G$ and $B \subset Y$ and $y \in Y$, we define

$$AB = \bigcup_{a \in A} aB = \{a \cdot b \,:\, a \in A,\ \ b \in B\} \subset Y,$$

and the (possibly empty) set

$$B_y = \{g \in G \,:\, g \cdot y \in B\} \subset G.$$

We note that

$$AB_y = (AB)_y \quad \text{and} \quad B_{g \cdot y} = B_y g^{-1} \quad \text{for all } g \in G.$$

2.2. **Means on** G

Let $\ell^\infty(G)$ denote the Banach space of real-valued bounded functions on G, equipped with the uniform norm. We let $\mathcal{M}$ denote the convex and weak*-compact subset of positive and unital functionals on $\ell^\infty(G)$, which we can identify with *finitely additive* probability measures on G via the (notation-abusive) formula $\mathfrak{m}(A) = \mathfrak{m}(\chi_A)$, where χ_A stands for the indicator function of the set $A \subset G$. If $\mathfrak{m} \in \mathcal{M}$, we write

$$\mathfrak{m}(\phi) = \int_G \phi(g)\, d\mathfrak{m}(g), \quad \text{for } \phi \in \ell^\infty(G),$$

even though the right-hand side is not an integral in the Lebesgue sense.

The (left) G-action on G itself induces an isometric action on $\ell^\infty(G)$ and a weak*-continuous action on $\mathcal{M}$ by

$$(g \cdot \phi)(t) = \phi(g^{-1}t) \quad \text{and} \quad (g \cdot \mathfrak{m})(\phi) = \mathfrak{m}(g^{-1} \cdot \phi),$$

for $\phi \in \ell^\infty(G)$ and $\mathfrak{m} \in \mathcal{M}$. We shall refer to the elements in $\mathcal{M}$ as **means** on G, and we define the (possibly empty) set

$$\mathcal{L}_G = \{\mathfrak{m} \in \mathcal{M} \,:\, g \cdot \mathfrak{m} = \mathfrak{m},\ \text{for all } g \in G\}$$

of **left-invariant means** on G. We note that G is amenable if and only if $\mathcal{L}_G$ is non-empty (see e.g. [22]). Let us from now on assume that G is amenable. We say that $\mathfrak{m} \in \mathcal{L}_G$ is **extremal** if it cannot be written as a non-trivial convex combination of other means in $\mathcal{L}_G$. By Krein-Milman's Theorem, extremal means always exist.

The following analogue of the Weak Ergodic Theorem will play an important role in the proofs to come.

Proposition 2.1 (Weak Ergodic Theorem)**.** *If $\mathfrak{m}$ is an extreme left-invariant mean on G, then*

$$\int_G \mathfrak{m}((g \cdot \phi)\, \psi)\, d\eta(g) = \mathfrak{m}(\phi)\mathfrak{m}(\psi), \quad \textit{for all } \phi, \psi \in \ell^\infty(G), \tag{2.1}$$

and for all $\eta \in \mathcal{L}_G$.

Proof. Fix an extreme left-invariant mean $\mathfrak{m}$ on G. It suffices to prove the proposition for $\psi \in \ell^\infty(G)$ and $\phi \in \ell^\infty(G)$ such that

$$0 \le \phi \le 1 \quad \text{and} \quad 0 < \mathfrak{m}(\phi) < 1. \tag{2.2}$$

Indeed, if $\mathfrak{m}(\phi) = 0$, then $\mathfrak{m}((g \cdot \phi)\psi) = 0$ for all $g \in G$ and $\psi \in \ell^\infty(G)$, in which case the proposition is trivial. Given ϕ as in (2.2) and $\eta \in \mathcal{L}_G$, we define the mean η_ϕ on G by

$$\eta_\phi(\psi) = \frac{1}{\mathfrak{m}(\phi)} \int_G \mathfrak{m}(\phi\, (g^{-1} \cdot \psi))\, d\eta(g), \quad \text{for } \psi \in \ell^\infty(G).$$

One readily verifies that $\eta_\phi, \eta_{1-\phi} \in \mathcal{L}_G$ and

$$\lambda = \lambda(\phi)\eta_\phi + \lambda(1-\phi)\eta_{1-\phi}.$$

Since λ is an extreme mean, we conclude that $\lambda = \eta_\phi$. Hence, since $\lambda((g\cdot\phi)\psi) = \lambda(\phi(g^{-1}\cdot\psi))$ for all $g \in G$, we have

$$\lambda(\psi) = \eta_\phi(\psi) = \frac{1}{\lambda(\phi)}\int_G \lambda(\phi\,(g^{-1}\cdot\psi))\,d\eta(g) = \frac{1}{\lambda(\phi)}\int_G \lambda((g\cdot\phi)\,\psi)\,d\eta(g),$$

which finishes the proof. □

2.3. **Pointed G-spaces**

Let X be a compact and metrizable space, equipped with an action of G by homeomorphisms, and suppose that there exists a point x_o whose G-orbit in X is dense. We refer to the pair (X, x_o) as a **compact pointed G-space**.

We denote by $C(X)$ the Banach space of real-valued continuous functions on X, equipped with the uniform norm, and by $\mathcal{P}(X)$ the space of (regular) Borel probability measures on X, identified with the convex and weak*-compact subset of positive and unital functionals in the dual space $C(X)^*$. We note that the G-action on X gives rise to an isometric G-action on $C(X)$ and a weak*-continuous action on $\mathcal{P}(X)$ by

$$(g\cdot f)(x) = f(g^{-1}\cdot x) \quad \text{and} \quad (g\cdot\mu)(f) = \mu(g^{-1}\cdot f),$$

for $f \in C(X)$ and $\mu \in \mathcal{P}(X)$. We define the (possibly empty) set

$$\mathcal{P}_G(X) = \{\mu \in \mathcal{P}(X) \,:\, g\cdot\mu = \mu, \ \text{ for all } g \in G\}$$

of G-**invariant** Borel probability measures. If G is AMENABLE, then $\mathcal{P}_G(X)$ is non-empty for every pointed G-space.

We say that $\mu \in \mathcal{P}_G(X)$ is **ergodic** if any G-invariant Borel subset of X is either μ-null or μ-conull, and **extremal** if it cannot be written as a non-trivial convex combination of elements in $\mathcal{P}_G(X)$. By Krein-Milman's Theorem, extremal measures always exist in $\mathcal{P}_G(X)$, and it is a classical fact (see e.g. Theorem 4.4 in [6]) that $\mu \in \mathcal{P}_G(X)$ is ergodic if and only if it is extremal.

If μ is a G-invariant Borel probability measure, then its *support* (the set of all points in X which admit open neighborhoods with positive μ-measures), here denoted by $\mathrm{supp}(\mu)$, is a closed G-invariant subset of X. The following lemma will be useful later in the text.

Lemma 2.2. *For every ergodic $\mu \in \mathcal{P}_G(X)$, there exists a μ-conull subset $X' \subset X$ such that*

$$\mathrm{supp}(\mu) = \overline{G\cdot x}, \quad \textit{for all } x \in X'.$$

Proof. We may without loss of generality assume that $\mathrm{supp}(\mu) = X$. Let (U_n) be a countable basis for the topology on X, and define

$$X' = \bigcap_n GU_n \subset X.$$

Since μ is ergodic, we have $\mu(GU_n) = 1$ for all n, and thus X' is μ-conull. By construction, every $x \in X'$ has a dense G-orbit. □

2.4. **Bebutov triples**

We denote by 2^G the space of all subsets of G endowed with the Tychonoff topology, which renders 2^G compact and metrizable, and G acts by homeomorphisms on 2^G by

$$g\cdot A = Ag^{-1}, \quad \text{for } g \in G \text{ and } A \in 2^G.$$

One readily checks that the set $U = \{A \in 2^G \,:\, e \in A\}$ is clopen, and $U_A = A$ for all $A \in 2^G$. Let $X = \overline{G\cdot A}$ and set $x_o = A$. Then (X, x_o) is a compact pointed G-space. We shall consistently

abuse notation and denote by A the (clopen) intersection $U \cap X$, so that $A_{x_o} = A$ (where A on the right hand side refers to the set in G). We refer to (X, x_o, A) as the **Bebutov triple** of the set $A \subset G$.

2.5. **The Bebutov map**

Let us from now on assume that G is amenable, so that $\mathcal{L}_G$ is non-empty, and let (X, x_o) be a compact pointed G-space. We note that we have a G-equivariant isometric, positive and unital map $S_{x_o} : C(X) \to \ell^\infty(G)$ defined by

$$(S_{x_o} f)(g) = f(g \cdot x_o), \quad \text{for } g \in G. \tag{2.3}$$

We refer to S_{x_o} as the **Bebutov map** of (X, x_o). Its transpose $S^*_{x_o}$ maps $\mathcal{M}$ into $\mathcal{P}(X)$, and $\mathcal{L}_G$ into $\mathcal{P}_G(X)$. The following proposition shows that this map is in fact onto.

Proposition 2.3. *The transpose $S^*_{x_o} : \mathcal{L}_G \to \mathcal{P}_G(X)$ is onto and maps the set of extremal means in $\mathcal{L}_G$ onto the set of ergodic measures in $\mathcal{P}_G(X)$.*

Proof. We begin by proving that $S^*_{x_o} : \mathcal{L}_G \to \mathcal{P}_G(X)$ is onto. Fix $\nu \in \mathcal{P}_G(X)$ and define

$$\lambda_\nu(S_{x_o} f) = \nu(f), \quad \text{for all } f \in C(X).$$

One checks that λ_ν defines positive and unital functional on the subspace $S_{x_o}(C(X)) \subset \ell^\infty(G)$ of norm one. By Hahn-Banach's Theorem, λ_ν extends to $\ell^\infty(G)$ with norm one and $\lambda_\nu(1) = 1$. We abuse notation and use λ_ν to denote this extension as well. We wish to prove that λ_ν is positive (and thus a mean on G). We note that if φ is a non-negative function on G, then the inequality

$$\|\|\varphi\|_\infty - \varphi\|_\infty \leq \|\varphi\|_\infty$$

holds. Since λ_ν has norm one,

$$\|\varphi\|_\infty - \lambda_\nu(\varphi) = \lambda_\nu(\|\varphi\|_\infty - \varphi) \leq \|\|\varphi\|_\infty - \varphi\|_\infty \leq \|\varphi\|_\infty,$$

and thus $\lambda_\nu(\varphi) \geq 0$. We conclude that the set $Q \subset \mathcal{M}$ defined by

$$Q = \{\lambda \in \mathcal{M} \,:\, \lambda\big|_{S_{x_o}(C(X))} = \lambda_\nu\}$$

contains λ_ν, and is thus a *non-empty* convex and weak*-compact, which is invariant under the affine and weak*-continuous G-action on $\mathcal{M}$. Since G is amenable, G must fix an element λ in Q (see e.g. [22]). By construction, $S^*_{x_o}\lambda = \nu$.

Let us now prove that extreme points in $\mathcal{L}_G$ are mapped to ergodic measures in $\mathcal{P}_G(X)$. Fix an extreme left invariant mean λ and set $\nu = S^*\lambda$. By Proposition 2.1,

$$\int_G \lambda((S_{x_o} f)(g^{-1} \cdot S_{x_o} f))\, d\eta(g) = \int_G S^*_{x_o}\lambda(f(g^{-1} \cdot f))\, d\eta(g) = \int_G \nu(f(g^{-1} \cdot f))\, d\eta(g) = \nu(f)^2,$$

for all $f \in C(X)$ and $\eta \in \mathcal{L}_G$. By a straightforward approximation argument, we conclude that

$$\int_G \nu(\chi_B(g^{-1} \cdot \chi_B))\, d\eta(g) = \nu(B)^2, \quad \text{for every Borel set } B \subset X.$$

In particular, if $B \subset X$ is G-invariant, then $\nu(B) = \nu(B)^2$ and thus $\nu(B)$ is either zero or one, and hence ν is ergodic. □

2.6. **Correspondence principles**

Given $\mu \in \mathcal{P}(X)$, a Borel set $A \subset X$ is **μ-Jordan measurable** if its *boundary* $\partial A = \overline{A} \setminus A^o$ is a μ-null set. It is well-known (see e.g. Proposition 2.3.3 in [24]) that $A \subset X$ is μ-Jordan measurable if and only if for every $\varepsilon > 0$ there are *continuous* real-valued functions f_- an f_+ on X such that

$$f_- \leq \chi_A \leq f_+ \quad \text{and} \quad \mu(f_+ - f_-) < \varepsilon.$$

Of course, every clopen set if μ-Jordan measurable.

Let $\mu \in \mathcal{P}_G(X)$ and suppose that $A \subset X$ is a μ-Jordan measurable set. Fix $\varepsilon > 0$ and let f_- and f_+ be as above. If $\lambda \in \mathcal{L}_G$ and $\mu = S^*_{x_o}\lambda$, then

$$\mu(f_-) = \lambda(S_{x_o}f_-) \leq \lambda(A_{x_o}) \leq \lambda(S_{x_o}f_+) = \mu(f_+),$$

and thus $|\lambda(A_{x_o}) - \mu(A)| \leq \varepsilon$. Since ε is arbitrary, we conclude that $\lambda(A_{x_o}) = \mu(A)$. The following corollary is now immediate.

Corollary 2.4. *Suppose that $\mathcal{P}_G(X) = \{\mu\}$. Then, for any μ-Jordan measurable set $A \subset X$, we have $\lambda(A_{x_o}) = \mu(A)$ for all $\lambda \in \mathcal{L}_G$.*

We note that although the Bebutov map $(S_{x_o}f)(g) = f(g \cdot x_o)$ is well-defined for *all* functions f on X, it is not true that if $S^*_{x_o}\lambda = \mu$ (as functionals on $C(X)$), then $\lambda(S_{x_o}f) = \mu(f)$ for all (say, Borel measurable) functions. Indeed, unless the support of μ is finite, the set $E = X \setminus G \cdot x_o$ is Borel measurable with full μ-measure, but $S_{x_o}\chi_E$ is identically zero on G. However, some bounds can be asserted for *semi-continuous* functions, as the (proof of the) following lemma shows.

Lemma 2.5. *If $\lambda \in \mathcal{L}_G$ and $\mu = S^*_{x_o}\lambda$, then $\mu(U) \leq \lambda(U_{x_o})$ for every open set $U \subset X$.*

Proof. Fix an open set $U \subset X$. Since X is metrizable, we can find an increasing sequence (f_n) in $C(X)$ such that $\chi_U = \sup_n f_n$. Hence, with $\mu = S^*_{x_o}\lambda \in \mathcal{P}(X)$, we have

$$\lambda(U_{x_o}) = \lambda(\sup_n S_{x_o}f_n) \geq \lambda(S_{x_o}f_m) = \mu(f_m), \quad \text{for every } m.$$

By monotone convergence (recall that μ is σ-additive, although λ is not), this inequality is preserved upon taking the limit $m \to \infty$. □

In the introduction of this paper, we defined, for a given subset $A \subset G$, its *upper Banach density* $d^*(A)$ by

$$d^*(A) = \sup\{\overline{d}_{(F_n)}(A) \,:\, (F_n) \text{ is a Følner sequence in } G\},$$

and its *lower Banach density* $d_*(A)$ by

$$d_*(A) = \inf\{\underline{d}_{(F_n)}(A) \,:\, (F_n) \text{ is a Følner sequence in } G\},$$

The following (well-known) alternative definition will be useful later in the text. We sketch a proof for completeness.

Proposition 2.6. *For every $A \subset G$, we have*

$$d^*(A) = \sup\{\lambda(A) \,:\, \lambda \in \mathcal{L}_G\} \quad \textit{and} \quad d_*(A) = \inf\{\lambda(A) \,:\, \lambda \in \mathcal{L}_G\}.$$

Furthermore, for every $A \subset G$, we can always find extremal $\lambda_+, \lambda_- \in \mathcal{L}_G$ such that $d^*(A) = \lambda_+(A)$ and $d_*(A) = \lambda_-(A)$.

Sketch of proof. Since $d^*(A) = 1 - d_*(A^c)$, it suffices to prove the first identity (to make referencing easier, let $m^*(A)$ denote the right hand side of this identity). We begin by showing that for every $A \subset G$, and $\varepsilon > 0$, there exists $\mathfrak{n} \in \mathcal{L}_G$ such that $d^*(A) \leq \mathfrak{n}(A) + \varepsilon$. This readily implies that $d^*(A) \leq m^*(A)$. We note that by the definition of d^*, we can find a Følner sequence (F_n) such that $d^*(A) \leq \overline{d}_{(F_n)}(A) + \varepsilon$. Define the sequence $(\mathfrak{n}_n)$ in $\mathcal{M}$ by

$$\mathfrak{n}_n(\phi) = \frac{1}{|F_n|} \sum_{g \in F_n} \phi(g), \quad \text{for } \phi \in \ell^\infty(G).$$

One readily checks any weak*-accumulation point $\mathfrak{n}$ belongs to $\mathcal{L}_G$ and $\overline{d}_{(F_n)}(A) = \mathfrak{n}(A)$, and thus $d^*(A) \leq \mathfrak{n}(A) + \varepsilon$.

Let us now show that $m^*(A) \leq d^*(A)$. Since the set $\mathcal{L}_G$ is convex and weak*-compact and the map $\mathfrak{n} \mapsto \mathfrak{n}(A)$ is weak*-continuous and affine, there exists at least one *extremal* $\mathfrak{n}$ in $\mathcal{L}_G$ such that $m^*(A) = \mathfrak{n}(A)$ (by Bauer's Maximum Principle, see e.g. [1]). We abuse notation and let (X, x_o, A) denote the Bebutov triple of A, and S_{x_o} the corresponding Bebutov map. We set $\mu = S^*_{x_o}\mathfrak{n}$, and note that μ is ergodic (by Proposition 2.3) and $\mathfrak{n}(A_{x_o}) = \mu(A)$.

Let (F_n) be a Følner sequence in G. It follows from the Mean Ergodic Theorem that the sequence (f_n) in $L^2(X, \mu)$ defined by

$$f_n(x) = \frac{1}{|F_n|} \sum_{g \in F_n} \chi_A(g \cdot x)$$

converges to $\mu(A)$ in the L^2-norm, and thus $f_{n_k}(x) \to \mu(A)$ for μ-almost every x, along some sub-sequence (n_k). Let us fix $x \in X$ for which convergence holds. Since $A \subset X$ is CLOPEN, we can find a sequence (g_{n_k}) in G such that

$$A_{g_{n_k} \cdot x_o} \cap F_{n_k} = A_x \cap F_{n_k}, \quad \text{for all } k,$$

and thus

$$f_{n_k}(x) = \frac{|A_x \cap F_{n_k}|}{|F_{n_k}|} = \frac{|A_{g_{n_k} \cdot x_o} \cap F_{n_k}|}{|F_{n_k}|} = \frac{|A_{x_o} \cap F_{n_k} g_{n_k}|}{|F_{n_k}|} \to \mu(A).$$

Note that $(F_{n_k} g_{n_k})$ is still a Følner sequence in G. Hence, if we use this sequence instead of (F_n) in the definition of $(\mathfrak{n}_n)$ above, we can extract a weak*-accumulation point $\mathfrak{n} \in \mathcal{L}_G$ such that $\mathfrak{n}(A_{x_o}) = \mu(A)$, which finishes the proof. □

We now arrive at the following corollary, which is often referred to as Furstenberg's Correspondence Principle (see e.g. [7]).

Corollary 2.7 (Furstenberg's Correspondence Principle)**.** *For every compact pointed G-space (X, μ) and Borel set $A \subset X$ which is μ-Jordan measurable for every $\mu \in \mathcal{P}_G(X)$, we have*

$$d^*(A_{x_o}) = \sup\{\mu(A) \,:\, \mu \in \mathcal{P}_G(X)\} \quad \textit{and} \quad d_*(A_{x_o}) = \inf\{\mu(A) \,:\, \mu \in \mathcal{P}_G(X)\}$$

Furthermore, for every such set $A \subset X$, we can always find ergodic measures $\mu_+, \mu_- \in \mathcal{P}_G(X)$ such that $d^*(A_{x_o}) = \mu_+(A)$ and $d_*(A_{x_o}) = \mu_-(A)$.

2.7. **Compactifications**

Let K be a compact metrizable group and suppose that G admits a homomorphism τ into K with dense image. We refer to (K, τ_K) as a **compactification** of G. Let e_K denote the identity element in K and note that (K, e_K) is a compact pointed G-space under the action

$$(g, k) \mapsto g \cdot k = \tau_K(g)k, \quad \text{for } g \in G \text{ and } k \in K.$$

We note that if $I \subset K$ is any subset and $t \in K$, then $I_t = \tau_K^{-1}(It^{-1})$. More generally, if $L < K$ is a closed (not necessarily normal) subgroup of K, the (pointed) quotient space $(K/L, L)$ is also

a compact pointed G-space under the action $(g, kL) \mapsto \tau_K(g)kL$. If $I \subset K/L$ is any subset, we can alternatively view it as a right L-invariant subset of K, and if $t = kL$, then $I_t = \tau_K^{-1}(It^{-1})$, which is well-defined by the right-L-invariance of I.

Since $\tau_K(G)$ is dense in K, it is not hard to show that $\mathcal{P}_G(K/L) = \{m_{K/L}\}$, where $m_{K/L}$ denotes the Haar probability measure on K/L. In particular, the following observation is an immediate consequence of Corollary 2.4.

Corollary 2.8. *If (K, τ_K) is a compactification of G, $L < K$ is a closed subgroup and $I \subset K/L$ is Jordan measurable, then*

$$d^*(\tau_{K/L}^{-1}(I)) = m_{K/L}(I) = d_*(\tau_{K/L}^{-1}(I)),$$

where $\tau_{K/L} = p \circ \tau_K : G \to K/L$ and $p : K \to K/L$ is the canonical quotient map.

Let us now point out an important property of compactifications of countable *amenable* groups, which will play a crucial role in the proof of Theorem 3.2.

Proposition 2.9. *If K is a compact group with a dense countable amenable subgroup, then the identity component of K is abelian.*

Proof. We shall argue by contradiction: Suppose that there exist two elements $x, y \in K^o$ such that $xyx^{-1}y^{-1} \neq e_K$. By Peter-Weyl's Theorem, we can find a positive integer n and a representation π of K into $U(n)$ such that $\pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} \neq e_{\pi(K)}$. We note that $\pi(x)$ and $\pi(y)$ belong to the identity component of the (possibly not connected) compact Lie group $\pi(K)$. Let Γ_π denote the image of G under $\pi \circ \tau_K$; by assumption Γ_π is a dense countable amenable subgroup of $\pi(K)$. By Theorem 6.5 (iii) in [11], $\pi(K)^o$ has finite index in $\pi(K)$, and a straightforward argument shows that $\Lambda := \Gamma_\pi \cap \pi(K)^o$ is a dense countable amenable subgroup of $\pi(K)^o$. In particular, the commutator subgroup $[\Lambda, \Lambda]$ is a dense amenable subgroup of $[\pi(K)^o, \pi(K)^o]$. By Theorem 6.18 in [11], the latter group is a semisimple and connected compact Lie group. At this point, Tits' Alternative [23] can be applied: a non-trivial semisimple and connected compact Lie group cannot contain a dense amenable subgroup. Hence $\pi(K)^o$ is abelian, which contradicts $\pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} \neq e_{\pi(K)}$. □

2.8. **Borel G-spaces and their Kronecker-Mackey factors**

Let (Z, η) be a standard probability space, equipped with an action of G by measurable maps which preserve η. We refer to (Z, η) as a **Borel G-space**, and we say that it is **ergodic** if any G-invariant measurable subset of Z is either η-null or η-conull. If (Z, η) and (W, ∂) are Borel G-spaces, $Z' \subset Z$ and $W' \subset W$ are conull G-invariant sets, and $q : Z' \to W'$ is a measurable G-equivariant map, such that

$$\eta(q^{-1}(B)) = \partial(B), \quad \text{for all measurable } B \subset W,$$

we say that (W, ∂) is a **factor** of (Z, η) with **factor map** q, and we say that (Z, η) and (W, ∂) are **isomorphic** if q in addition is a bi-measurable bijection.

We see that if $\mathcal{B}_Z$ and $\mathcal{B}_W$ denote the σ-algebras on Z and W respectively, then $\mathcal{F}_q = q^{-1}(\mathcal{B}_W)$ is a G-invariant sub-σ-algebra of $\mathcal{B}_Z$. Conversely (see e.g. Theorem 6.5 in [6]), if $\mathcal{F}$ is any G-invariant sub-σ-algebra of $\mathcal{B}_Z$, then there exists a (unique up to isomorphisms) factor (W, ∂) and a factor map from (Z, η) to (W, ∂) such that $\mathcal{F} = q^{-1}(\mathcal{B}_W)$ modulo null sets. For this reason, we shall refer to G-invariant sub-σ-algebras of $\mathcal{B}_Z$ as **factors** as well, and to the corresponding Borel G-space (W, ∂) as the **factor Borel G-space** of $\mathcal{F}$.

Let (Z, η) and (W, ∂) be Borel G-spaces. We define their **direct product** as the Borel space $(Z \times W, \eta \otimes \partial)$ equipped with the diagonal action

$$(g, (z, w)) \mapsto g \cdot (z, w) = (g \cdot z, g \cdot w), \quad \text{for } g \in G,\ z \in Z \text{ and } w \in W.$$

We stress that even if both (Z,η) and (W,∂) are ergodic, $(Z\times W,\eta\otimes\eta)$ may not be ergodic. We denote by $\mathcal{E}_G(Z)$ the sub-σ-algebra of $\mathcal{B}_Z$ consisting of G-invariant Borel sets, and we denote by $\mathcal{K}$ the *smallest* sub-σ-algebra of $(Z\times Z,\eta\otimes\eta)$ such that $\mathcal{E}_G(Z\times Z)$ is contained in $\mathcal{K}\otimes\mathcal{K}$. We refer to $\mathcal{K}$ as the **Kronecker-Mackey factor** of (Z,η). The following proposition shows that the corresponding factor Borel G-space is quite special.

Proposition 2.10. *Let (Z,η) be an ergodic Borel G-space, $\mathcal{K}\subset\mathcal{B}_Z$ the Kronecker-Mackey factor of (Z,η) and (W,∂) the corresponding factor Borel G-space. Then there exists a metrizable compactification (K,τ_K) of G and a closed subgroup $L<K$ such that (W,∂) is isomorphic to $(K/L,m_{K/L})$, where G acts on K/L as in Subsection 2.7*

We shall refer to (K,L,τ_K) as the **Kronecker-Mackey triple** associated to (Z,η).

Proof. By a classical result of Mackey (Theorem 1 in [17]), it suffices to show that the sub-representation $L^2(Z,\mathcal{K},\eta)$ of the regular G-representation $L^2(Z,\eta)$, consisting of $\mathcal{K}$-measurable functions, decomposes into a direct sum of FINITE-DIMENSIONAL sub-representations. Let $\mathcal{L}\subset\mathcal{B}_Z$ denote the G-invariant sub-σ-algebra with respect to which all elements in $L^2(Z,\eta)$ with a finite-dimensional linear span are measurable. We shall prove that $\mathcal{E}_G(Z\times Z)\subset\mathcal{L}\otimes\mathcal{L}$, which by the definition of $\mathcal{K}$ forces $\mathcal{K}\subset\mathcal{L}$, and thus $L^2(Z,\mathcal{K},\eta)\subset L^2(Z,\mathcal{L},\eta)$. By definition, the right hand side does decompose into finite-dimensional sub-representations, which finishes the proof of the proposition.

To establish the inclusion $\mathcal{E}_G(Z\times Z)\subset\mathcal{L}\otimes\mathcal{L}$, we shall prove that any G-invariant function in $L^2(Z\times Z,\eta\otimes\eta)$ is measurable with respect to $\mathcal{L}\otimes\mathcal{L}$. Since every G-invariant function f can be written on the form $f=f_1+f_2$ where f_1,f_2 are G-invariant, and satisfy

$$f_1(z,w)=\overline{f_1(w,z)}\quad\text{and}\quad f_2(z,w)=-\overline{f_2(w,z)},$$

and the function $f_3=if_2$ satisfies the same equation as f_1, we see that it suffices to show that any G-invariant function f in $L^2(Z\times Z,\eta\otimes\eta)$ such that $f(z,w)=\overline{f(w,z)}$ almost everywhere, is $\mathcal{L}\otimes\mathcal{L}$-measurable.

Note that if f is any such function, then the operator $T_f:L^2(Z,\eta)\to L^2(Z,\eta)$ defined by

$$(T_f\phi)(z)=\int_Z f(z,w)\,\phi(w)\,d\eta(w),\quad\text{for }\phi\in L^2(Z,\eta),$$

is compact, self-adjoint and G-equivariant. Hence, by the Spectral Theorem for compact and self-adjoint operators, we have

$$T_f\phi=\sum_n \lambda_n\langle\phi,\psi_n\rangle\,\psi_n,\quad\text{for all }\phi\in L^2(Z,\eta),$$

where (ψ_n) and (λ_n) are the eigenvectors and eigenvalues of T_f respectively, and the sum converges in the L^2-sense. In particular, we have

$$f=\sum_n \lambda_n\,\psi_n\otimes\overline{\psi_n},$$

where the sum is taken in the weak topology on $L^2(Z\times Z,\eta\otimes\eta)$. Furthermore, the eigenspaces corresponding to non-zero λ_n's are finite-dimensional, and since T_f is G-equivariant, each such subspace is a sub-representation of the regular representation. Since only non-zero eigenvalues appear in the sum, and each corresponding eigenfunction ψ_n belongs to a finite-dimensional sub-representation of $L^2(Z,\eta)$, we conclude that each ψ_n is measurable with respect to $\mathcal{L}$, and thus f is measurable with respect to $\mathcal{L}\otimes\mathcal{L}$. □

2.9. **Joining containment**

Let (X, μ) and (Y, ν) be Borel G-spaces. A G-invariant Borel probability measure ξ on $X \times Y$ such that

$$\xi(A \times Y) = \mu(A) \quad \text{and} \quad \xi(X \times B) = \nu(B), \quad \text{for all Borel sets } A \subset X \text{ and } B \subset Y$$

is called a **joining** of (X, μ) and (Y, ν). We note that the product measure $\mu \otimes \nu$ is always a joining, and we denote by $\mathcal{J}_G(\mu, \nu)$ the set of all joinings of (X, μ) and (Y, ν). It is a fundamental fact, see. e.g. Theorem 6.2 in [8], that ergodic joinings always exist provided that (X, μ) and (Y, ν) are both ergodic. If $A \subset X$ and $B \subset Y$ are Borel sets, and $\xi \in \mathcal{J}_G(\mu, \nu)$, we say that A is **joining contained** in B, and we write $A \subseteq_\xi B$ if

$$\xi(\{(x, y) \,:\, A_x \subset B_y\}) = 1.$$

Equivalently, $A \subseteq_\xi B$ if $\xi(A \times B^c) = 0$. In particular, if $A \subseteq_\xi B$, then

$$\mu(A) = \xi(A \times Y) = \xi(A \times B) \leq \xi(X \times B) = \nu(B), \tag{2.4}$$

The following simple result will be useful later on.

Lemma 2.11. *Let (K, τ_K) be a metrizable compactification and $L < K$ a closed subgroup. Let (Z, η) be an ergodic Borel G-space and suppose that $(K/L, m_{K/L})$ is a factor of (Z, η) with factor map q. If $A \subset Z$ and $I \subset K/L$ are Borel sets, and $A \subset q^{-1}(I)$ modulo null sets, then there exists an ergodic $\xi \in \mathcal{J}_G(\eta, m_K)$ with the property that $A \subseteq_\xi p^{-1}(I)$, where $p : K \to K/L$ is the canonical quotient map.*

Sketch of proof. Let ξ_o denote the joining of (Z, η) and $(K/L, m_{K/L})$ defined by

$$\xi_o(A \times I) = \eta(A \cap q^{-1}(I)), \quad \text{for Borel sets } A \subset Z \text{ and } I \subset K/L.$$

Since η is ergodic, so is ξ_o. We have a natural quotient map from $Z \times K$ onto $Z \times K/L$ given by $(z, k) \mapsto (z, p(k))$. We can argue as in the proof of Proposition 2.3 to find a G-invariant measure ξ on $Z \times K$ which projects onto ξ_o. By construction, we have $\xi(A \times p^{-1}(I)^c) = \xi_o(A \times I^c)$, so if $A \subset I$ modulo null sets, then $\xi(A \times p^{-1}(I)^c) = 0$. We note that the same identity holds for almost every ergodic component of ξ (see e.g. Theorem 4.8 in [6]) so we may just as well assume that ξ is ergodic (we stress that the choice of ergodic component may depend on A). Since the projection of ξ onto K is G-invariant, it must coincide with m_K, and thus ξ is an ergodic joining of (Z, η) and (K, m_K). □

2.10. **Thickness and syndeticity**

Let G be a countable amenable group. Recall from the introduction that a subset $B \subset G$ is

- **thick** if for every finite set $F \subset G$, there is $s \in G$ such that $Fs \subset B$.
- **syndetic** if there exists a finite set $F \subset G$ such that $FB = G$.

The following alternative characterization of thickness will be useful.

Lemma 2.12. *A subset $B \subset G$ is thick if and only if $d^*(B) = 1$.*

Proof. Let (F_n) be a Følner sequence in G. If $B \subset G$ is thick, then we can find (s_n) such that $F_n s_n \subset B$ for all n, and thus $\overline{d}_{(F_n s_n)}(B) = 1$ (and hence $d^*(B) = 1$).

On the other hand, if $d^*(B) = 1$, then one readily checks that $d^*(\bigcap_{f \in F} fB) = 1$ for every finite set $F \subset G$, which shows that each such intersection is non-empty, and thus B is thick. □

One readily verifies that $B \subset G$ is *not* syndetic if and only if B^c is thick. Hence the previous lemma implies the following result.

Lemma 2.13. *A subset $B \subset G$ is syndetic if and only if $d_*(B) > 0$.*

3. Outlines of the proofs of Theorem 1.10 and Theorem 1.11

Let us now re-formulate Theorem 1.10 and Theorem 1.11 in a way which will fit better with the way that proofs are set up later in the text. For the convenience of the reader, we sketch below the proofs of the reductions that we will use.

In what follows, let G be a countable amenable group and let (X, x_o) be a compact pointed G-space. We shall also fix an ergodic G-invariant Borel probability measure μ on X and an ergodic p.m.p. G-space (Y, υ). In the proofs of Theorem 1.10 and Theorem 1.11 below, one should think of (X, x_o) as part of the Bebutov triple (X, x_o, A) associated to the set $A \subset G$, and μ is an ergodic Borel probability measure on X such that $\mu(A) = d^*(A_{x_o})$ (where A on the right hand side is a clopen subset of X).

Theorem 3.1. *If $A \subset X$ is open, $C \subset Y$ is a Borel set with positive υ-measure and*

$$\upsilon(A_{x_o}^{-1}C) < 1 \quad \textit{and} \quad \upsilon(A_{x_o}^{-1}C) < \mu(A) + \upsilon(C),$$

then there exist $A_o \subset A_{x_o}$ and a proper periodic set $P \subset G$ such that $d^*(A_o) \geq \mu(A)$ and $A_o \subset P$. If all finite quotients of G are abelian, then we can take $A_o = A_{x_o}$ and P can be chosen such that

$$d^*(P) < \mu(A) + \frac{1}{[G : \mathrm{Stab}_G(P)]}. \tag{3.1}$$

Theorem 3.2. *If $A \subset X$ is open, $C \subset Y$ is Borel and*

$$\upsilon(A_{x_o}^{-1}C) = \mu(A) + \upsilon(C) < 1,$$

then at least one of the following holds:

(1) There is $A_o \subset A_{x_o}$ with $d^*(A_o) \geq \mu(A)$ which is contained in a proper periodic set.

(2) There exists a finite-index subgroup $G_o < G$ such that $\upsilon(G_oC) < 1$.

(3) The set A_{x_o} is contained in a Sturmian set with the same upper Banach density as A_{x_o}.

3.1. Proof of Theorem 1.10 assuming Theorem 3.1

Let $A \subset X$ be a clopen set. By Corollary 2.7, there exists an ergodic $\mu \in \mathcal{P}_G(X)$ such that $d^*(A_{x_o}) = \mu(A)$. Let (Y, υ) be an ergodic Borel p.m.p. G-space and suppose that $B \subset Y$ is a Borel set with positive measure such that

$$\upsilon(A_{x_o}B) < 1 \quad \text{and} \quad \upsilon(A_{x_o}B) < \mu(A) + \upsilon(B).$$

We define $C = (A_{x_o}B)^c$ and note that $A_{x_o}^{-1}C \subset B^c$, and thus

$$\upsilon(C) > 0 \quad \text{and} \quad \upsilon(A_{x_o}^{-1}C) \leq 1 - \upsilon(B) < \mu(A) + \upsilon(C).$$

By Theorem 3.1, there exist a subset $A_o \subset A_{x_o}$ with $d^*(A_o) \geq \mu(A)$ and a proper periodic set $P \subset G$ such that $A_o \subset P$. Since $\mu(A) = d^*(A_{x_o}) \geq d^*(A_o) \geq \mu(A)$, we conclude that the set A_o has the same upper Banach density as A_{x_o}, and thus A_{x_o} is not spread-out.

Finally, if all finite quotients of G are abelian, then the last assertion of Theorem 3.1 tells us that we can take $A_o = A_{x_o}$ and P can be chosen such that (3.1) holds.

3.2. Proof of Theorem 1.11 assuming Theorem 3.2

Let $A \subset X$ be a clopen set. By Proposition 2.3, there exists an ergodic $\mu \in \mathcal{P}_G(X)$ such that $d^*(A_{x_o}) = \mu(A)$. Let (Y, υ) be an ergodic Borel p.m.p. G-space and suppose that $B \subset Y$ is a Borel set with positive measure such that

$$\upsilon(A_{x_o}B) = \mu(A) + \upsilon(B) < 1.$$

We define $C = (A_{x_o}B)^c$ and note that $\nu(C) > 0$ and $A_{x_o}^{-1}C \subset B^c$, and thus

$$\nu(A_{x_o}^{-1}C) \leq 1 - \nu(B) = \mu(A) + \nu(C).$$

If the inequality is strict, then Theorem 3.1 implies that A_{x_o} cannot be spread-out. Since $\nu(B) > 0$ we have $\nu(A_{x_o}^{-1}C) < 1$. Let us from now on assume that the set A_{x_o} is spread-out, so that

$$\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1.$$

By Theorem 3.2 we conclude that either the set A_{x_o} is contained in a Sturmian subset with the same upper Banach density as A_{x_o}, OR there exists a finite-index subgroup $G_o < G$ such that $\nu(G_oC) < 1$. In the latter case,

$$AB = C^c \supset (G_oC)^c =: Z.$$

We note that Z is a Borel set with positive measure which is invariant under G_o.

4. Outlines of the proofs of Theorem 1.1 and Theorem 1.5

In this section we show how to deduce Theorem 1.1 and Theorem 1.5 from Theorem 1.10 and Theorem 1.11 respectively. This will be done in several steps, so we kindly ask the reader for some patience.

As preparation for the proofs, let G be a countable amenable group and let us fix a Følner sequence (F_n) in G once and for all. We define a sequence $(\mathfrak{m}_n)$ of means on G by

$$\mathfrak{m}_n(\varphi) = \frac{1}{|F_n|}\sum_{g\in G}\varphi(g), \quad \text{for } \varphi \in \ell^\infty(G).$$

Since the set of means on G is weak*-compact, the set $\mathcal{F}$ of weak*-cluster points (of sub-nets, not necessarily sub-sequences) of the sequence $(\mathfrak{m}_n)$ is non-empty. It follows by the Følner property of (F_n) that each such weak*-cluster point is left-invariant, and thus $\mathcal{F} \subset \mathcal{L}_G$. Furthermore, one readily checks that

$$\overline{d}_{(F_n)}(B) = \sup\{\mathfrak{m}(B) \,:\, \mathfrak{m} \in \mathcal{F}\} \tag{4.1}$$

and

$$\underline{d}_{(F_n)}(B) = \inf\{\mathfrak{m}(B) \,:\, \mathfrak{m} \in \mathcal{F}\} \tag{4.2}$$

for every $B \subset G$. Since $\mathcal{F}$ is weak*-closed by construction and since for every fixed $B \subset G$, the map $\mathfrak{m} \mapsto \mathfrak{m}(B)$ is weak*-continuous on $\mathcal{L}_G$, there exist means $\mathfrak{m}_+$ and $\mathfrak{m}_-$ in $\mathcal{F}$ such that

$$\mathfrak{m}_+(B) = \overline{d}_{(F_n)}(B) \quad \text{and} \quad \mathfrak{m}_-(B) = \underline{d}_{(F_n)}(B). \tag{4.3}$$

4.1. Lower bounds on invariant means of product sets

Theorem 4.1. *If $A \subset G$ is spread-out, $B \subset G$ is syndetic and AB is not thick, then*

$$\mathfrak{m}(AB) \geq d^*(A) + \mathfrak{m}(B),$$

for every left-invariant mean $\mathfrak{m}$ on G. If all finite quotients of G are abelian, then instead of assuming that A is spread-out, it suffices to assume that A is large and not contained in a proper periodic subset $P \subset G$ such that (1.2) *holds.*

Theorem 4.2. *If $A \subset G$ is spread-out, $B \subset G$ is syndetic, AB does* NOT *contain a piecewise periodic set and there exists a left-invariant mean $\mathfrak{m}$ on G such that*

$$\mathfrak{m}(AB) = d^*(A) + \mathfrak{m}(B) < 1,$$

then A is contained in a Sturmian set with the same upper Banach density as A.

4.2. **Proof of Theorem 1.1 assuming Theorem 4.1**

Let $\mathcal{F}$ be as above, and suppose that $A \subset G$ is spread-out, $B \subset G$ is syndetic and AB is not thick. By (4.3), we can find $\mathfrak{n}_+$ and $\mathfrak{n}_-$ in $\mathcal{F}$ such that

$$\mathfrak{n}_+(B) = \overline{d}_{(F_n)}(B) \quad \text{and} \quad \mathfrak{n}_-(AB) = \underline{d}_{(F_n)}(AB).$$

By (4.1) and (4.2) we further have

$$\overline{d}_{(F_n)}(AB) \geq \mathfrak{n}_+(AB) \quad \text{and} \quad \mathfrak{n}_-(B) \geq \underline{d}_{(F_n)}(B).$$

By Theorem 4.1,

$$\mathfrak{n}_+(AB) \geq d^*(A) + \mathfrak{n}_+(B) \quad \text{and} \quad \mathfrak{n}_-(AB) \geq d^*(A) + \mathfrak{n}_-(B),$$

from which the first part of Theorem 1.1 now follows. If all finite quotients of G are abelian, then instead of assuming that A is spread-out, it suffices to assume that A is large and not contained in a proper periodic subset $P \subset G$ such that (1.2) holds.

4.3. **Proof of Theorem 1.5 assuming Theorem 4.2**

Let $\mathcal{F}$ be as above, and suppose that $A \subset G$ is spread-out, $B \subset G$ is syndetic and AB does not contain a piecewise periodic set. In particular, AB is not thick. Suppose that either

$$\overline{d}_{(F_n)}(AB) = d^*(A) + \overline{d}_{(F_n)}(B) < 1 \quad \text{or} \quad \underline{d}_{(F_n)}(AB) = d^*(A) + \underline{d}_{(F_n)}(B) < 1.$$

By (4.3), we can find $\mathfrak{n}_+$ and $\mathfrak{n}_-$ in $\mathcal{F}$ such that

$$\mathfrak{n}_+(B) = \overline{d}_{(F_n)}(B) \quad \text{and} \quad \mathfrak{n}_-(AB) = \underline{d}_{(F_n)}(AB).$$

By (4.1) and (4.2) we further have

$$\overline{d}_{(F_n)}(AB) \geq \mathfrak{n}_+(AB) \quad \text{and} \quad \mathfrak{n}_-(B) \geq \underline{d}_{(F_n)}(B).$$

Hence, either

$$\mathfrak{n}_+(AB) \leq d^*(A) + \mathfrak{n}_+(B) \quad \text{or} \quad \mathfrak{n}_-(AB) \leq d^*(A) + \mathfrak{n}_-(B).$$

On the other hand, these upper bounds are also lower bounds by Theorem 4.1, so we conclude that either

$$\mathfrak{n}_+(AB) = d^*(A) + \mathfrak{n}_+(B) < 1 \quad \text{or} \quad \mathfrak{n}_-(AB) = d^*(A) + \mathfrak{n}_-(B) < 1.$$

By Theorem 4.2, A is contained in a Sturmian subset of G with the same upper Banach density as A.

4.4. **Interlude: A few words about ergodic decomposition**

In this interlude, we explain the relevance of the assumptions on the sets A, B and AB above in the dynamical setting of Theorem 1.1 and Theorem 1.5.

In what follows, let G be a countable amenable group and let (Y, y_o) be a compact pointed G-space. Let us fix a left-invariant mean on G once and for all, and denote by ν the image in $\mathcal{P}_G(Y)$ of $\mathfrak{n}$ under the adjoint of the Bebutov map of (Y, y_o). If $A \subset G$ and $B \subset Y$ is a clopen subset, then by Lemma 2.5, we have

$$\mathfrak{n}(B_{y_o}) = \nu(B) \quad \text{and} \quad \mathfrak{n}(AB_{y_o}) \geq \nu(AB). \tag{4.4}$$

We stress that ν may not be ergodic. However, it is a very useful fact in ergodic theory that ν can always be "decomposed" into "ergodic components"; more precisely (see e.g. Theorem 4.8 in [6]), there exists a unique Borel probability measure κ on $\mathcal{P}_G(Y)$, which is supported on the set of ERGODIC G-invariant Borel probability measures on Y such that

$$\nu(D) = \int_{\mathcal{P}_G(Y)} \nu'(D)\, d\kappa(\nu'), \quad \text{for every Borel set } D \subset Y. \tag{4.5}$$

Let $B \subset Y$ be a clopen set. By Lemma 2.7, we have

$$\inf\{\nu'(B) \,:\, \nu' \in \operatorname{supp}(\kappa)\} \geq d_*(B_{y_o}).$$

Furthermore, since the action set $AB \subset Y$ is open and $(AB)_{y_o} = AB_{y_o}$, we have by Lemma 2.7 and Lemma 2.5

$$\sup\{\nu'(AB) \,:\, \nu' \in \operatorname{supp}(\kappa)\} \leq d^*(AB_{y_o}).$$

Hence, if we assume that $B_{y_o} \subset G$ is SYNDETIC and AB_{y_o} is NOT THICK, then Lemma 2.12 and Lemma 2.13 tell us that we must have

$$\inf\{\nu'(B) \,:\, \nu' \in \operatorname{supp}(\kappa)\} > 0 \quad \text{and} \quad \sup\{\nu'(AB) \,:\, \nu' \in \operatorname{supp}(\kappa)\} < 1.$$

We conclude by Theorem 1.10 that if $A \subset G$ is spread-out (or, if all finite quotients of G are abelian, that A is not contained in proper periodic subset P which satisfies (1.2)), B_{y_o} is syndetic and AB_{y_o} is not thick, then

$$\nu'(AB) \geq d^*(A) + \nu'(B), \quad \text{for } \kappa\text{-a.e. } \nu'. \tag{4.6}$$

Finally, let us assume that A is spread-out, B_{y_o} is syndetic and there exists at least one ν' in the support of κ such that

$$\nu'(AB) = d^*(A) + \nu'(B) < 1.$$

By Theorem 1.11, this means that A is either contained in a Sturmian subset of G with the same upper Banach density as A, or the action set AB contains a Borel set Z which is invariant under a finite-index subgroup. In the latter case, AB_{y_o} must contain a piecewise periodic set by Lemma 10.4 (applied to the open set $U = AB \subset Y$).

4.5. **Proof of Theorem 4.1 assuming Theorem 1.10**

Assume that $A \subset G$ is spread-out, or, if all finite quotients of G are abelian, that A is not contained in proper periodic subset P which satisfies (1.4)). Further assume that $B \subset G$ is syndetic and AB is not thick.

We abuse notation and write (Y, y_o, B) for the Bebutov triple of B. Let $\mathfrak{m}$ be a left-invariant mean on G, and denote by ν the image of $\mathfrak{m}$ under the adjoint of the Bebutov map. If κ is the ergodic decomposition of ν, then the discussion in the interlude above shows that

$$\nu'(AB) \geq d^*(A) + \nu'(B), \quad \text{for } \kappa\text{-a.e. } \nu',$$

and thus, by (4.5),

$$\nu(AB) = \int_{\mathcal{P}_G(Y)} \nu'(AB)\, d\kappa(\nu) \geq d^*(A) + \int_{\mathcal{P}_G(Y)} \nu'(B)\, d\kappa(\nu') = d^*(A) + \nu(B),$$

Finally, by (4.4), we conclude that

$$\mathfrak{m}(AB_{y_o}) \geq \nu(AB) \geq d^*(A) + \nu(B) = d^*(A) + \mathfrak{m}(B_{y_o}),$$

which finishes the proof.

4.6. **Proof of Theorem 4.2 assuming Theorem 1.11**

Assume that $A \subset G$ is spread-out, $B \subset G$ is syndetic and AB does not contain a piecewise periodic subset. We also assume that there exists a left-invariant mean $\mathfrak{m}$ on G such that

$$\mathfrak{m}(AB) = d^*(A) + \mathfrak{m}(B) < 1. \tag{4.7}$$

We abuse notation and denote by (Y, y_o, B) the Bebutov triple of B. Let ν be the image of $\mathfrak{m}$ under the adjoint of the Bebutov map. If κ is the ergodic decomposition of ν, then by (4.4), (4.5), (4.6) and (4.7), we have

$$\int_{\mathcal{P}_G(Y)} \big(d^*(A) + \nu(B)\big)\, d\kappa(\nu') = \mathfrak{m}(AB_{y_o}) \geq \nu(AB) \geq \int_{\mathcal{P}_G(Y)} \big(d^*(A) + \nu'(B)\big)\, d\kappa(\nu').$$

We conclude that the inequalities are in fact identities, and thus

$$\nu'(AB) = d^*(A) + \nu'(B) < 1, \quad \text{for } \kappa\text{-a.e. } \nu'.$$

By the last part of the discussion in the interlude above (recall that we assume that AB_{y_o} does not contain a piecewise periodic set), this implies that A must be contained in a Sturmian set with the same upper Banach density as A.

5. Correspondences for action sets and the proofs of Theorem 3.1 and Theorem 3.2

Let us now formulate our two main technical ingredients in this paper, and deduce Theorem 3.1 and Theorem 3.2 from them, using some results about products of Borel sets in compact groups.

In what follows, let (X, x_o) be a compact pointed G-space and let μ be an ergodic G-invariant Borel probability measure on X. Let (Y, ν) be an ergodic Borel G-space. If K is a compact group, we say that a Borel set $I \subset K$ is **spread-out** if every Borel set $I' \subset I$ with the same Haar measure as I projects onto every finite quotient of K. Equivalently, $I'U = K$ for every *open* subgroup U of K.

We note that a proper subset of a finite group can never be spread-out, and if K is totally disconnected, then every spread-out set in K must be dense. Indeed, if K is totally disconnected, then open subgroups form a neighborhood basis of the identity, so if $I \subset K$ is a spread-out set which is *not* dense, then there exists an open subgroup U of K and $k \in K$ such that $I \cap kU = \emptyset$, contradicting the fact that $IU = K$. Finally, if $I, J \subset K$ are Borel sets, and M is a factor group of K with factor map $p : K \to M$, we say that (I, J) **reduces** to a pair (I_o, J_o) in M if

$$I \subset p^{-1}(I_o) \quad \text{and} \quad J \subset p^{-1}(J_o) \quad \text{and} \quad m_K(I^{-1}J) = m_M(I_o^{-1}J_o). \tag{5.1}$$

Our first result shows that if $A \subset X$ is open and $C \subset Y$ is Borel, then the sets A and C are joining contained in some Borel subsets I and J of a compact metrizable group K, while the action set $A_{x_o}^{-1}C$ in Y is "larger" than the product set $I^{-1}J$ in K.

Proposition 5.1 (Correspondence Principle I)**.** *There exist*

- *a metrizable compactification (K, τ_K) of G,*
- *ergodic joinings $\xi_\mu \in \mathcal{J}_G(\mu, m_K)$ and $\xi_\nu \in \mathcal{J}_G(\nu, m_K)$,*

with the following properties: For every open set $A \subset X$ and Borel set $C \subset Y$, there are Borel sets $I, J \subset K$ such that

$$A \subseteq_{\xi_\mu} I \quad \text{and} \quad C \subseteq_{\xi_\nu} J$$

and

$$\nu(A_{x_o}^{-1}C) \geq \mu \otimes \nu(G(A \times C)) \geq m_K(I^{-1}J).$$

Furthermore, if $\nu(A_{x_o}^{-1}C) < 1$, and

- *if there is no subset $A_o \subset A_{x_o}$ with $d^*(A_o) \geq \mu(A)$, which is contained in a proper periodic set, then $I \subset K$ is spread-out.*
- *if there is no finite-index subgroup $G_o < G$ such that $\nu(G_oC) < 1$, then $J \subset K$ is spread-out.*

Our second result upgrades the joining containment to "local containment" provided that the pair (I, J) from Proposition 5.1 reduces to "nice" subsets in some factor group of K.

Proposition 5.2 (Correspondence Principle II)**.** *Let $A \subset X$ be an open set and $C \subset Y$ a Borel set, and let (K, τ_K) and $I, J \subset K$ be as in Proposition 5.1. Let M be a factor group of K with canonical factor map $p : K \to M$, and suppose that (I, J) reduces to a Borel pair (I_o, J_o) in M such that I_o is Jordan measurable in M. Then there exists a subset $A_o \subset A_{x_o}$ and $t \in M$ such that*

$$d^*(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset \tau_M^{-1}(I_o t),$$

and for all $s \in A_{x_o} \setminus A_o$, we have

$$\nu(A_{x_o}^{-1}C) - \nu(C) \geq m_M(I_o^{-1}J_o) - m_M(J_o) + m_M(\tau_M(s)^{-1}J_o \setminus t^{-1}I_o^{-1}J_o),$$

where $\tau_M = p \circ \tau_K$.

The proof of Correspondence Principle I, modulo the two last assertions, will be outlined in the next section. The proofs of the remaining assertions and Correspondence Principle II will be outlined in Section 10.

5.1. **Proof of Theorem 3.1**

Let $A \subset X$ be an open set and $C \subset Y$ a Borel set, and suppose that

$$\nu(A_{x_o}^{-1}C) < 1 \quad \text{and} \quad \nu(A_{x_o}^{-1}C) < \mu(A) + \nu(C).$$

By Proposition 5.1, there exist

- a metrizable compactification (K, τ_K) of G,
- Borel sets $I, J \subset K$ with $m_K(I) \geq \mu(A)$ and $m_K(J) \geq \nu(C)$

such that

$$m_K(I^{-1}J) \leq \nu(A_{x_o}^{-1}C) < \mu(A) + \nu(C) \leq m_K(I) + m_K(J). \tag{5.2}$$

At this point, we recall the following classical results by Kemperman and Kneser.

Theorem 5.3 (Theorem 1, [14])**.** *Let K be a compact Hausdorff group and let $I, J \subset K$ be Borel sets. If*

$$m_K(I^{-1}J) < 1 \quad \text{and} \quad m_K(I^{-1}J) < m_K(I) + m_K(J),$$

then (I, J) reduces to a pair (I_o, J_o) of proper subsets of a finite factor group M of K. In particular, neither I nor J can be spread-out.

Furthermore, if the inclusion $s^{-1}J_o \subset I_o^{-1}J_o$ holds for some $s \in M$, then $s \in I_o$.

Remark 5.4. The last assertion in Theorem 5.3 is not explicitly stated in [14], so we collect here the necessary steps in its deduction. Let M be finite group. Given a pair (I_o, J_o) of subsets of M, we define

$$I_1^{-1} := \bigcap_{y \in J_o} I_o^{-1}J_o y^{-1},$$

and note that $I_o \subset I_1$ and $I_o^{-1}J_o = I_1^{-1}J_o$. In particular, if the Borel pair (I, J) in K reduces to (I_o, J_o) in M, then it also reduces to (I_1, J_o). Furthermore, by construction, if $s \in M$ is such that $s^{-1}J_o \subset I_o^{-1}J_o = I_1^{-1}J_o$, then $s \in I_1$.

Theorem 5.5 (Satz 1, [16])**.** *Let K, I and J be as in Theorem 5.3. If every finite factor group of K is* ABELIAN*, then M and (I_o, J_o) can be chosen so that $\mathrm{Stab}_M(I_o)$ is trivial and*

$$m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) - m_M(\{e_M\}).$$

Furthermore, if the inclusion $s^{-1}J_o \subset I_o^{-1}J_o$ holds for some $s \in M$, then $s \in I_o$.

Remark 5.6. We stress that the identity for (I_o, J_o) in the theorem does NOT hold for arbitrary finite groups. Explicit examples (with trivial stabilizers) were first constructed by Olson in [21]. The failure of this identity is the main reason why we cannot establish global "periodic" containment in Theorem 1.1 for general amenable groups.

Let us now return to the pair (I, J) in (5.2). By Theorem 5.3, this pair must reduce to a pair (I_o, J_o) of proper subsets of a *finite* factor group M of K. Let $p : K \to M$ denote the factor map. Of course, I_o is Jordan measurable in M. By Proposition 5.2, we can find $A_o \subset A_{x_o}$ and $t \in M$ such that

$$d^*(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subseteq \tau_M^{-1}(I_o t),$$

and for all $s \in A_{x_o} \setminus A_o$, we have (since $m_M(I_o) \geq m_K(I)$)

$$m_M(I_o) \geq \mu(A) > \nu(A_{x_o}^{-1}C) - \nu(C) \geq m_M(I_o^{-1}J_o) - m_M(J_o) + m_M(\tau_M(s)^{-1}J_o \setminus t^{-1}I_o^{-1}J_o), \tag{5.3}$$

where $\tau_M = p \circ \tau_K$. Note that since M is finite, $P = \tau_M^{-1}(I_o t)$ is a proper periodic set, and thus we have established the first part of Theorem 3.1.

Let us now assume that all finite quotients of G are abelian. Suppose that N is a finite factor of K and $q : K \to N$ the corresponding factor map. Then $r = q \circ \tau_K$ is factor map from G onto N, so by our assumption, N is abelian. We conclude that every finite factor group of K is abelian, and thus Theorem 5.5 tells us that we can choose M and (I_o, J_o) above such that I_o has a trivial stabilizer in M and

$$m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) - m_M(\{e_M\}).$$

If we plug this into (5.3), we conclude that

$$m_M(\{e_M\}) > m_M(\tau_M(s)^{-1}J_o \setminus t^{-1}I_o^{-1}J_o), \quad \text{for all } s \in A_{x_o} \setminus A_o,$$

and thus the right hand side must be zero, from which we deduce that $\tau_M(s)^{-1}J_o \subset t^{-1}I_o^{-1}J_o$ for all $s \in A_{x_o} \setminus A_o$.

By the second part of Theorem 5.3, we see that this forces $\tau_M(s)t^{-1} \in I_o$ for all $s \in A_{x_o} \setminus A_o$. Since A_o is already contained in $\tau_M^{-1}(I_o t)$, we conclude that $A_{x_o} \subset P = \tau_M^{-1}(I_o t)$ as well. Since I_o has a trivial stabilizer in M, we see that $[G : \mathrm{Stab}_G(P)]^{-1} = m_M(\{e_M\})$ and, by (5.3),

$$d^*(P) = m_M(I_o) = m_M(I_o^{-1}J_o) - m_M(J_o) + m_M(\{e_M\}) < \mu(A) + \frac{1}{[G : \mathrm{Stab}_G(P)]},$$

which finishes the proof.

5.2. **Proof of Theorem 3.2**

Let $A \subset X$ be an open set and $C \subset Y$ a Borel set, and suppose that

$$\nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) < 1.$$

By Proposition 5.1, there exist

- a metrizable compactification (K, τ_K) of G,
- Borel sets $I, J \subset K$ with $m_K(I) \geq \mu(A)$ and $m_K(J) \geq \nu(C)$

such that

$$m_K(I^{-1}J) \leq \nu(A_{x_o}^{-1}C) = \mu(A) + \nu(C) \leq m_K(I) + m_K(J).$$

If at least one of the inequalities above is strict, then Theorem 5.3 implies that neither I nor J can be spread-out, and thus the second part of Proposition 5.1 tells us that

- there exists $A_o \subset A_{x_o}$ with $d^*(A_o) \geq \mu(A)$ which is contained in a proper periodic subset, and
- there exists a finite-index subgroup $G_o < G$ such that $\nu(G_oC) < 1$.

Let us henceforth assume that neither of these conclusions hold, so that both I and J are spread-out and thus

$$m_K(I^{-1}J) = m_K(I) + m_K(J) < 1. \tag{5.4}$$

By Proposition 2.9, every compactification of G (being a countable amenable group) must have an ABELIAN identity component. Since K is metrizable, the following theorem by the first-named author applies.

Theorem 5.7 (Theorem 1.8, [3]). *Let K be a compact metrizable group with an abelian identity component, and suppose that $I, J \subset K$ are spread-out Borel sets such that*

$$m_K(I^{-1}J) = m_K(I) + m_K(J) < 1.$$

Let M denote either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$, where the multiplicative group $\{-1, 1\}$ acts on $\mathbb{T}$ by multiplication, and let $I', J' \subset \mathbb{T}$ be the unique closed and symmetric intervals such that $m_{\mathbb{T}}(I') = m_K(I)$ and $m_{\mathbb{T}}(J') = m_K(J)$. Then there exist $a, b \in M$ such that (I, J) reduces to either

$$(I_o, J_o) = (I' + a, J' + b) \quad \text{or} \quad (I_o, J_o) = ((I' \rtimes \{-1, 1\})a, (J' \rtimes \{-1, 1\})b), \tag{5.5}$$

depending on whether $M = \mathbb{T}$ or $M = \mathbb{T} \rtimes \{-1, 1\}$.

Remark 5.8. If M is either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$, and (M, τ) is a compactification of G, and (I_o, J_o) is as in (5.5), then $S := \tau^{-1}(I_o)$ is a Sturmian set in G. Since I_o is Jordan measurable in M, Corollary 2.8 implies that $d^*(S) = d_*(S) = m_M(I_o)$.

Let us now return to the pair (I, J) in (5.4) above. By the previous theorem, it must reduce to a pair (I_o, J_o) of the form (5.5) in either

$$M = \mathbb{T} \quad \text{or} \quad M = \mathbb{T} \rtimes \{-1, 1\}.$$

Since I_o is clearly Jordan measurable, Proposition 5.2 tells us that there exist a subset $A_o \subset A_{x_o}$ and $t \in M$ such that

$$d^*(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subseteq \tau_M^{-1}(I_o t),$$

and for all $s \in A_{x_o} \setminus A_o$, we have

$$m_M(I_o) \geq \mu(A) = \nu(A_{x_o}^{-1}C) - \nu(C) \geq m_M(I_o^{-1}J_o) - m_M(J_o) + m_M(\tau_M(s)^{-1}J_o \setminus t^{-1}I_o^{-1}J_o). \tag{5.6}$$

Since $m_M(I_o^{-1}J_o) = m_M(I_o) + m_M(J_o) < 1$, we conclude that $m_M(I_o) = \mu(A)$ and

$$m_M(\tau_M(s)^{-1}J_o \setminus t^{-1}I_o^{-1}J_o) = 0, \quad \text{for all } s \in A_{x_o} \setminus A_o.$$

It is now straightforward to check that this implies $\tau_M(s)t^{-1} \in I_o$ for all $s \in A_{x_o} \setminus A_o$. Since A_o is already contained in the Sturmian set $S = \tau_M^{-1}(I_o t)$ and $d^*(S) = m_M(I_o) = \mu(A)$, we conclude that A_{x_o} is also contained in S. If we collect all of the identities and lower bounds above, we get

$$d^*(S) \geq d^*(A_{x_o}) \geq \mu(A) = m_M(I_o) = d^*(S).$$

We conclude that the set A_{x_o} is contained in a Sturmian set with the same upper Banach density as A_{x_o}.

5.3. **Appendix: Ergodicity of spread-out sets in finitely generated torsion groups**

Let G be a countable amenable group. We say that a subset $A \subset G$ is **ergodic** if for every ergodic Borel G-space (Y, ν) and for every Borel set $B \subset Y$ with positive ν-measure, we have $\nu(AB) = 1$. By definition, G itself is always an ergodic set. In this appendix, we prove the following result.

Proposition 5.9. *If G is a finitely generated amenable* TORSION *group, then every spread-out set in G is ergodic.*

We first note that if K is a compact group and $I \subset K$ is a dense subset, then $m_K(I^{-1}J) = 1$ for every Borel set $J \subset K$ with positive Haar measure. Indeed, if it were not the case, then for some $J \subset K$ with positive measure, there would be another Borel set $D \subset K$ with positive measure such that $I^{-1} \cap DJ^{-1} = \emptyset$. However, as is well-known (Steinhaus' Lemma), DJ^{-1} must have non-empty interior, and thus intersects I^{-1} non-trivially (being a dense set).

Let $A \subset G$ be a spread-out set and suppose that $B \subset Y$ is a Borel set with positive measure such that $\nu(AB) < 1$. Then $C = (AB)^c$ has positive measure, and $\nu(A^{-1}C) < 1$. Let (X, x_o, A) denote the Bebutov triple of A. Since A is large, there exists by Corollary 2.7, an ergodic $\mu \in \mathcal{P}_G(X)$ such that $\mu(A) = d^*(A_{x_o})$. By Correspondence Principle I, there exists a metrizable compactification (K, τ_K) of G and Borel sets $I, J \subset K$ such that

$$m_K(I) \geq \mu(A) > 0 \quad \text{and} \quad 1 > \nu(A_{x_o}^{-1}C) \geq m_K(I^{-1}J).$$

By the last part of Correspondence Principle I, if A_{x_o} is spread-out, then so is I. Hence, if we can prove that K is totally disconnected, then I must be dense (the argument is given in the introduction of this section), and thus the first part of this appendix shows that $m_K(I^{-1}J) = 1$, which is a contradiction. What remains is then to prove the following lemma of independent interest.

Lemma 5.10. *Let G be a finitely generated torsion group and suppose that (K, τ_K) is a compactification of G. Then K is totally disconnected.*

Proof. By Corollary 2.36 in [11] we can find a family (N_a) of closed normal subgroups of K and a family (n_a) of integers such that

$$\bigcap_a N_a = \{e_K\} \quad \text{and} \quad K_a := K/N_a \subset U(n_a) \text{ for all } a,$$

where $U(n_a)$ denotes the unitary group of dimension n_a. Let $\Gamma = \tau_K(G)$, which is again a finitely generated torsion group, and note that for every a, the quotient $\Gamma_a = \Gamma/\Gamma \cap N_a$ is dense in $K_a \subset U(n_a)$. By the Jordan-Schur Theorem, Γ_a (being a finitely generated torsion subgroup of a linear group) must be finite, and thus K_a is finite as well. We conclude that N_a is open for every a, which shows that the family (N_a) is a neighborhood basis of K, and thus K is totally disconnected. □

6. Proof of Correspondence Principle I

In this section, we break down the proof of Proposition 5.1 (Correspondence Principle I) into three propositions, whose proofs will be postponed to later sections.

Throughout the section, let G be a countable amenable group and let (X, x_o) be a compact metrizable pointed G-space. We fix an ergodic G-invariant Borel probability measure μ on X, and an ergodic Borel G-space (Y, ν).

6.1. **Step I: Symmetrization**

The following proposition is established in Section 7 below.

Proposition 6.1. *There exist an ergodic joining $\eta \in \mathcal{J}_G(\mu, \nu)$ such that for every open set $A \subset X$ and Borel set $C \subset Y$, we have*

$$\nu(A_{x_o}^{-1} C) \geq \mu \otimes \nu(G(A \times C)) = \eta \otimes \eta(G(A' \times C')),$$

where

$$A' = A \times Y \quad \text{and} \quad C' = X \times C.$$

6.2. **Step II: Finding a compactification**

We set $Z = X \times Y$ and let η be as in Proposition (6.1) so that (Z, η) is an ergodic Borel G-space. Recall the definition of a Kronecker-Mackey triple from Subsection 2.8. The following proposition will be established in Section 8.

Proposition 6.2. *Let (K, L, τ_K) be Kronecker-Mackey triple of (Z, η), and let $q : Z \to K/L$ denote the corresponding factor map. Then, for all Borel sets $A', C' \subset Z$, there are Borel sets $I', J' \subset K/L$ such that*

$$A' \subset q^{-1}(I') \quad \text{and} \quad C' \subset q^{-1}(J'),$$

modulo η-null sets, and

$$\eta \otimes \eta(G(A' \times C')) = m_{K/L} \otimes m_{K/L}(G(I' \times J')).$$

If $p : K \to K/L$ is the canonical quotient map, then Lemma 2.11 shows that we can find an ergodic joining ξ of (Z, η) and (K, m_K) such that

$$A' \subseteq_{\xi} p^{-1}(I') \quad \text{and} \quad C' \subseteq_{\xi} p^{-1}(J')$$

Let us from now on abuse notation and write I' and J' for the lifts $p^{-1}(I')$ and $p^{-1}(J')$, which we think of as right-L-invariant Borel sets in K. Let ξ_μ and ξ_ν denote the projections of ξ onto $X \times K$ and $Y \times K$ respectively. One readily checks that ξ_μ and ξ_ν are ergodic joinings in $\mathcal{J}_G(\mu, m_K)$ and $\mathcal{J}_G(\nu, m_K)$ respectively, and if $A' = A \times Y$ and $C' = X \times C$, we have

$$A \subseteq_{\xi_\mu} I' \quad \text{and} \quad C \subseteq_{\xi_\nu} J'.$$

6.3. **Step III: Trimming sets in direct products**

Let us briefly summarize what we have done so far. Given (X, x_o), μ and (Y, ν) we have found a metrizable compactification (K, τ_K) of G and ergodic joinings $\xi_\mu \in \mathcal{J}_G(\mu, m_K)$ and $\xi_\nu \in \mathcal{J}_G(\nu, m_K)$ such that for any open set $A \subset X$ and Borel set $C \subset Y$, there are Borel sets $I', J' \subset K$ such that

$$A \subseteq_{\xi_\mu} I' \quad \text{and} \quad C \subseteq_{\xi_\nu} J' \quad \text{and} \quad \nu(A_{x_o}^{-1} C) \geq m_K \otimes m_K(\tau_K(G)(I' \times J')),$$

where we think of $\Gamma := \tau_K(G)$ as a dense subgroup of the diagonal $\Delta(K)$ in $K \times K$. The following proposition, which will be established in Section 9, finishes the proof of Correspondence Principle I (modulo the two last assertions), upon noting that if $I \subset I'$ and $J \subset J'$ are Borel sets with the same measures as I' and J', then

$$A \subseteq_{\xi_\mu} I \quad \text{and} \quad C \subseteq_{\xi_\nu} J$$

as well.

Proposition 6.3. *Let K be a compact metrizable group and suppose that $\Gamma < \Delta(K)$ is a dense subgroup. Then, for all Borel measurable sets $I', J' \subset K$, there are Borel measurable sets $I \subset I'$ and $J \subset J'$ with the same measures as I' and J' respectively such that*

$$m_K \otimes m_K(\Gamma(I' \times J')) = m_K(I^{-1} J).$$

7. Proof of Proposition 6.1

Let (X, x_o) be a compact pointed G-space and fix an *ergodic* G-invariant Borel probability measure μ on X. Let (Y, ν) be a Borel G-space.

Lemma 7.1. *If $A \subset X$ is open, then for every Borel set $C \subset Y$,*

$$\nu(A_{x_o}^{-1}C) \geq \nu(A_x^{-1}C), \quad \textit{for all } x \in X. \tag{7.1}$$

Proof. Fix $x \in X$ and $\varepsilon > 0$. Since ν is a σ-additive measure, we can find a finite set $F \subset A_x$ such that $\nu(F^{-1}C) \geq \nu(A_x^{-1}C) - \varepsilon$. Since $F \subset A_x$ and $A \subset X$ is open, the set

$$U := \{z \in X \,:\, F \subset A_z\} = \bigcap_{f \in F} f^{-1}A$$

is a non-empty open subset of X. Since x_o has a dense G-orbit, there exists at least one $g \in G$ such that $g \cdot x_o \in U$, and thus $F \subset A_{g \cdot x_o} = A_{x_o} g^{-1}$. We note that

$$\nu(A_{x_o}^{-1}C) = \nu((A_{g \cdot x_o})^{-1}C) \geq \nu(F^{-1}C) \geq \nu(A_x^{-1}C) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is finished. □

Lemma 7.2. *If $A \subset X$ and $C \subset Y$ are Borel sets, then there exists a μ-conull Borel set $X' \subset X$ such that*

$$\nu(A_x^{-1}C) = \int_X \nu(A_z^{-1}C)\, d\mu(z) = \mu \otimes \nu(G(A \times C)), \quad \textit{for all } x \in X'. \tag{7.2}$$

Proof. One readily checks that

$$\varphi(x) = \nu(A_x^{-1}C) = \int_Y \chi_{G(A \times C)}(x, y)\, d\nu(y),$$

is Borel measurable on X and G-invariant, and thus constant μ-almost everywhere by ergodicity of μ. This constant must be equal to its μ-integral, and thus

$$\int_X \varphi(x)\, d\mu(x) = \int_X \Big(\int_Y \chi_{G(A \times C)}(x, y)\, d\nu(y) \Big)\, d\mu(x) = \mu \otimes \nu(G(A \times C)),$$

by Fubini's Theorem. □

Let η be an ergodic joining of (X, μ) and (Y, ν). Given $A \subset X$ and $C \subset Y$ we define

$$A' = A \times Y \quad \text{and} \quad C' = X \times C.$$

We note that the map $\pi : (X \times Y)^2 \to X \times Y$ given by

$$\pi((x_1, y_1), (x_2, y_2)) = (x_1, y_2), \quad \text{for } x_1, x_2 \in X \text{ and } y_1, y_2 \in Y$$

is G-equivariant with respect to the diagonal action of G, and the push-forward of $\eta \otimes \eta$ under π equals $\mu \otimes \nu$. Furthermore,

$$A' \times C' = \pi^{-1}(A \times C).$$

and thus

$$\mu \otimes \nu(G(A \times C)) = \eta \otimes \eta(G(A' \times C')),$$

which finishes the proof.

8. Proof of Proposition 6.2

Throughout this section, let G be a countable (not necessarily amenable) group and let (Z, η) be an ergodic Borel G-space. Let $\mathcal{B}_Z$ denote its Borel σ-algebra. Recall that the Kronecker-Mackey factor $\mathcal{K} \subset \mathcal{B}_Z$ is the smallest G-invariant sub-σ-algebra of $\mathcal{B}_Z$ such that $\mathcal{E}_G(Z \times Z) \subset \mathcal{K} \otimes \mathcal{K}$. By the discussion in the introduction of Subsection 2.8 and Proposition 2.10, to prove Proposition 6.2, it suffices to establish the following lemma.

Lemma 8.1. *If $A, C \subset Z$ are Borel sets, then there are $\mathcal{K}$-measurable sets $I, J \subset Z$ such that $A \subset I$ and $C \subset J$ modulo η-null sets, and*

$$\eta \otimes \eta(G(A \times C)) = \eta \otimes \eta(G(I \times J)).$$

8.1. **Proof of Lemma 8.1**

If (X, μ) is a Borel G-space and $\mathcal{F} \subset \mathcal{B}_X$ is a factor (i.e. a G-invariant sub-σ-algebra of $\mathcal{B}_X$), then we say that a $\mathcal{F}$-measurable subset $J \subset X$ is a **$\mathcal{F}$-shadow** of a Borel set $C \subset X$ if

$$\mu(C) = \int_J \mathbb{E}[\chi_C | \mathcal{F}]\, d\mu \quad \text{and} \quad \int_D \mathbb{E}[\chi_C | \mathcal{F}]\, d\mu > 0,$$

for every Borel set $D \subset J$ of positive μ-measure. In particular, $\mu(C \setminus J) = 0$. We note that every Borel set $C \subset X$ admits a $\mathcal{F}$-shadow for any given factor $\mathcal{F} \subset \mathcal{B}_X$. Indeed, let f be a pointwise realization of $\mathbb{E}[\chi_C | \mathcal{F}]$, and let J be its positivity set. We leave it to the reader to verify that J is a $\mathcal{F}$-shadow of C.

The following lemma is key.

Lemma 8.2. *Let $\mathcal{F} \subset \mathcal{B}_X$ be a factor and suppose that $\mathcal{E}_G(X) \subset \mathcal{F}$. For every Borel set $C \subset X$ and $\mathcal{F}$-shadow J of C, we have $\mu(GC) = \mu(GJ)$.*

Proof. Define $E = GJ \setminus GC \subset X$, which clearly belongs to $\mathcal{E}_G(X)$ and hence to $\mathcal{F}$. We need to prove that $\mu(E) = 0$. First note that

$$0 = \mu(E \cap C) = \int_E \mathbb{E}[\chi_C \,|\, \mathcal{F}]\, d\mu,$$

which forces $\mu(E \cap J) = 0$ since J is a $\mathcal{F}$-shadow of C. Indeed, if $D = E \cap J \subset J$ is not μ-null, then

$$0 < \int_D \mathbb{E}[\chi_C | \mathcal{F}]\, d\mu \leq \int_E \mathbb{E}[\chi_C | \mathcal{F}]\, d\mu = 0,$$

which is a contradiction. We conclude that, since E is G-invariant and G is countable, we have $\mu(E \cap GJ) = \mu(E) = 0$, which finishes the proof. □

Let us now turn to the proof of Lemma 8.1, retaining the notation from the beginning of this section. Let $A, C \subset Z$ be Borel sets, and consider the (not necessarily ergodic) Borel G-space

$$(X, \mu) = (Z \times Z, \eta \otimes \eta).$$

One can readily check that if I and J denote the $\mathcal{K}$-shadows of A and C respectively, then $I \times J$ is a $\mathcal{K} \otimes \mathcal{K}$-shadow of $A \times C$. By definition, we have $\mathcal{E}_G(X) \subset \mathcal{F} := \mathcal{K} \otimes \mathcal{K}$, and thus, by Lemma 8.2

$$\eta \otimes \eta(G(A \times C)) = \eta \otimes \eta(G(I \times J)),$$

which finishes the proof.

9. Proof of Proposition 6.3

9.1. Dirac sequences and balanced sets

Let H be a compact and second countable group and let m_H denote the unique Haar probability measure on H. We denote by e the identity element in H.

Definition 9.1 (Dirac sequence)**.** A sequence (B_j) of closed subsets of H is called *Dirac* if

$$\bigcap_j B_j = \{e\} \quad \text{and} \quad B_{j+1} \subset B_j, \quad \text{for all } j.$$

We note that every sub-sequence of a Dirac sequence is again a Dirac sequence.

Remark 9.2. Dirac sequences always exist: Indeed, since H is assumed to be second countable, we can find a decreasing sequence (U_j) of open neighborhoods of the identity in H such that $U_j \subset \overline{U}_j \subset U_{j+1}$ and $\bigcap_j U_j = \{e\}$. One can readily check that $B_j = \overline{U}_j$ forms a Dirac sequence in H.

Definition 9.3 (Balanced set)**.** Let $D \subset H$ be a Borel set and let (B_j) be a Dirac sequence of closed subsets of H. We say that D is *balanced with respect to* (B_j) if

$$\lim_j \frac{m_H(D \cap sB_j)}{m_H(B_j)} = 1, \quad \text{for all } s \in D.$$

If D is balanced with respect to *some* Dirac sequence in H, then we simply say that D is *balanced*.

Proposition 9.4. *Let (B_j) be a Dirac sequence in H. Then, for every Borel set $D \subset H$ with positive measure, there exists a conull Borel subset $D' \subset D$ which is balanced with respect to some subsequence (B_{j_k}).*

Proposition 9.5. *Let $L < H$ be a closed subgroup and suppose that $\Gamma < L$ is a dense Borel subgroup. If $D \subset H$ is a balanced Borel set with positive measure, then $m_H(\Gamma D) = m_H(LD)$.*

9.2. Proof of Proposition 6.3 assuming Proposition 9.4 and Proposition 9.5

Suppose that K is a compact and countable group, $\Gamma < \Delta(K)$ is a dense countable subgroup and $I', J' \subset K$ are Borel sets. Note that if $I \subset I'$ and $J \subset J'$ are Borel sets with the same measures as I and J respectively, then, since Γ is countable, we have

$$m_K \otimes m_K(\Gamma(I' \times J')) = m_K \otimes m_K(\Gamma(I \times J)).$$

It suffices to show that I and J can be chosen so that

$$m_K \otimes m_K(\Gamma(I \times J)) = m_K \otimes m_K(\Delta(K)(I \times J)). \tag{9.1}$$

Indeed, consider the multiplication map $(s,t) \mapsto s^{-1}t$. By the uniqueness of Haar probability measures on compact groups, we see that it maps $m_K \otimes m_K$ to m_K, and the pre-image of product set $I^{-1}J$ is $\Delta(K)(I \times J)$ in $K \times K$. In particular,

$$m_K \otimes m_K(\Delta(K)(I \times J)) = m_K(I^{-1}J).$$

Let us now prove (9.1). In order to align the notation with the one of Proposition 9.4 and Proposition 9.5, we set

$$H = K \times K \quad \text{and} \quad L = \Delta(K) < H.$$

Fix a Dirac sequence (B'_j) in K. By Proposition 9.4 we can find a (common) sub-sequence (B'_{j_k}) and subsets $I \subset I'$ and $J \subset J'$ with the same Haar measures as I' and J' which are balanced with respect to (B'_{j_k}). If we set

$$D = I \times J \subset H \quad \text{and} \quad B_k = B'_{j_k} \times B'_{j_k} \subset H,$$

then it is readily checked that (B_k) is a Dirac sequence in H, and that D is balanced with respect to (B_k). By Proposition 9.5, we conclude that $m_H(\Gamma D) = m_K(HD)$, and thus

$$m_H(\Gamma D) = m_K \otimes m_K(\Gamma(I \times J)) = m_H(LD) = m_K \otimes m_K(\Delta(K)(I \times J)).$$

9.3. **Proof of Proposition 9.4**

Lemma 9.6. *Let (B_j) be a Dirac sequence in H and define*

$$\rho_j(s) = \frac{\chi_{B_j}(s)}{m_H(B_j)}, \quad \textit{for } s \in H.$$

Then, $\lim_j f * \rho_j = f$ in the norm topology on $L^1(H, m_H)$, for every $f \in L^1(H, m_H)$.

Proof. It suffices to establish the lemma in the case when f is a *continuous*, and hence *uniformly continuous*, function on H. Fix $\varepsilon > 0$ and find an open neighborhood U of the identity in H such that

$$\sup_{t \in H} \left| f(ts) - f(t) \right| < \varepsilon, \quad \text{for all } s \in U.$$

Then,

$$\sup_{t \in H} \left| (f * \rho_j)(t) - f(t) \right| \leq \int_U \rho_j(s) \left| f(ts) - f(t) \right| dm_H(s) < \varepsilon,$$

for every j such that $B_j \subset U$, which shows that $f * \rho_j \to f$ *uniformly* on H as $j \to \infty$. □

We now turn to the proof of Proposition 9.4. Let $D \subset H$ be a Borel set with positive measure and define $f = \chi_D$. Let (B_j) be a Dirac sequence in H. By Lemma 9.6, we have $\lim_j f * \rho_j = f$ in the norm topology on $L^1(H, m_H)$, and thus we can find a subsequence (j_k) and a conull Borel set $H' \subset H$ such that the $\lim_k f * \rho_{j_k}(s) = f(s)$ for all $s \in H'$. We define the set $D' = D \cap H'$ and note that

$$\lim_k (f * \rho_{j_k})(s) = \lim_k \frac{m_H(D' \cap sB_{j_k})}{m_H(B_{j_k})} = \chi_D(s) = 1,$$

for all $s \in D'$, which shows that D' is balanced with respect to (B_{j_k}).

9.4. **Proof of Proposition 9.5**

Let (B_j) be a Dirac sequence in H and suppose that $D \subset H$ is a Borel set with positive measure which is balanced with respect to (B_j). Fix a closed subgroup $L < K$ and a dense subgroup $\Gamma < L$ and define the left Γ-invariant Haar measurable set $C = LD \setminus \Gamma D$. We wish to prove that C is a null set. We argue by contradiction. Assume that $m_H(C) > 0$ and define the left Γ-invariant functions $f_j(s) = m_H(C \cap sB_j)$ on H and note that each f_j is continuous. Since $\Gamma < L$ is dense, we conclude that f_j is left L-invariant for every j.

Fix $0 < \varepsilon < \frac{1}{2}$ and use Proposition 9.4 to find a subset $C' \subset C$ with the same Haar measure as C and which is balanced with respect to some sub-sequence (B_{j_k}). We note that every $s \in C'$ can be written on the form $s = ld$, for some $l \in L$ and $d \in D$. Hence, since f_j is left L-invariant and $C', D \subset H$ are balanced with respect to (B_{j_k}), we have

$$m_H(C \cap sB_{j_k}) = m_H(C \cap dB_{j_k}) \geq (1 - \varepsilon) m_H(B_{j_k}),$$

and

$$m_H(D \cap dB_{j_k}) \geq (1 - \varepsilon) m_H(B_{j_k})$$

for large enough k. In particular, since $2\varepsilon < 1$, we have

$$m_H(C \cap D \cap dB_{j_k}) \geq (1 - 2\varepsilon) m_H(B_{j_k}) > 0,$$

for large enough k. Hence $C \cap D$ is non-empty, which is a contradiction.

10. Proofs of Proposition 5.2 and the last part of Proposition 5.1

Let us briefly summarize the setting of Correspondence Principle I and II. As usual, G is a countable amenable group, and we have fixed a compact pointed G-space (X, x_o) and an ergodic Borel G-space (Y, ν) once and for all. Furthermore, we have fixed an *ergodic* G-invariant Borel probability measure μ on X, an open set $A \subset X$ with $\mu(A) > 0$ and a Borel set $C \subset Y$.

By Correspondence Principle I, we can find

- a metrizable compactification (K, τ_K) of G and Borel sets $I, J \subset K$
- ergodic joinings $\xi_\mu \in \mathcal{J}_G(\mu, m_K)$ and $\xi_\nu \in \mathcal{J}_G(\nu, m_K)$ such that
$$A \subseteq_{\xi_\mu} I \quad \text{and} \quad C \subseteq_{\xi_\nu} J$$
and
$$\nu(A_{x_o}^{-1} C) \geq \mu \otimes \nu(G(A \times C)) \geq m_K(I^{-1} J).$$

We wish to prove that if (I, J) reduces to a pair (I_o, J_o) in a factor group M of K, with factor map $p : K \to M$, and $I_o \subset M$ is Jordan measurable, then we can find a subset $A_o \subset A_{x_o}$ and $t \in M$ such that
$$d^*(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset \tau_M^{-1}(I_o t),$$
and for all $s \in A_{x_o} \setminus A_o$, we have
$$\nu(A_{x_o}^{-1} C) - \nu(C) \geq m_M(I_o^{-1} J_o) - m_M(J_o) + m_M(\tau_M(s)^{-1} J_o \setminus t^{-1} I_o^{-1} J_o),$$
where $\tau_M = p \circ \tau_K$. The last inequality can be equivalently written as
$$\nu(A_{x_o}^{-1} C) - \nu(C) \geq m_M(I_o^{-1} J_o) - m_M(\tau_M(s)^{-1} J_o \cap t^{-1} I_o^{-1} J_o), \tag{10.1}$$
for all $s \in A_{x_o} \setminus A_o$.

By the definition of reduction of pairs, see (5.1), we have $I \subseteq p^{-1}(I_o)$, and thus $A \subseteq_\xi I_o$ as well, where $\xi = (\mathrm{id} \times p)_* \xi_\mu$ is an ergodic joining of (X, μ) and (M, m_M), where G acts on the latter space via τ_M. We note that if I_o is Jordan measurable in M, so that $U := I_o^o$ has the same measure as I_o, then $A \subseteq_\xi U$. We recall that if $w \in M$, then $U_w = \tau_M^{-1}(U w^{-1})$.

Let W be a compact metrizable space equipped with an action of G by homeomorphisms. We say that a point $w \in W$ is **almost automorphic** if whenever (g_n) is a sequence in G such that $g_n \cdot w \to w'$ for some point $w' \in W$, then $g_n^{-1} \cdot w' \to w$ as well. We say that W is an **isometric G-space** if every point is almost automorphic. One readily checks that M, with the G-action induced by τ_M, is isometric.

In what follows, let (X, x_o), μ and (Y, ν) be as above, and let W be a compact G-space and ϑ an ergodic G-invariant Borel probability measure on W. The following lemma now implies Correspondence Principle II (Proposition 5.2) (with $W = K$, $U = I_o^o$ and $V = J_o$).

Lemma 10.1. *Let $A \subset X$ and $U \subset W$ be open sets and let $C \subset Y$ and $V \subset W$ be Borel sets. Suppose that there are $\xi \in \mathcal{J}_G(\mu, \vartheta)$ and $\rho \in \mathcal{J}_G(\nu, \vartheta)$ with ξ ergodic such that*
$$A \subseteq_\xi U \quad \textit{and} \quad C \subseteq_\rho V.$$
If W is isometric, then we can find

- *a left-invariant mean λ on G, and*

- *a subset* $A_o \subset A_{x_o}$ *and* $w_o \in W$ *such that*

$$\mathfrak{m}(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset U_{w_o},$$

and with the property that for all $s \in A_{x_o} \setminus A_o$,

$$\nu(A_{x_o}^{-1}C) - \nu(C) \geq \mu \otimes \nu(G(A \times C)) - \vartheta(s^{-1}V \cap U_{w_o}^{-1}V).$$

10.1. **Proof of Lemma 10.1**

Lemma 10.2. *If* $A \subset X$ *and* $U \subset W$ *are open sets such that* $A \subseteq_\xi U$, *and there exists a point* (x, w) *in* $X \times W$ *with* w *almost automorphic such that* $\overline{G(x, w)} = \operatorname{supp}(\xi)$, *then we can find*

- *a left-invariant mean* $\mathfrak{m}$ *on* G, *and*
- *a subset* $A_o \subset A_{x_o}$ *and* $w_o \in W$ *such that*

$$\mathfrak{m}(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset U_{w_o}.$$

Furthermore, we can choose A_o *so that whenever* (Y, ν) *is a p.m.p.* G*-space and* $C \subset Y$ *is a Borel set, then* $\nu(A_o^{-1}C) \geq \nu(A_x^{-1}C)$.

Proof. Since x_o has a dense G-orbit in X, we can find a sequence (g_n) in G such that $g_n \cdot x_o \to x$. Let w_o be a cluster point of the sequence $(g_n^{-1} \cdot w)$ in W. Since w is assumed to be almost automorphic, there exists a sub-sequence (g_{n_k}) such that $g_{n_k} \cdot w_o \to w$, and thus

$$Z := \overline{G \cdot (x_o, w_o)} \supset \overline{G \cdot (x, w)} = \operatorname{supp}(\xi). \tag{10.2}$$

We set $z_o := (x_o, w_o)$ and $V = (A \times U) \cap Z$. Then z_o has a dense G-orbit in Z and V is open subset of Z. If we define $A_o := (A \times U)_{z_o}$, then

$$A_o \subset A_{x_o} \quad \text{and} \quad A_o \subset U_{w_o}$$

by construction. In view of the inclusion (10.2), we may consider ξ as a G-invariant Borel probability measure on Z. Since z_o has a dense G-orbit in Z, we can by Proposition 2.3 find a left-invariant mean $\mathfrak{m}$ on G such that $S_{z_o}^* \mathfrak{m} = \xi$, where S_{z_o} denotes the Bebutov map of (Z, z_o). Since V is open in Z, we have $\mathfrak{m}(V_{z_o}) \geq \xi(V)$ by Lemma 2.5, and thus

$$\mathfrak{m}(V_{z_o}) \geq \xi(V) = \xi(A \times U) = \mu(A),$$

since $A \subseteq_\xi U$ and thus $\xi(A \times U^c) = 0$. Since $A_o = V_{z_o}$ and $A_x = V_{(x,w)}$, the last assertion of the lemma follows from Lemma 7.1 applied to (Z, z_o), the open set $V \subset Z$ and $z = (x, w)$. □

Proof of Lemma 10.1

Since ξ is ergodic, there exists by Lemma 2.2 a ξ-conull Borel set $T \subset X \times W$ such that $\operatorname{supp}(\xi) = \overline{G \cdot (x, w)}$ for all $(x, w) \in T$. Furthermore, by Lemma 7.1 and Lemma 7.2, we have

$$\nu(A_{x_o}^{-1}) \geq \nu(A_x^{-1}C) = \mu \otimes \nu(G(A \times C)), \quad \text{for } \mu\text{-a.e. } x \in X.$$

We can now fix $(x, w) \in T$ such that

$$\operatorname{supp}(\xi) = \overline{G \cdot (x, w)} \quad \text{and} \quad \nu(A_{x_o}^{-1}C) \geq \nu(A_x^{-1}C) = \mu \otimes \nu(G(A \times C)). \tag{10.3}$$

Since W is isometric, w is an almost automorphic point. Since $A \subseteq_\xi U$, Lemma 10.2 tells us that we can find

- a left-invariant mean $\mathfrak{m}$ on G, and
- subset $A_o \subset A_{x_o}$ and $w_o \in W$ such that

$$\mathfrak{m}(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset U_{w_o} \quad \text{and} \quad \nu(A_o^{-1}C) \geq \nu(A_x^{-1}C). \tag{10.4}$$

Pick $s \in A_{x_o} \setminus A_o$, and note that

$$\nu(A_{x_o}^{-1}C) \geq \nu(A_o^{-1}C \cup s^{-1}C) = \nu(A_o^{-1}C) + \nu(C) - \nu(s^{-1}C \cap A_o^{-1}C).$$

Since $C \subseteq_\rho V$, we have

$$s^{-1}C \subseteq_\rho s^{-1}V \quad \text{and} \quad A_o^{-1}C \subseteq_\rho A_o^{-1}V \subset U_{w_o}^{-1}V, \tag{10.5}$$

and thus $s^{-1}C \cap A_o^{-1}C \subseteq_\rho s^{-1}V \cap U_{w_o}^{-1}V$. In particular, by (2.4), we have

$$\nu(s^{-1}C \cap A_o^{-1}C) \leq \vartheta(s^{-1}V \cap U_{w_o}^{-1}V).$$

Using the bounds in (10.3), (10.4) and (10.5), we can conclude that

$$\nu(A_{x_o}^{-1}C) - \nu(C) \geq \mu \otimes \nu(G(A \times C)) - \vartheta(s^{-1}V \cap U_{w_o}^{-1}V).$$

Since $s \in A_{x_o} \setminus A_o$ is arbitrary, we are done.

10.2. **A small variation of Lemma 10.2**

The proof of Lemma 10.2 can be readily modified to yield the following result (where the positions of the spaces X and W have been permuted).

Lemma 10.3. *If $I \subset X$ and $U \subset W$ are open sets such that $I \subseteq_\xi U$, and there exists a point (x, w) in $X \times W$ with x almost automorphic such that $\overline{G(x, w)} = \operatorname{supp}(\xi)$, then we can find*

- *a left-invariant mean $\mathfrak{n}$ on G, and*
- *a subset $I_o \subset I_{x_o}$ and $w_o \in W$ such that*

$$\mathfrak{n}(I_o) \geq \mu(I) \quad \text{and} \quad I_o \subset U_{w_o}.$$

The following corollary of Lemma 10.3 was used in the proof of Theorem 4.2.

Lemma 10.4. *Let (Y, y_o) be a compact pointed G-space and let ν be an ergodic G-invariant Borel probability measure on Y. Let $U \subset Y$ be an open set, and suppose that there exist a finite-index subgroup $G_o < G$ and a G_o-invariant Borel set $Z \subset U$ with positive ν-measure. Then U_{y_o} contains a piecewise periodic set.*

Sketch of proof. Upon possibly passing to further finite-index subgroups, we may without loss of generality assume that G_o is normal in G. Let $\mathcal{F}$ denote the factor of (Y, ν) consisting of all left-translates of Z. Since G_o has finite index in G, we note that $\mathcal{F}$ is finite, and thus the corresponding factor space is isomorphic to a *finite* homogeneous space, say K/L, where K is a finite group and L a subgroup thereof, and G acts on K/L by left translations induced by a (surjective) homomorpism $\tau : G \to K$. We conclude there exists $I \subset K/L$ such that $q^{-1}(I) = Z$ modulo null sets, where $q : Y \to K/L$ is the corresponding factor map. In particular, we have $I \subseteq_\xi U$, where $\xi(J \times B) = \nu(q^{-1}(J) \cap B)$, and thus Lemma 10.3, applied to $(X, \mu) = (K/L, m_{K/L})$ and $W = Y$, implies that we can find a subset $I_o \subset U_{y_o}$ and a left-invariant mean $\mathfrak{n}$ such that

$$I_o \subset I_L \quad \text{and} \quad \mathfrak{n}(I_o) \geq m_{K/L}(I) = \mathfrak{n}(I_L).$$

In particular, I_o is a piecewise periodic subset of U_{y_o}. □

10.3. **Proof of the last part of Proposition 5.1**

Recall that a Borel set $D \subset K$ is *spread-out* if every Borel set $D' \subset D$ with the same measure as D projects onto every finite quotient of K. Equivalently, $D \subset K$ is *not* spread-out if there exists $D' \subset D$ with $m_K(D') = m_K(D)$ and an open subgroup $L < K$ such that $m_K(LD') < 1$. Note that the set LD' is a proper clopen subset of K.

Let us first assume that $J \subset K$ is not spread-out. We can find a Borel set $J' \subset J$ with $m_K(J') = m_K(J)$ and an open subgroup $L < K$ such that $m_K(LJ') < 1$. Let $G_o = \tau_K^{-1}(L)$ and note that G_o is a finite-index subgroup of G. Since $C \subseteq_{\xi_\upsilon} J$, we have $G_oC \subseteq_{\xi_\upsilon} LJ'$ and thus $\upsilon(G_oC) \leq m_K(LJ') < 1$. This finishes the proof of the last assertion in Proposition 5.1.

Let us now assume that $I \subset K$ is not spread-out. As in the previous paragraph, we can conclude that $A \subseteq_\xi U$, where U is a proper open subset of K which is invariant under $\tau_K(G_o)$ for some finite-index subgroup $G_o < G$. By Lemma 10.2 applied to $W = K$, we can find

- a left-invariant mean $\mathfrak{n}$ on G, and
- a subset $A_o \subset A_{x_o}$ and $w_o \in K$ such that

$$\mathfrak{n}(A_o) \geq \mu(A) \quad \text{and} \quad A_o \subset P := \tau_K^{-1}(Uw_o^{-1}).$$

Since U is invariant under $\tau_K(G_o)$, we conclude that P is a proper periodic subset of G, which finishes the proof of the second to last assertion in Proposition 5.1.

11. Counterexample Machine

Definition 11.1 (Contracting triple)**.** Let G be a countable group, $N \triangleleft G$ is a normal subgroup and $\Lambda < N$ is a *proper* subgroup. We say that the triple (G, Λ, N) is **contracting** if for every finite set $F \subset N$, there exists $g \in G$ such that $gFg^{-1} \subset \Lambda$.

The following two examples of contracting triples will be useful to keep in mind. In both cases, G is a two-step solvable group.

Example 4. Let $p \geq 2$ be an integer, and define the groups

$$G = \mathbb{Z}\Big[\frac{1}{p}\Big] \rtimes \mathbb{Z} \quad \text{and} \quad \Lambda = \mathbb{Z} \rtimes \{0\} \quad \text{and} \quad N = \mathbb{Z}\Big[\frac{1}{p}\Big] \rtimes \{0\},$$

where $\mathbb{Z}$ acts on $\mathbb{Z}[1/p]$ by multiplication by multiplicative powers of p. We claim that (G, Λ, N) is contracting: Indeed, let $F \subset \mathbb{Z}[1/p]$ be a finite set, and pick a large enough integer N such that $p^N F \subset \mathbb{Z}$. Then,

$$(0, N)(F \rtimes \{0\})(0, -N) = (p^N F \rtimes \{0\}) \subset \Lambda.$$

Example 5. Let $\mathbb{Q}$ denote the additive group of rational numbers, and let $\mathbb{Q}_+^*$ denote the multiplicative group of *positive* rational numbers. Define the groups

$$G = \mathbb{Q} \rtimes \mathbb{Q}_+^* \quad \text{and} \quad \Lambda = \mathbb{Z} \rtimes \{0\} \quad \text{and} \quad N = \mathbb{Q} \rtimes \{1\}.$$

We claim that the triple (G, Λ, N) is contracting. Indeed, let $F \subset \mathbb{Q}$ be a finite set, and pick a large enough positive integer q such that $qF \subset \mathbb{Z}$. Then,

$$(0, q)(F \rtimes \{0\})(0, q^{-1}) = (qF \rtimes \{0\}) \subset \Lambda.$$

In what follows, suppose that G is a countable group with two *abelian* subgroups N and L such that N is normal in G and

$$N \cap L = \{e_G\} \quad \text{and} \quad NL = G.$$

In other words, G is the semi-direct product of N and L, and one readily checks that G is a two-step solvable group and thus amenable. We further assume that there is a FINITELY GENERATED subgroup Λ of N such that (G, Λ, N) is a contracting triple. Note that the two examples given above are of this form with

$$L = \{0\} \rtimes \mathbb{Z} \quad \text{and} \quad L = \{0\} \rtimes \mathbb{Q}_+^*.$$

We denote by d_G^* and d_L^* the upper Banach densities, and by d_*^G and d_*^L the lower Banach densities, on G and L respectively.

The following proposition is the main technical result in this section. We shall use it below to establish Theorem 1.12 and Theorem 1.13.

Proposition 11.2 (Counterexample machine)**.** *There exist $S, T \subset G$ such that*

$$d_G^*(S) = d_G^*(T) = 1 \quad \textit{and} \quad e_G \notin T$$

with the following properties: If $A_o, B_o \subset L$, and

$$A = NA_o \cap S \quad \textit{and} \quad B = \big(NB_o \cap T\big) \sqcup \{e_G\},$$

then $d_G^*(AB) \leq d_L^*(A_oB_o)$.

11.1. **Proof of Proposition 11.2**

Let G, N, Λ and L be as above.

Proposition 11.3. *If $S = L\Lambda$, then*

$$d_G^*(S) = 1 \quad \textit{and} \quad d_*^G(S^{-1}S) = 0. \tag{11.1}$$

Let S be as in Proposition 11.3 and define $T = (S^{-1}S)^c$. Then,

$$d_G^*(T) = 1 \quad \text{and} \quad e_G \notin T \quad \text{and} \quad ST \cap S = \emptyset.$$

Given subsets $A_o, B_o \subset L$, we define

$$A = NA_o \cap S \quad \text{and} \quad B = (NB_o \cap T) \sqcup \{e_G\}.$$

Since N is normal in G, we have

$$AB \subset \big(NA_oB_o \cap ST\big) \sqcup (NA_o \cap S).$$

By Proposition 2.6, there exists an *extreme* left-invariant mean $\mathfrak{m}$ on G such that $d_G^*(AB) = \mathfrak{m}(AB)$. Hence,

$$d_G^*(AB) = \mathfrak{m}(AB) \leq \mathfrak{m}(NA_oB_o \cap ST) + \mathfrak{m}(NA_o \cap S). \tag{11.2}$$

The following lemma will now be useful.

Lemma 11.4. *Suppose that $C \subset G$ is left N-invariant and $D \subset G$ is left L-invariant. Then,*

$$\mathfrak{m}(C \cap D) = \mathfrak{m}(C)\,\mathfrak{m}(D),$$

for every extreme left-invariant mean $\mathfrak{m}$ on G.

Proof. Let $\mathfrak{m}$ be a left-invariant mean on G, and suppose that $C \subset G$ is left N-invariant and $D \subset G$ is left L-invariant. Since $G = LN$, we can write every $g \in G$ on the form ln with $l \in L$ and $n \in N$. We conclude that

$$\mathfrak{m}((g \cdot \chi_C)\,\chi_D) = \mathfrak{m}(gC \cap D) = \mathfrak{m}(nC \cap l^{-1}D) = \mathfrak{m}(C \cap D), \quad \text{for all } g \in G.$$

If $\mathfrak{m}$ in addition is an extreme left-invariant mean, then Proposition 2.1 tells us that

$$\int_G \mathfrak{m}(gC \cap D)\, d\eta(g) = \mathfrak{m}(C)\,\mathfrak{m}(D), \quad \text{for all } \eta \in \mathcal{L}_G.$$

Hence, $\mathfrak{m}(C \cap D) = \mathfrak{m}(C)\,\mathfrak{m}(D)$. □

Let us apply this lemma to the pairs

$$(C, D) = (NA_oB_o, ST) \quad \text{and} \quad (C, D) = (NA_o, S).$$

We conclude that

$$\mathfrak{m}(NA_oB_o \cap ST) = \mathfrak{m}(NA_oB_o)\mathfrak{m}(ST) \quad \text{and} \quad \mathfrak{m}(NA_o \cap S) = \mathfrak{m}(NA_o)\mathfrak{m}(S).$$

By plugging these identities into (11.2), we get

$$d_G^*(AB) \leq \mathfrak{m}(NA_oB_o)\mathfrak{m}(ST) + \mathfrak{m}(NA_o)\mathfrak{m}(S).$$

Since N is normal, we see that $\mathfrak{m}'(E) = \mathfrak{m}(NE)$ defines a left L-invariant mean on L. Since L is abelian, $\mathfrak{m}'$ is automatically right-invariant, and thus

$$\mathfrak{m}(NA_o) = \mathfrak{m}'(A_o) \leq \mathfrak{m}'(A_oB_o) = \mathfrak{m}(NA_oB_o).$$

Since $ST \cap S = \emptyset$, we get

$$d_G^*(AB) \leq \mathfrak{m}(NA_oB_o)(\mathfrak{m}(ST) + \mathfrak{m}(S)) \leq \mathfrak{m}'(A_oB_o) \leq d_L^*(A_oB_o),$$

which finishes the proof.

11.2. **Proof of Proposition 11.3**

By assumption, the triple (G, Λ, N) is contracting. We recall that this means that for every finite set $F_N \subset N$, we can find $g \in G$ such that

$$gF_Ng^{-1} \subset \Lambda.$$

Since N is abelian, we can always choose $g \in L$. Indeed, since $G = NL = LN$, every $g \in G$ can be written on the form ln with $l \in L$ and $n \in N$, and thus

$$gF_Ng^{-1} = l(nF_Nn^{-1})l^{-1} = lF_Nl^{-1} \subset \Lambda.$$

Let $S = L\Lambda \subset G$. In order to prove that $d_G^*(S) = 1$, it suffices by Lemma 2.12 to show that S is thick, that is to say, we need to show that for every finite set $F \subset G$, there exists $g \in G$ such that $Fg^{-1} \subset S$.

We may without loss of generality assume that $F = F_LF_N$, where $F_L \subset L$ and $F_N \subset N$ are finite sets. Since N is abelian, we can find $l \in L$ such that $lF_Nl^{-1} \subset \Lambda$. Hence,

$$Fl^{-1} \subset F_LF_Nl^{-1} = F_Ll^{-1}(lF_Nl^{-1}) \subset L\Lambda = S.$$

Since F was chosen arbitrary, this shows that S is thick.

In order to show that $d_*(S^{-1}S) = 0$, it suffices by Lemma 2.13 to prove that $S^{-1}S$ is *not* syndetic. We shall argue by contradiction. Suppose that $S^{-1}S$ is syndetic, and choose finite subsets $F_N \subset N$ and $F_L \subset L$ such that $F_NF_LS^{-1}S = G$. In particular, upon intersecting with N, we get

$$F_NF_L\Lambda L\Lambda \cap N = N.$$

Since $N \cap L = \{e_G\}$ and $\Lambda < N$, we note that for a fixed $l \in L$,

$$l\Lambda L\Lambda \cap N = l\Lambda l^{-1}\Lambda,$$

and thus

$$F_NF_L\Lambda L\Lambda \cap N = F_N(F_L\Lambda L\Lambda \cap N) = F_N\Big(\bigcup_{l\in F_L} l\Lambda l^{-1}\Big)\Lambda = N. \tag{11.3}$$

Let $\Lambda_o \subset \Lambda$ be a finite generating set for Λ. It is now immediate from (11.3) that the *finite* set

$$N_o := F_N \cup \Big(\bigcup_{l\in F_L} l\Lambda_o l^{-1}\Big) \cup \Lambda_o \subset N$$

generates N. However, if (G, Λ, N) is any contracting triple, then N cannot be finitely generated. Indeed, suppose that N were finitely generated, and let N_o be a finite generating set for N. Since (G, Λ, N) is contracting, we can find $g \in G$ such that $gN_og^{-1} \subset \Lambda$, and thus $gNg^{-1} = N \subset \Lambda$, which contradicts our assumption that Λ is a proper subgroup of N.

11.3. **Proof of Theorem 1.12**

Let (G, Λ, N) be as in Example 4, and let $L = \{0\} \rtimes \mathbb{Z}$. Let S and T be as in Proposition 11.3, and set $L_2 = \{0\} \rtimes 2\mathbb{Z}$. Fix $0 < \varepsilon < \frac{1}{2}$ and let $I_o \subset \mathbb{T}$ be a closed interval of Haar measure 2ε. If $a \in \mathbb{T}$ is an irrational number, then one readily verifies that the set

$$C_o = \{0\} \rtimes \{n \in \mathbb{Z} \,:\, na \in I_o\} \subset L,$$

satisfies $d^*_L(C_o) = 2\varepsilon$, and

$$L_o(L_2 \cap C_o) = \begin{cases} L_2 & \text{if } L_o \subset L_2 \\ L & \text{if } L_o \subsetneq L_2, \end{cases}$$

for any non-trivial (hence finite-index) subgroup $L_o \subset L \cong \mathbb{Z}$.

We now set

$$A = NL_2 \cap S \quad \text{and} \quad B = (Nr(L_2 \cap C_o) \cap T) \sqcup \{(0,0)\},$$

where $r = (0,1)$. Note that $rL_2 = \{0\} \rtimes (2\mathbb{Z} + 1)$. In the notation of Proposition 11.3, we have chosen $A_o = L_2$ and $B_o = r(L_2 \cap C_o)$. One readily checks that

$$d^*_G(A) = \frac{1}{2} \quad \text{and} \quad d^*_G(B) = \frac{1}{2} d^*_L(C_o) = \varepsilon < \frac{1}{2},$$

and hence, by Proposition 11.3,

$$d^*_G(AB) \leq d^*_L(L_2(L_2 \cap C_o)) = d^*_L(L_2) = \frac{1}{2}.$$

Suppose that B is contained in a proper periodic set $P \subset G$ with $\mathrm{Stab}_G(P) = G_o$ (which has finite index in G). We may assume that $P = G_oB \neq G$. Let $L_o < L$ be a subgroup such that $NG_o = NL_o$. Then L_o is a non-trivial (hence finite-index) subgroup of L. The following lemma will now be useful.

Lemma 11.5. *Let (K, τ) be a metrizable compactification of a countable group H, and let $U \subset K$ be an open set. Then, for every thick set $T \subset H$ and finite-index subgroup $H_o < H$, we have*

$$H_o(\tau^{-1}(U) \cap T) = H_o\tau^{-1}(U).$$

Proof. Let us fix a finite-index subgroup $H_o < H$, and define

$$D = \tau^{-1}(U) \quad \text{and} \quad D_+ = \{s \in H_o\backslash H \,:\, D \cap H_os \neq \emptyset\}.$$

We note that $H_oD = H_oD_+$.

We leave it to the reader to verify that if $s \in D_+$, then $D \cap H_os$ is in fact syndetic in H, and thus intersects every thick set $T \subset H$ non-trivially. Let us fix a thick set $T \subset H$. We note that

$$H_o(D \cap T) = \bigsqcup_{s \in D_+} H_o(D \cap H_os \cap T) = H_oD_+,$$

and thus $H_o(D \cap T) = H_oD_+ = H_oD$, which finishes the proof. □

We apply this lemma as follows. First note that if $(L/L_2 \times \mathbb{T}, \tau_o)$ denotes the compactification of L given by

$$\tau_o(l) = (l + 2\mathbb{Z}, la), \quad \text{for } l \in L \cong \mathbb{Z},$$

and $U = \{1\} \times (I_o + a)$, then $r(L_2 \cap C_o) = \tau_o^{-1}(U)$. Since N is normal in G, we can extend τ_o to a homomorphism $\tau : G \to L/L_2 \times \mathbb{T}$ by $\tau(n, l) = \tau_o(l)$. We now note that

$$G_oB = G_o(Nr(L_2 \cap C_o) \cap T) \cup G_o = G_o(\tau^{-1}(U) \cap T) \cup G_o,$$

and thus, by the previous lemma,

$$G_oB = G_o\tau^{-1}(U) \cup G_o.$$

In particular, we get

$$G_oB = NrL_o(L_2 \cap C_o) \cup G_o = NrL_2 \cup G_o,$$

if $L_o \subset L_2$, and

$$G_oB = NrL_o(L_2 \cap C_o) \cup G_o = NL = G,$$

if $L_o \subsetneq L_2$.

Since $P = G_oB$ is assumed to be proper, we conclude that only the first case can occur. We have $G_o \subset NL_o$ and since $L_o \subset L_2$, the intersection $NrL_2 \cap NL_o$ is empty. Hence, for every left-invariant mean $\mathfrak{m}$ on G, we have

$$\mathfrak{m}(P) = \mathfrak{m}(G_oB) = \mathfrak{m}(NrL_2) + \mathfrak{m}(G_o) = \frac{1}{2} + \frac{1}{[G : \mathrm{Stab}_G(P)]} > d_G^*(B) + \frac{1}{[G : \mathrm{Stab}_G(P)]},$$

which finishes the proof of Theorem 1.12.

11.4. **Proof of Theorem 1.13**

Let (G, Λ, N) be as in Example 4, and let $L = \{0\} \rtimes \mathbb{Q}_+^*$. Let S and T be as in Proposition 11.3, and let $(\mathbb{T}, \tau_o)$ be the compactification of L given by

$$\tau_o(0, q) = \log q \mod 1, \quad \text{for } (0, q) \in L.$$

We denote by $\tau : G \to \mathbb{T}$ the extension of τ_o given by $\tau(n, q) = \tau_o(0, q)$. Let $I \subset \mathbb{T}$ be a closed interval with $m_{\mathbb{T}}(I) < 1/3$, which does NOT contain 0, and set

$$A_o = B_o = \tau_o^{-1}(I)$$

so that

$$NA_o = NB_o = \tau^{-1}(I),$$

and

$$A = NA_o \cap S \quad \text{and} \quad B = (NB_o \cap T) \sqcup \{(0, 1)\}.$$

One readily checks that $d_G^*(A) = d_G^*(B) = m_{\mathbb{T}}(I)$, and

$$AB \subset NA_oB_o \cup NA_o = \tau^{-1}((I + I) \cup I).$$

By Proposition 11.2, we have

$$d_G^*(AB) \le d_L^*(A_oB_o) = 2m_{\mathbb{T}}(I) = d_G^*(A) + d_G^*(B) < 1. \tag{11.4}$$

We claim that A is spread-out. Indeed, if it were not, then we could find a set $A' \subset A$ with $d^*(A') = d^*(A)$ which is contained in a proper periodic set. We can fix $\mathfrak{m} \in \mathcal{L}_G$ such that

$$\mathfrak{m}(A') = d^*(A) \quad \text{and} \quad \mathfrak{m}(A) = m_{\mathbb{T}}(I).$$

We can clearly write A' of the form $\tau^{-1}(I) \cap R$ for some set $R \subset G$ such that $\tau^{-1}(I) \cup R = G$. Since

$$1 = \mathfrak{m}(\tau^{-1}(I) \cup R) = m_{\mathbb{T}}(I) + \mathfrak{m}(R) - \mathfrak{m}(\tau^{-1}(I) \cap R) = \mathfrak{m}(R),$$

we conclude that R is thick by Lemma 2.12. By assumption, $A' = \tau^{-1}(I) \cap R$ is contained in a proper periodic subset $\mathcal{Q}$ of G, and thus $G_oA' \neq G$ for the finite-index stabilizer of $\mathcal{Q}$. However, by Lemma 11.5,

$$G_oA' = G_o(\tau^{-1}(I) \cap R) = G_o\tau^{-1}(I) = \tau^{-1}(\tau(G_o)I),$$

Since $\mathbb{T}$ is connected, the right hand side must equal G, which is a contradiction, and thus A is spread-out.

Since A is spread-out in G, Scholium (1.3) shows that the inequality in (11.4) cannot be strict, and thus

$$d^*_G(AB) = d^*_G(A) + d^*_G(B) < 1.$$

In particular, AB is not thick by Lemma 2.12. In fact, AB cannot contain a piecewise periodic set, say $Q \cap T$ where Q is a (proper) periodic set and T is thick. We note that if this is the case, then

$$\overline{\tau(Q \cap T)} \subset \overline{\tau(AB)} = (I + I) \cup I \neq \mathbb{T},$$

so our assertion follows from the following simple lemma applied to the connected compactification $(\mathbb{T}, \tau)$ of G above.

Lemma 11.6. *If (K, τ_K) is a connected compactification of a countable group H, then, for every periodic set $Q \subset H$ and thick set $T \subset H$, we have $\overline{\tau_K(Q \cap T)} = K$.*

Proof. Let (F_n) be an exhaustion of H by finite sets, and let $Q \subset H$ be periodic and T thick. We can find a sequence (h_n) in H such that $F_n h_n \subset T$, and upon passing to a sub-sequence we may assume that the sequence $h_n H_o$ is constant, say equal to hH_o, where H_o is a finite-index normal subgroup of H, contained in the stabilizer of Q. Furthermore, if we fix an invariant metric d_K on K, and $\varepsilon > 0$, we can restrict to a further sub-sequence, and assume that $d_K(\tau_K(h_n), t) < \varepsilon$ for some $t \in K$, and for all sufficiently large n in this sub-sequence. Then,

$$\tau_K(Q \cap T) \supset \tau_K(Q \cap F_n h_n) = \tau_K(Qh^{-1} \cap F_n)\tau_K(h_n), \quad \text{for all } n.$$

Since (F_n) exhaust G, we have

$$\overline{\tau_K(Q \cap T)} B_\varepsilon(t)^{-1} \supset \overline{\tau_K(Qh^{-1})},$$

where $B_\varepsilon(t)$ denotes the closed ball (with respect to the metric d_K) of radius ε around t. Since H_o has finite-index in G, we see that $\overline{\tau_K(H_o)} < K$ is an open subgroup of K, and thus equal to K by connectivity. Since Q is invariant under H_o, we see that $\tau_K(Qh^{-1})$ must be dense in K, and thus

$$\overline{\tau_K(Q \cap T)} B_\varepsilon(t)^{-1} = K, \quad \text{for all } \varepsilon > 0.$$

We leave it to reader to show that this implies that $\tau_K(Q \cap T)$ is dense in K, which finishes the proof. □

Let us now show that B is not contained in a Sturmian set with the same upper Banach density as B. We shall argue by contradiction. Let (M, τ_M) be a compactification of G, where M denotes either $\mathbb{T}$ or $\mathbb{T} \rtimes \{-1, 1\}$, and fix a symmetric interval $J' \subset \mathbb{T}$ of Haar measure equal to $d^*(B)$, and an element $a \in M$. Let

$$J = J' + a \quad \text{or} \quad I = (J' \rtimes \{-1, 1\})a,$$

depending on whether $M = \mathbb{T}$ or $M = \mathbb{T} \rtimes \{-1, 1\}$, and suppose that $B \subset \tau_M^{-1}(J)$. Since $e_G \in B$, we note that $e_M \in J$. In both cases, J equals the closure of its interior.

Let us now define the compactification (K, τ_K) of G by $\tau_K(g) = (\tau(g), \tau_M(g))$ for $g \in G$, where K denotes the closure of the image of τ in $\mathbb{T} \times M$, and set

$$C = (I \times M) \cap K \quad \text{and} \quad D = (\mathbb{T} \times J) \cap K.$$

One readily checks that

$$m_K(C) = m_{\mathbb{T}}(I) \quad \text{and} \quad m_K(D) = m_M(J) \quad \text{and} \quad \tau_K^{-1}(C) \cap T \subset \tau_K^{-1}(D).$$

In particular, $m_K(C) = m_K(D)$, and $\tau_K^{-1}(C^o \setminus D) \cap T$ is empty. Since $C^o \setminus D$ is open in K and $\tau_K(G)$ is dense, either $C^o \subset D$ or $\tau_K^{-1}(C^o \setminus D)$ is syndetic. Since T is thick, and thus intersects every syndetic set non-trivially, we see that only the first alternative can hold. We leave it to the reader to verify that $\overline{C^o} = C$, which then implies (since D is closed) that $C \subset D$.

Since $m_K(C) = m_K(D)$, the open set $D^o \setminus C \subset K$ must be empty, and thus $D^o \subset C$. Since D equals the closure of D^o, and C is closed, we conclude that $C = D$. However, since $e_M \in J$, we have $e_K \in D$, and thus $e_K \in C$, which then implies that $e_{\mathbb{T}} \in I$. We have assumed is not the case. This contradiction implies that B cannot be contained in the Sturmian set $\tau_M^{-1}(J)$.

Appendix: Necessity of the conditions in Theorem 1.1 and Theorem 1.5

In this appendix we focus on the (additively written) abelian group $G = (\mathbb{Z}, +)$ and the Følner sequence $([-n, n])$. In order to spare sub-indices, we denote by $\underline{d}$ the corresponding lower asymptotic density, that is to say, for $A \subset \mathbb{Z}$, we define

$$\underline{d}(A) = \varliminf_{n\to\infty} \frac{|A \cap [-n, n]|}{2n + 1}.$$

In what follows, we shall in this setting address the necessity of the conditions in Theorem 1.1 and Theorem 1.5. We give below four examples (two examples per theorem), how different attempts to weaken the hypotheses in these theorems fail. All of the examples are constructed by similar procedures. To avoid repeating the same construction four times, we collect here some notation that will be used throughout the appendix.

The basic parameter is a proper closed interval I of the one-dimensional torus $\mathbb{T}$, which we shall think of as the quotient $\mathbb{R}/\mathbb{Z}$. Let $a \in \mathbb{T}$ be irrational, and define the Sturmian set

$$C = \{n \in \mathbb{Z} \,:\, na \in I\}.$$

In the examples below we shall specify I, and thereafter reserve the letter C for the set above. One can readily check that C is not contained in a proper periodic set, and

$$\underline{d}(C \cap \mathbb{N}) = \underline{d}(C \cap (-\mathbb{N})) = \frac{1}{2} m_{\mathbb{T}}(I). \tag{11.5}$$

and

$$\underline{d}(C) = \underline{d}((C + C) \cap \mathbb{N}) = \underline{d}((C + C) \cap (-\mathbb{N})) = m_{\mathbb{T}}(I). \tag{11.6}$$

Finally, C does not *contain* a piecewise periodic set, nor does its sumset $C + C$ as long as we assume that $m_{\mathbb{T}}(I) < 1/2$.

11.5. **Weakening the conditions in Theorem 1.1: First attempt**

Let us begin by showing that the assumption that the sumset $A + B$ is *not* thick cannot be left out in Theorem 1.1.

Proposition 11.7. *There exist $A, B \subset \mathbb{Z}$ such that*

- *A is not contained in a proper periodic set,*
- *B is syndetic,*
- *$A + B$ is thick,*

and

$$\underline{d}(A + B) < d^*(A) + \underline{d}(B) < 1.$$

Proof. Suppose that $m_{\mathbb{T}}(I) < \frac{1}{3}$, and define the sets

$$A = C \cap \mathbb{N} \quad \text{and} \quad B = C \cup \mathbb{N}.$$

One readily check that A is not contained in a proper periodic set, B is syndetic and $A+B$ is thick. Furthermore, by (11.5) and (11.6)

$$d^*(A) = m_{\mathbb{T}}(I) \quad \text{and} \quad \underline{d}(B) = \frac{1}{2}(1 + m_{\mathbb{T}}(I)),$$

and, since $A + B \subset (C + C) \cup \mathbb{N}$, we have

$$\underline{d}(A+B) \leq \underline{d}((C+C) \cap (-\mathbb{N})) + \frac{1}{2} \leq m_{\mathbb{T}}(I) + \frac{1}{2} < d^*(A) + \underline{d}(B) < 1.$$

□

11.6. **Weakening the conditions in Theorem 1.1: Second attempt**

Let us now show that the assumption that the set B is syndetic in Theorem 1.1 cannot be replaced with the (weaker) condition of having positive lower asymptotic density.

Proposition 11.8. *There exist $A, B \subset \mathbb{Z}$ such that*

- *A is not contained in a proper periodic set,*
- *B is not syndetic, but $\underline{d}(B) > 0$,*
- *$A + B$ is not thick,*

and

$$\underline{d}(A+B) < d^*(A) + \underline{d}(B) < 1.$$

Proof. Suppose that $m_{\mathbb{T}}(I) < \frac{1}{2}$ and define

$$A = B = C \cap \mathbb{N}.$$

We note that A is not contained in a proper periodic set, B is not syndetic and by (11.5)

$$d^*(A) = m_{\mathbb{T}}(I) \quad \text{and} \quad \underline{B} = \frac{1}{2} m_{\mathbb{T}}(I).$$

Furthermore, since $A + B \subset (C + C) \cap \mathbb{N}$, and $m_{\mathbb{T}}(I) < \frac{1}{2}$, we see that the sumset $A + B$ is not thick. By (11.6),

$$\underline{d}(A+B) \leq m_{\mathbb{T}}(I) < d^*(A) + \underline{d}(B) < 1,$$

which finishes the proof. □

11.7. **Weakening the conditions in Theorem 1.5: First attempt**

Let us now turn to Theorem 1.5 and show that the assumption that the set B is syndetic cannot be removed.

Proposition 11.9. *There exist $A, B \subset \mathbb{Z}$ such that*

- *A is spread-out,*
- *B is not syndetic, but $\underline{d}(B) > 0$,*
- *$A + B$ does not contain a piecewise periodic set,*

and

$$\underline{d}(A+B) = d^*(A) + \underline{d}(B) < 1,$$

but A is not contained in a Sturmian subset with the same upper Banach density as A.

Proof. Suppose that $I \subset \mathbb{T}$ and $n \in \mathbb{Z}$ satisfies

$$(I + I) \cap (I + na) = \emptyset.$$

In particular, $m(I) < 1/3$. Define the sets

$$A = (C \cap \mathbb{N}) \cup \{n\} \quad \text{and} \quad B = C \cap \mathbb{N},$$

and note that A is spread-out and B is not syndetic. Furthermore,

$$d^*(A) = m_{\mathbb{T}}(I) \quad \text{and} \quad \underline{d}(B) = \frac{1}{2} m_{\mathbb{T}}(I).$$

Since $A + B \subset (C + n) \cup (C + C)$, and the latter set does not contain a proper piecewise periodic set, neither does $A + B$. It is straightforward, albeit tedious, to verify that

$$\underline{d}(A + B) = m_{\mathbb{T}}(I) + \frac{1}{2} m_{\mathbb{T}}(I) = d^*(A) + \underline{d}(B) < 1.$$

□

11.8. **Weakening the conditions in Theorem 1.5: Second attempt**

Our last example is quite technical, and we shall only provide a rough sketch. The point here is to demonstrate the necessity of the assumption that the sumset $A+B$ does not contain a piecewise periodic set (in particular, it is not thick).

Proposition 11.10. *There exist $A, B \subset \mathbb{Z}$ such that*

- *A is spread-out,*
- *B is syndetic,*
- *$A + B$ is thick,*

and

$$\underline{d}(A + B) = d^*(A) + \underline{d}(B) < 1,$$

but A is not contained in a Sturmian subset with the same upper Banach density as A.

Rough sketch of proof. It can be shown, with quite a lot of work, that there exists a thick set $T \subset \mathbb{N}$ such that

$$\underline{d}_{([1,n])}(T) = \frac{1}{10} \quad \text{and} \quad \underline{d}_{([1,n])}(T + T) = \frac{2}{10},$$

and with the property that the sequence (F_n) of finite subsets of $\mathbb{N}$ given by

$$F_n := [1, n] \setminus (T + T) \quad \text{is Følner.}$$

Let $I \subset \mathbb{T}$ be an interval with $m_{\mathbb{T}}(I) = 4/9$ such that there exists $m \in T$ with

$$m_{\mathbb{T}}((I + I) \cup (I + \tau(m))) = (2 + \frac{1}{24}) m_{\mathbb{T}}(I).$$

The existence of such I and m can be proved using the fact that $\tau(T) \subset \mathbb{T}$ is dense, and utilizing the flexibility to translate a given interval of measure $1/4$ around in $\mathbb{T}$. Finally, once these sets and numbers have been chosen, we define the sets

$$A = (C \cap T) \cup \{m\} \quad \text{and} \quad B = C \cup T.$$

We note that A is spread-out, B is syndetic *and* thick, and thus $A + B$ is thick as well. It is now possible, but once again quite involved, to check that

$$\underline{d}(A + B) = d^*(A) + \underline{d}(B) < 1.$$

□

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