

# ON BOHR SETS OF INTEGER-VALUED TRACELESS MATRICES

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ABSTRACT. In this paper we show that any Bohr-zero non-periodic set  $B$  of traceless integer valued matrices, denoted by  $\Lambda$ , intersects non-trivially the conjugacy class of any matrix from  $\Lambda$ . As a corollary, we obtain that the family of characteristic polynomials of  $B$  contains all characteristic polynomials of matrices from  $\Lambda$ . The main ingredient used in this paper is an equidistribution result of Bourgain-Furman-Lindenstrauss-Mozes [6].

## 1. INTRODUCTION

Let us denote by  $\Lambda = \text{Mat}_d^0(\mathbb{Z})$ ,  $d \geq 2$ , the set of integer valued  $d \times d$  matrices with zero trace, and by  $\mathbb{T}^n$ ,  $n \geq 1$ , the  $n$ -dimensional torus  $\mathbb{R}^n/\mathbb{Z}^n$ . Let  $G$  be a countable abelian group. A set  $B \subset G$  is called a *non-periodic Bohr set* if there exist a homomorphism  $\tau : G \rightarrow \mathbb{T}^n$ , for some  $n \geq 1$ , with  $\overline{\tau(G)} = \mathbb{T}^n$ , and an open set  $U \subset \mathbb{T}^n$  satisfying  $B = \tau^{-1}(U)$ . If the open set  $U$  contains the zero element of  $\mathbb{T}^n$ , then the set  $B$  is called a *Bohr-zero set*. We will also denote by  $SL_d(\mathbb{Z})$  the group of  $d \times d$  integer-valued matrices of determinant one.

The main result of this paper is the following.

**Main Theorem.** *Let  $d \geq 2$ , and  $B \subset \text{Mat}_d^0(\mathbb{Z})$  be a Bohr-zero non-periodic set. Then for any matrix  $C \in \text{Mat}_d^0(\mathbb{Z})$  there exists a matrix  $A \in B$  and a matrix  $g \in SL_d(\mathbb{Z})$  such that  $C = g^{-1}Ag$ .*

The same result has been also proved independently by Björklund and Bulinski [4]. They use the recent works of Benoist-Quint [2] and [3], instead of the work of Bourgain-Furman-Lindenstrauss-Mozes as the main ingredient in the proof.

**Corollary 1.1.** *Let  $d \geq 2$ , and  $B \subset \text{Mat}_d^0(\mathbb{Z})$  be a Bohr-zero non-periodic set. The set of characteristic polynomials of the matrices in  $B$  coincides with the set of all characteristic polynomials of the matrices in  $\text{Mat}_d^0(\mathbb{Z})$ .*

The following number-theoretic statement conjectured by B. Green and T. Sanders is an immediate implication of Corollary 1.1.

**Corollary 1.2.** *Let  $B \subset \mathbb{Z}$  be a Bohr-zero non-periodic set. Then the set of the discriminants over  $B$  defined by*

$$D := \{xy - z^2 \mid x, y, z \in B\}$$

*satisfies that  $D = \mathbb{Z}$ .*

At this point we will define Furstenberg's system corresponding to a set  $B$  of positive density in a countable abelian group  $G$ . Recall, we say that  $B$  has *positive density* if *upper Banach density* of  $B$  is positive:

$$d^*(B) = \sup_{\lambda \in \mathcal{F}} \lambda(1_B) > 0,$$

where  $\mathcal{F}$  is the set of all  $\Lambda$ -invariant means on  $\ell^\infty(G)$ , i.e., non-negative normalised  $G$ -invariant linear functionals on  $\ell^\infty(G)$ . Since  $G$  is abelian, this implies that  $\mathcal{F} \neq \emptyset$ .

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Furstenberg in his seminal paper [8] constructed an (ergodic)  $G$ -measure-preserving system<sup>1</sup>  $(X, \eta, \sigma)$  and a clopen set  $\tilde{B} \subset X$  such that

- $d^*(B \cap (B + h)) \geq \eta(\tilde{B} \cap \sigma(h)\tilde{B})$ , for any  $h \in G$ .
- $\eta(\tilde{B}) = d^*(B)$ .

We will denote Furstenberg's system corresponding to  $B$  by  $X_B = (X, \eta, \sigma, \tilde{B})$ . Next, we will define the notion of the spectral measure corresponding to a set  $B$  of a countable abelian group  $G$  of positive density and its Furstenberg's system  $X_B = (X, \eta, \sigma, \tilde{B})$ . Denote by  $1_{\tilde{B}}$  the indicator function of the set  $\tilde{B}$ . Then by Bochner's spectral theorem [7] there exists a non-negative finite Borel measure  $\nu$  on  $\hat{G}$  (the dual of  $G$ ) which satisfies:

$$\langle 1_{\tilde{B}}, \sigma(h)1_{\tilde{B}} \rangle = \int_{\hat{G}} \chi(h) d\nu(\chi), \text{ for } h \in G.$$

The measure  $\nu$  will be called the *spectral measure of the set  $B$  and its Furstenberg's system  $X_B$* , and we will denote by  $\hat{\nu}(h)$  the right hand side of the last equation. We are at the position to state the main technical claim of the paper.

**Theorem 1.1.** *Let  $d \geq 2$ , and let  $B \subset \text{Mat}_d^0(\mathbb{Z})$  be a set of positive density such that the spectral measure<sup>2</sup> of  $B$  has no atoms at non-trivial characters having finite torsion. Then for every  $C \in \text{Mat}_d^0(\mathbb{Z})$  there exist  $A \in B - B$  and  $g \in \text{SL}_d(\mathbb{Z})$  with  $C = g^{-1}Ag$ .*

Theorem 1.1 is the strengthening of the following result that has been proved in [5] by use of the equidistribution result of Benoist-Quint [1].

**Theorem 1.2.** *Let  $d \geq 2$ , and let  $B \in \text{Mat}_d^0(\mathbb{Z})$  be a set of positive density. Then there exists  $k \geq 1$  such that for any matrix  $C \in k\text{Mat}_d^0(\mathbb{Z})$  there exists  $A \in B - B$  and  $g \in \text{SL}_d(\mathbb{Z})$  with  $C = g^{-1}Ag$ .*

We would like to finish the introduction by stating the piecewise version of Main Theorem. We recall that a set  $B \subset \Lambda$  called *piecewise Bohr set* if there is a Bohr set  $B_0 \subset \Lambda$  and a (thick) set  $T \subset \Lambda$  of upper Banach density one, i.e.,  $d^*(T) = 1$  such that  $B = B_0 \cap T$ . Moreover, if the set  $B_0$  is non-periodic Bohr-zero, then the set  $B$  will be called *piecewise Bohr-zero non-periodic*. Theorem 1.1 implies the following result.

**Theorem 1.3.** *Let  $d \geq 2$ , and let  $B \subset \text{Mat}_d^0(\mathbb{Z})$  be a piecewise Bohr non-periodic set. Then for every  $C \in \text{Mat}_d^0(\mathbb{Z})$  there exist  $A \in B - B$  and  $g \in \text{SL}_d(\mathbb{Z})$  with  $C = g^{-1}Ag$ .*

Let us show that Theorem 1.3 implies Main Theorem.

**Proof of Main Theorem.** Let  $B \subset \Lambda$  be a Bohr-zero non-periodic set. Notice that there exists  $B_0 \subset \Lambda$  a Bohr-zero non-periodic set with the property that

$$B_0 - B_0 \subset B.$$

Now, we apply Theorem 1.3 for the set  $B_0$ , and as a conclusion obtain the statement of the theorem.  $\square$

*Organisation of the paper.* In Section 2 we establish the consequences of the equidistribution result of Bourgain-Furman-Lindenstrauss-Mozes [6] related to the adjoint action of

<sup>1</sup>A triple  $(X, \eta, \sigma)$  is a  $G$ -measure-preserving system, if  $X$  is a compact metric space on which acts  $G$  by a measurable action denoted by  $\sigma$ ,  $\eta$  is a Borel probability measure on  $X$ , and the action of  $G$  preserves  $\eta$ . A  $G$ -measure-preserving system is *ergodic* if any  $G$ -invariant measurable set has measure either zero or one.

<sup>2</sup>We assume the existence of some Furstenberg's system  $X_B$  corresponding to the set  $B$ , such that the associated spectral measure satisfies the requirement of the theorem.

$SL_d(\mathbb{Z})$  on  $Mat_d^0(\mathbb{R})/Mat_d^0(\mathbb{Z})$ . In Section 3 we prove Theorems 1.1, and 1.3. In Section 4 we prove main technical propositions stated in Sections 2 and 3.

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## 2. CONSEQUENCES OF THE WORK OF BOURGAIN-FURMAN-LINDENSTRAUSS-MOZES

We start by recalling the property of strong irreducibility of an action of a discrete group. Let  $\Gamma$  be a countable group, and  $V$  be a finite dimensional real space. We say that an action  $\rho : \Gamma \rightarrow \text{End}(V)$  is *strongly irreducible* if for every finite index subgroup  $H$  of  $\Gamma$ , the restriction of the action of  $\rho$  to  $H$  is irreducible. We also will be using the notion of a proximal element. An operator  $T \in \text{End}(V)$  will be called *proximal*, if there is only one eigenvalue of the largest absolute value, and corresponding to it eigenspace is one-dimensional.

Assume that a countable group  $\Gamma$  acts on a compact Borel measure space  $(X, \nu)$ . Let  $\mu$  be a probability measure on  $\Gamma$ . Then the convolution measure  $\mu * \nu$  on  $X$  is defined by:

$$\int_X f d(\mu * \nu) = \int_X \left( \sum_{g \in \Gamma} f(gx) \mu(g) \right) d\nu(x), \text{ for any } f \in C(X).$$

We will denote the Dirac probability measure at a point  $x \in X$  by  $\delta_x$ . For every  $k \geq 2$ , we define the probability measure  $\mu^{*k}$  on  $\Gamma$  by

$$\mu^{*k}(g) = \sum_{g_1 \dots g_k = g} \mu(g_1) \mu(g_2) \dots \mu(g_k).$$

The main ingredient in the proofs of all our main results is the following seminal equidistribution statement due to Bourgain-Furman-Lindenstrauss-Mozes [6].

**Theorem 2.1** (Corollary B in [6]). *Let  $\Gamma < SL_n(\mathbb{Z})$  be a subgroup which acts totally irreducibly on  $\mathbb{R}^n$ , and having a proximal element. Let  $\mu$  be a finite generating probability measure on  $\Gamma$ . Let  $x \in \mathbb{T}^n$  be a non-rational point. Then the measures  $\mu^{*k} * \delta_x$  converge in weak\*-topology as  $k \rightarrow \infty$  to Haar measure on  $\mathbb{T}^n$ .*

In this note, the acting group will be  $\Gamma = SL_d(\mathbb{Z})$ . The group  $\Gamma$  acts by the conjugation on the real vector space  $V = Mat_d^0(\mathbb{R})$  of real valued  $d \times d$  matrices with zero trace. So, an element  $g \in SL_d(\mathbb{Z})$  acts on  $v \in V$  by  $Ad(g)v = g^{-1}vg$ , and such action called the *adjoint action* of  $SL_d(\mathbb{Z})$ . Notice that  $V$  is isomorphic to  $\mathbb{R}^{d^2-1}$ . The next claim will allow us to apply Theorem 2.1 in our setting.

**Proposition 2.1.** *The adjoint action of  $SL_d(\mathbb{Z})$  on  $Mat_d^0(\mathbb{R})$  is strongly irreducible, and  $SL_d(\mathbb{Z})$  contains an element which acts proximally.*

Let us denote by  $A_d = V/\Lambda$ . Notice that  $A_d$  is isomorphic to  $\mathbb{T}^{d^2-1}$ , and it is the dual group of  $\Lambda$ . The adjoint action of  $SL_d(\mathbb{Z})$  leaves  $\Lambda$  invariant. Therefore,  $SL_d(\mathbb{Z})$  also acts on  $A_d$ . Proposition 2.1 implies by Corollary B from [6] the following statement.

**Proposition 2.2.** *Let  $\mu$  be a probability measure on  $SL_d(\mathbb{Z})$  with finite generating support. Let  $x \in A_d$  be a non-rational point. Then the measures  $\mu^{*k} * \delta_x$  converge as  $k \rightarrow \infty$  in the weak\* topology to the normalised Haar measure on  $A_d$ .*

We will be using Proposition 2.2 to prove the following claim.

**Proposition 2.3.** *Let  $\mu$  be a probability measure on  $SL_d(\mathbb{Z})$  with finite generating support. Let  $\nu$  be a probability measure on  $A_d$  with no atoms at rational points. Then the measures  $\mu^{*k} * \nu$  converge as  $k \rightarrow \infty$  in the weak\* topology to the normalised Haar measure on  $A_d$ .*

## 3. PROOFS OF THEOREMS 1.1, AND 1.3

**Proof of Theorem 1.1.** Recall that we denote by  $\Lambda = \text{Mat}_d^0(\mathbb{Z})$ , and by  $\Gamma = \text{SL}_d(\mathbb{Z})$ . Let  $B \subset \Lambda$  be a set of positive density with Furstenberg's system  $X_B = (X, \eta, \sigma, \tilde{B})$  and such that the spectral measure of  $B$  has no atoms at non-trivial characters. We make the identification of the dual space of  $\Lambda$  with the torus  $A_d$  by corresponding for every  $x \in A_d$  the character  $\chi_x$  on  $\Lambda$  given by:

$$\chi_x(h) = \exp(2\pi i x \cdot h), \text{ for } h \in \Lambda.$$

Notice that the trivial character on  $\Lambda$  corresponds to the zero element  $o_{A_d}$  of  $A_d$ , and characters having finite torsion correspond to the rational points of  $A_d$ . Denote by  $\nu$  the spectral measure of  $B$ , i.e., for every  $h \in \Lambda$  we have

$$(1) \quad \langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle = \int_{A_d} \exp(2\pi i x \cdot h) d\nu(x).$$

By the assumptions of the theorem,  $\nu$  has no atoms at the rational points. Then  $\Gamma$  acts on  $\Lambda$  by the conjugation. We will show that for every  $h \in \Lambda$  there exists  $g \in \Gamma$  such that

$$\hat{\nu}(g^{-1}hg) = \langle 1_{\tilde{B}}, \sigma(g^{-1}hg) 1_{\tilde{B}} \rangle > 0.$$

This will imply the claim of the theorem by the first property of Furstenberg's system  $X_B$ . Assume, that on the contrary, that there exists  $h \in \Lambda$  such that for all  $g \in \Gamma$  we have

$$(2) \quad \hat{\nu}(g^{-1}hg) = 0.$$

It follows from the first property of the spectral measure listed below that the equation 2 holds for a non-zero  $h \in \Lambda$ . By the assumptions on  $B$ , we know that

- $\nu(\{o_{A_d}\}) = \eta(\tilde{B})^2 > 0$ .
- For every rational non-zero point  $x \in A_d$  we have  $\nu(\{x\}) = 0$ .

Indeed, the second property is given to us by the assumptions. To prove the first one, notice that for any Følner sequence<sup>3</sup>  $(F_n)$  in  $\Lambda$ :

$$(3) \quad \frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle = \int_{A_d} \frac{1}{|F_n|} \sum_{h \in F_n} \exp(2\pi i x \cdot h) d\nu(x) \rightarrow \nu(\{o_{A_d}\}), \text{ as } N \rightarrow \infty.$$

In the last transition, we have used Lebesgue's dominated convergence theorem and the easy claim that for any non-trivial character  $\chi$  on  $\Lambda$ , and a Følner sequence  $(F_n)$  in  $\Lambda$  we have:

$$\frac{1}{|F_n|} \sum_{h \in F_n} \chi(h) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

By ergodicity of Furstenberg's system and von-Neumann's ergodic theorem it follows that the left hand side of (3) satisfies

$$\frac{1}{|F_n|} \sum_{h \in F_n} \langle 1_{\tilde{B}}, \sigma(h) 1_{\tilde{B}} \rangle \rightarrow \eta(\tilde{B})^2, \text{ as } n \rightarrow \infty.$$

This finishes the proof of the second property of the spectral measure  $\nu$ .

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<sup>3</sup>A sequence of finite sets  $(F_n)$  in  $\Lambda$  is called Følner if it is asymptotically  $\Lambda$ -invariant, i.e. for every  $h \in \Lambda$  we have  $\frac{|F_n \cap (F_n + h)|}{|F_n|} \rightarrow 1$ , as  $n \rightarrow \infty$ . For any countable abelian group the family of Følner sequences is non-empty.

Let  $\mu$  be a probability measure on  $SL_d(\mathbb{Z})$  having a finite generating support. By Proposition 2.3 the measures  $\mu^{*k} * \nu$  converge as  $k \rightarrow \infty$  in weak\* topology to

$$\eta(\tilde{B}) \left(1 - \eta(\tilde{B})\right) m_{A_d} + \eta(\tilde{B})^2 \delta_{o_{A_d}},$$

where  $m_{A_d}$  stands for the normalised Haar measure on  $A_d$ . Notice that  $\Gamma$  also acts on  $A_d$  by  $g \cdot x = g^t x (g^t)^{-1}$ , for  $g \in \Gamma$ . The action of  $\Gamma$  on  $A_d$  and the adjoint action of  $\Gamma$  on  $\Lambda$  are related by the following:

$$(g \cdot x) \cdot h = x \cdot Ad(g)h, \text{ for every } g \in \Gamma, h \in \Lambda, x \in A_d.$$

Since

$$\begin{aligned} \widehat{\mu^{*k} * \nu}(h) &= \int_{A_d} \exp(2\pi i x \cdot h) d(\mu^{*k} * \nu)(x) = \\ &= \int_{A_d} \left( \sum_{g \in \Gamma} \exp(2\pi i (g \cdot x) \cdot h) \mu^{*k}(g) \right) d\nu(x) = \\ &= \sum_{g \in \Gamma} \left( \int_{A_d} \exp(2\pi i x \cdot (g^{-1}hg)) d\nu(x) \right) \mu^{*k}(g) = \sum_{g \in \Gamma} \widehat{\nu}(g^{-1}hg) \mu^{*k}(g). \end{aligned}$$

Recall, we assumed that there exists a non-zero  $h \in \Lambda$  such that  $\widehat{\nu}(g^{-1}hg) = 0$ , for all  $g \in \Gamma$ . Therefore, we have  $\widehat{\mu^{*k} * \nu}(h) = 0$ , for all  $k \geq 1$ . On other hand, since  $\widehat{m_{A_d}}(h) = 0$ , and  $\widehat{\delta_{o_{A_d}}}(h) = 1$ , we have:

$$\widehat{\mu^{*k} * \nu}(h) \rightarrow \eta(\tilde{B})^2 > 0, \text{ as } k \rightarrow \infty.$$

Thus, we have a contradiction. This finishes the proof of the theorem.  $\square$

**Proof of Theorem 1.3.** Theorem 1.3 follows immediately from Theorem 1.1 by use of the following statement which will be proved in the next section.

**Proposition 3.1.** *Let  $B \subset Mat_d^0(\mathbb{Z})$  be a non-periodic piecewise Bohr set corresponding to a Jordan measurable<sup>4</sup> open set in a finite-dimensional torus. There exists a spectral measure associated with  $B$  that does not have atoms at non-zero rational points of  $A_d$ .*

Indeed, let  $B \subset \Lambda$  be a piecewise non-periodic Bohr set given by  $B = \tau^{-1}(U) \cap T$ , where  $\tau : \Lambda \rightarrow \mathbb{T}^n$  is a homomorphism with a dense image,  $U \subset \mathbb{T}^n$  is an open set, and  $T \subset \Lambda$  is a set with  $d^*(T) = 1$ . Then  $U$  contains an open ball  $U_o$ , and  $m_{\mathbb{T}^n}(\partial U_o) = 0$ , where  $m_{\mathbb{T}^n}$  denotes the Haar normalised measure on  $\mathbb{T}^n$ . Denote by  $B' = \tau^{-1}(U_o) \cap T \subset B$ . Then the statement of Theorem 1.3 for the non-periodic piecewise Bohr set  $B'$  follows from Proposition 3.1. The latter implies the statement of the Theorem for the set  $B$ .  $\square$

#### 4. PROOFS OF PROPOSITIONS 2.1, 2.3, AND 3.1

**4.1. Proof of Proposition 2.1.** It is proved in [5] [Corollary 5.4] that the adjoint action of  $SL_d(\mathbb{Z})$  on  $Mat_d^0(\mathbb{R})$  is strongly irreducible. Therefore, it is remained to prove that there is at least one element of  $SL_d(\mathbb{Z})$  which acts on  $Mat_d^0(\mathbb{R})$  proximally. The next claim finishes the proof of Proposition 2.1.

<sup>4</sup>A set  $A$  in a topological space  $X$  equipped with a measure  $m_X$  is *Jordan measurable* if  $m_X(\partial A) = 0$ , where  $\partial A = \overline{A} \setminus \overset{\circ}{A}$ .

**Proposition 4.1.** *The matrix*

$$B_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

*acts (by conjugation) on  $Mat_2^0(\mathbb{R})$  proximally. For  $d \geq 3$ , the matrix*

$$B_d = \left[ \begin{array}{cc|c} 1 & -1 & \mathbf{0}_{2 \times (d-2)} \\ -1 & 2 & \\ \hline \mathbf{0}_{(d-2) \times (d-2)} & & Id_{(d-2) \times (d-2)} \end{array} \right]$$

*acts proximally on  $Mat_d^0(\mathbb{R})$ .*

*Proof.* It is straightforward to check that the operator  $B_2 : Mat_2^0(\mathbb{R}) \rightarrow Mat_2^0(\mathbb{R})$  can be written in the matrix form as<sup>5</sup>

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of this operator is  $\chi_{B_2}(\lambda) = (1 - \lambda)(\lambda^2 - \lambda - 1)$ . Since all eigenvalues are distinct by their absolute value, it follows that the operator acts proximally.

In the case  $d \geq 3$ , notice that the action of  $B_d$  on  $Mat_d^0(\mathbb{R})$  is decomposed into 4 orthogonal spaces. The actions on the  $2 \times 2$  upper left corner,  $2 \times (d-2)$  upper right corner,  $(d-2) \times 2$  bottom left corner, and the identity action on the bottom right  $(d-2) \times (d-2)$  corner. Correspondingly, the dimensions of the spaces are  $4, 2 \cdot (d-2), (d-2) \cdot 2$ , and  $(d-2)^2 - 1$ .

The 4-dimensional left upper corner part can be written in the matrix form as

$$\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 4 & -1 & 2 \\ 1 & -1 & 1 & -1 \\ -1 & 2 & -1 & 2 \end{bmatrix}.$$

Its characteristic polynomial is  $(\lambda - 1)^2(\lambda^2 - 7\lambda + 1)$ . Therefore there is a unique highest eigenvalue by the absolute value equal to  $\frac{7+3\sqrt{5}}{2}$ , and it has multiplicity one.

The operator  $B_d$  acts on the upper right corner in the following way

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_{d-2} & y_{d-2} \end{bmatrix}^t \rightarrow \begin{bmatrix} 2x_1 + y_1 & x_1 + y_1 \\ 2x_2 + y_2 & x_2 + y_2 \\ \dots & \dots \\ 2x_{d-2} + y_{d-2} & x_{d-2} + y_{d-2} \end{bmatrix}^t.$$

It is clear that it has two eigenvalues with multiplicity  $d-2$ . These eigenvalues correspond to the eigenvalues of the matrix

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

These eigenvalues are the roots of the characteristic polynomial of the matrix  $C$  which are  $\frac{3 \pm \sqrt{5}}{2}$ .

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<sup>5</sup>We use the identification between  $Mat_2^0(\mathbb{R})$  and  $\mathbb{R}^3$ , by

$$\begin{bmatrix} x & y \\ z & -x \end{bmatrix} \rightarrow [x, y, z]^t$$

The operator  $B_d$  acts on the bottom left corner in the following way:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_{d-2} & y_{d-2} \end{bmatrix} \rightarrow \begin{bmatrix} x_1 - y_1 & -x_1 + 2y_1 \\ x_2 - y_2 & -x_2 + 2y_2 \\ \dots & \dots \\ x_{d-2} - y_{d-2} & -x_{d-2} + 2y_{d-2} \end{bmatrix}.$$

Therefore it has two eigenvalues of the matrix  $C^{-1}$  each one having multiplicity  $d - 2$ . It is immediate to check that  $C^{-1}$  has the same characteristic polynomial as  $C$ , therefore the eigenvalues of the operator  $B_d$  acting on the bottom left corner are  $\frac{3 \pm \sqrt{5}}{2}$ , each one having multiplicity  $d - 2$ .

As the conclusion of the previous considerations we find the the eigenvalues of the operator  $B_d$  are  $\frac{7+3\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 1, \frac{7-3\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$  with corresponding multiplicities equal to  $1, 2(d-2), [(d-2)^2 - 1] + 2, 1, 2(d-2)$ . This implies that  $B_d$  acts proximally on  $Mat_d^0(\mathbb{R})$ .  $\square$

**4.2. Proof of Proposition 2.3.** Let  $\nu$  be a probability measure on  $A_d$  with no atoms at rational points, and let  $\mu$  be a probability measure on  $\Gamma = SL_d(\mathbb{Z})$  with a finite generating support. By Proposition 2.2 for every  $x \in \text{supp}(\nu)$  the measures  $\mu^{*k} * \delta_x$  converge in weak\*-topology as  $k \rightarrow \infty$  to the Haar measure on  $A_d$ . Let  $f$  be a continuous function on  $A_d$ . Then for every  $x \in \text{supp}(\nu)$  we have that  $f_k(x) := \int f d(\mu^{*k} * \delta_x) \rightarrow \int f$ . We have to show that

$$\int_{A_d} f d(\mu^{*k} * \nu) \rightarrow \int f.$$

By Egorov's theorem, for every  $\varepsilon > 0$ , there exists  $X' \subset A_d$  with  $\nu(X') \geq 1 - \varepsilon$  and  $K(\varepsilon)$  with the property that for every  $x \in X'$  and every  $k \geq K(\varepsilon)$  we have

$$\left| f_k(x) - \int f \right| < \varepsilon.$$

Notice that

$$\begin{aligned} \int f d\mu^{*k} * \nu &= \sum_{g \in \Gamma} \int f(gx) \mu^{*k}(g) d\nu(x) = \int \left( \sum_{g \in \Gamma} f(gx) \mu^{*k}(g) \right) d\nu(x) \\ &= \int \left( \int f d(\mu^{*k} * \delta_x) \right) d\nu(x). \end{aligned}$$

Let  $\delta > 0$ . Denote by  $M = \|f\|_\infty$ , and take  $\varepsilon > 0$  so small that  $\varepsilon M < 2\delta$ , and  $\varepsilon < \delta$ . Then we have

$$\left| \int f d(\mu^{*k} * \nu) - \int f \right| < (1 - \varepsilon)\varepsilon + \varepsilon M < 3\delta,$$

for  $k \geq K(\varepsilon)$ . Since  $\delta$  can be chosen arbitrary small, we have shown that

$$\int f d(\mu^{*k} * \nu) \rightarrow \int f.$$

This finishes the proof because the function  $f$  was an arbitrary continuous function on  $A_d$ .

**4.3. Proof of Proposition 3.1.** Recall that  $\Lambda = Mat_d^0(\mathbb{Z})$ . We are given a piecewise Bohr non-periodic set  $B \subset \Lambda$  corresponding to a Jordan measurable open set in a finite dimensional torus. This means that  $B = B_o \cap T$ , where  $T \subset \Lambda$  with  $d^*(T) = 1$ , and  $B_o \subset \Lambda$  given via a homomorphism  $\tau : \Lambda \rightarrow \mathbb{T}^n$ , for some  $n \geq 1$  with the dense image, and an open Jordan measurable set  $U_o \subset \mathbb{T}^n$  such that

$$B_o = \tau^{-1}(U_o).$$

We will construct an ergodic Furstenberg's  $\Lambda$  system  $X_B = (X, \eta, \sigma, \tilde{B})$  corresponding to the set  $B$ , and will show that the spectral measure of the function  $1_{\tilde{B}}$  has no atoms at the rational non-zero points of  $A_d := \hat{\Lambda}$ .

Let  $X = \mathbb{T}^n$ ,  $\eta$  be the Haar normalised measure on  $X$ ,  $\sigma_h(x) := x + \tau(h)$  for  $x \in X, h \in \Lambda$ , and  $\tilde{B} = U_o$ . We will denote by  $X_B := (X, \eta, \sigma, \tilde{B})$ . It remains to show that

- For every  $h \in \Lambda$  we have  $d^*(B \cap (B + h)) \geq \eta(\tilde{B} \cap \sigma_h(\tilde{B}))$ .
- $\eta(\tilde{B}) = d^*(B)$ .
- The spectral measure of  $1_{\tilde{B}}$  has no atoms at non-zero rational points of  $A_d$ .

The first two properties will follow from the statement that for every  $h \in \Lambda$ :

$$d^*(B \cap (B + h)) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})).$$

First, notice that for every  $h \in \Lambda$  the set  $U_o \cap \sigma_h(U_o)$  is Jordan measurable. By unique ergodicity of  $X_B$ , for every Følner sequence  $(F_k)$  in  $\Lambda$  and any  $h \in \Lambda$  we have

$$\frac{1}{|F_k|} \sum_{h \in F_k} \frac{|B_o \cap (B_o + h) \cap F_k|}{|F_k|} \rightarrow \int_X 1_{U_o \cap \sigma_h(U_o)}(x) d\eta(x) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})), \text{ as } k \rightarrow \infty.$$

The latter will imply that for every  $h \in \Lambda$

$$\eta(\tilde{B} \cap \sigma_h(\tilde{B})) \geq d^*(B \cap (B + h)).$$

On the other hand, for any Følner sequence  $(F_k)$  which lies inside the thick set  $T$  we will have for every  $h \in \Lambda$  by a similar argument as before

$$\frac{1}{|F_k|} \sum_{h \in F_k} \frac{|B \cap (B + h) \cap F_k|}{|F_k|} \rightarrow \int_X 1_{U_o \cap \sigma_h(U_o)}(x) d\eta(x) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})), \text{ as } k \rightarrow \infty.$$

This establishes that for every  $h \in \Lambda$ :

$$d^*(B \cap (B + h)) = \eta(\tilde{B} \cap \sigma_h(\tilde{B})).$$

It remains to prove that the spectral measure corresponding to  $1_{\tilde{B}}$  and the system  $X_B$  has no atoms at non-zero rational points of  $A_d$ . We will be abusing the notation and will also use  $T$  to denote the Koopman operator on  $L^2(X)$  corresponding to  $\sigma$ . Let us list two important properties of the system  $X_B$ :

- (1)  $X_B$  is *totally ergodic*, i.e., every subgroup  $H < \Lambda$  of a finite index acts ergodically on  $X_B$ .
- (2) For  $f \geq 0, f \in L^2(X)$ , the spectral measure  $\mu_f$  of  $f$  defined by

$$\widehat{\mu_f}(h) := \int_{A_d} \exp(2\pi h \cdot x) d\mu_f(x) = \langle f, T_h f \rangle$$

is non-negative.

The first property follows from Lemma 4.2, while the second property is standard fact, see [7]. To prove Lemma 4.2 we will need the following result.

**Lemma 4.1.** *Let  $H < \Lambda$  be a subgroup of a finite index. Then for every point  $x \in X$ , the  $H$ -orbit of  $x$ , i.e.,  $\{\sigma_h(x) \mid h \in H\}$ , is dense in  $X$ .*

*Proof.* Lemma follows from two facts that utilise the connectivity of  $X$ , and Baire Category theorem:

- For every subgroup  $H < \Lambda$  the closed subgroup  $\overline{\tau(H)} < X$  nowhere dense.
- Finite union of nowhere dense sets cannot cover  $X$ .

□



**Lemma 4.2.** *Let  $H < \Lambda$  be a subgroup of a finite index. The restriction of the  $\Lambda$ -action of  $X$  to  $H$  is uniquely ergodic.*

*Proof.* It follows from Lemma 4.1 that any  $H$ -invariant Borel probability measure on  $X$  is also  $X$ -invariant. The uniqueness of the Haar normalised measure on  $X$  implies the statement of the lemma.  $\square$

Let  $f \in L^2(X)$ , then by the ergodicity of  $X_B$  (property (1)) it follows that for any Følner sequence  $(F_k)_{k \geq 1}$  of finite sets in  $\Lambda$  we have

$$\frac{1}{|F_k|} \sum_{h \in F_k} \langle f, T_h f \rangle \rightarrow |\langle f, 1 \rangle|^2, \text{ as } k \rightarrow \infty.$$

On the other hand, it is a standard identity:

$$\frac{1}{|F_k|} \sum_{h \in F_k} \exp(2\pi i h \cdot x) \rightarrow \begin{cases} 1, & x = o_{A_d} \\ 0, & x \neq o_{A_d}. \end{cases}$$

It follows from Lebesgue's dominated convergence theorem that

$$(4) \quad |\langle f, 1 \rangle|^2 = \mu_f(o_{A_d}).$$

Let  $x_0 \in A_d$  be a non-zero rational point with the least common denominator equal to  $q$ . Then the stabiliser of  $x_0$  in  $\Lambda$  is  $H_{x_0} = q\Lambda$ . Using the ergodicity of  $H_{x_0}$  action on  $X_B$  (property (1)), we obtain

$$\frac{1}{|F_k|} \sum_{h \in F_k} \langle f, T_{qh} f \rangle \rightarrow |\langle f, 1 \rangle|^2, \text{ as } k \rightarrow \infty.$$

On the other hand, we have

$$\frac{1}{|F_k|} \sum_{h \in F_k} \exp(2\pi i h \cdot (qx)) \rightarrow \begin{cases} 1, & qx = o_{A_d} \\ 0, & qx \neq o_{A_d}. \end{cases}$$

Therefore, by Lebesgue's dominated convergence theorem we obtain

$$(5) \quad |\langle f, 1 \rangle|^2 = \sum_{qx = o_{A_d}} \mu_f(\{x\}).$$

If we know in addition that  $f \geq 0$ , then by property (2), the spectral measure  $\mu_f$  is non-negative. Therefore, by use of equations (4) and (5) we get that for all non-zero points  $x \in A_d$  with  $qx = o_{A_d}$  we have

$$\mu_f(\{x\}) = 0.$$

In particular, we have that  $\mu_f(\{x_0\}) = 0$ . This finishes the proof of Proposition 2.3, if we choose  $f = 1_{\tilde{B}}$ .

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