CASIMIR ELEMENTS FOR CERTAIN POLYNOMIAL CURRENT LIE ALGEBRAS

Alexander Molev

Centre for Mathematics and its Applications
School of Mathematical Sciences
Australian National University
Canberra, ACT 0200, Australia
e-mail: molev@pell.anu.edu.au

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Abstract
We consider the polynomial current Lie algebra $\mathfrak{gl}(n)[x]$ corresponding to the general linear Lie algebra $\mathfrak{gl}(n)$, and its factor-algebra $\mathfrak{g}_m$ by the ideal $\sum_{k\geq m}\mathfrak{gl}(n)x^k$. We construct two families of algebraically independent generators of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_m)$ by using the quantum determinant and the quantum contraction for the Yangian of level m.

0. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Denote by φ the canonical isomorphism $\varphi : \mathrm{S}(\mathfrak{g}) \to \operatorname{gr} \mathrm{U}(\mathfrak{g})$ of the symmetric algebra $\mathrm{S}(\mathfrak{g})$ to the graded algebra $\operatorname{gr} \mathrm{U}(\mathfrak{g})$ associated with the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. The restriction of φ to the subalgebra $\mathrm{I}(\mathfrak{g})$ of $\mathfrak{g}$-invariants in $\mathrm{S}(\mathfrak{g})$ yields an isomorphism

$$\varphi : \mathrm{I}(\mathfrak{g}) \to \operatorname{gr} \mathrm{Z}(\mathfrak{g}), \tag{0.1}$$

where $\mathrm{Z}(\mathfrak{g})$ denotes the center of $\mathrm{U}(\mathfrak{g})$.

If the Lie algebra $\mathfrak{g}$ is reductive, each of the algebras $\mathrm{I}(\mathfrak{g})$ and $\mathrm{Z}(\mathfrak{g})$ admits a family of algebraically independent generators (see, e.g., Dixmier [Di], Ch. 7.3, 7.4).

For some non-reductive Lie algebras $\mathfrak{g}$ an analogous property still takes place. A class of such Lie algebras was investigated by Rais and Tauvel in [RT]. In that paper one considers the polynomial current Lie algebra $\mathfrak{g}[x] = \mathfrak{g} \otimes \mathbb{C}[x]$ corresponding to a semi-simple complex Lie algebra $\mathfrak{g}$, and, given a positive integer m, one defines the factor-algebra $\mathfrak{g}_m$ of $\mathfrak{g}[x]$ by the ideal $\sum\limits_{k \geq m} \mathfrak{g} x^k$. One of the main results of [RT] is a construction of a family of algebraically independent generators of the algebra $\mathrm{I}(\mathfrak{g}_m)$ of $\mathfrak{g}_m$-invariants in $\mathrm{S}(\mathfrak{g}_m)$.

In the present paper we study the Lie algebra $\mathfrak{g}_m$ corresponding to the complex reductive Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$, that is, $\mathfrak{g}_m$ is the factor-algebra of $\mathfrak{gl}(n)[x]$ by the ideal $I_m = \sum\limits_{k \geq m} \mathfrak{gl}(n) x^k$. (Note that the construction of [RT] can be easily transfered to this case as well.)

Our aim is to construct families of algebraically independent generators of the center $\mathrm{Z}(\mathfrak{g}_m)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_m)$ (that is, 'to quantize' the construction of [RT]). We give explicit expressions for the generators in terms of the basis elements of $\mathfrak{g}_m$. Using isomorphism (0.1) we thus obtain a family of algebraically independent generators of the algebra $\mathrm{I}(\mathfrak{g}_m)$.

The main results are formulated in Section 1. The proofs are based on some properties of the algebra $\mathrm{Y}_m(n) = \mathrm{Y}_m(\mathfrak{gl}(n))$ called the *Yangian of level* m for the Lie algebra $\mathfrak{gl}(n)$ (see [C], [Dr]).

First, we prove a Poincaré–Birkhoff–Witt-type theorem for the algebra $\mathrm{Y}_m(n)$. Then we show that $\mathrm{Y}_m(n)$ admits a filtration such that the corresponding graded algebra is isomorphic to $\mathrm{U}(\mathfrak{g}_m)$.

Further, we use the fact (see, e.g., [MNO]) that the coefficients of both the *quantum determinant* $\operatorname{qdet} \mathcal{T}(u) \in \mathrm{Y}_m(n)[u]$ and the *quantum contraction* $z(u) \in \mathrm{Y}_m(n)[u]$ belong to the center of the algebra $\mathrm{Y}_m(n)$. So, taking the images of these coefficients in the graded algebra $\operatorname{gr} \mathrm{Y}_m(n) \simeq \mathrm{U}(\mathfrak{g}_m)$ we get two families of central elements in $\mathrm{U}(\mathfrak{g}_m)$.

Finally, using the results of [G] and [RT] we apply an analogue of the Harish-Chandra homomorphism for the algebra $\mathrm{U}(\mathfrak{g}_m)$ to prove that the images of the coefficients of the quantum determinant are algebraically independent and generate the center of $\mathrm{U}(\mathfrak{g}_m)$. To prove this property for the images of the coefficients of the quantum contraction we apply the *quantum Liouville formula* [MNO].

As a corollary, we obtain that the coefficients of the quantum determinant, as well as those of the quantum contraction, are algebraically independent generators of the center of $\mathrm{Y}_m(n)$.

This work can be regarded as a generalization of [M] where an analogous approach was applied to the construction of Casimir elements and computing their Harish-Chandra images for the classical Lie algebras of series A–D.

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1. Construction of Casimir elements

Denote by E_{ij}, $(i,j=1,\dots,n)$ the standard basis of the general linear Lie algebra $\mathfrak{gl}(n)$. Then the elements $E_{ij}^{(k)}:=E_{ij}x^k$ with $1\le i,j\le n$ and $0\le k\le m-1$ form a basis of the Lie algebra $\mathfrak{g}_m=\mathfrak{gl}(n)[x]/I_m$; see Introduction. Let u be a formal variable. To construct the first family of Casimir elements, introduce the following $\mathrm{U}(\mathfrak{g}_m)$-valued polynomials in u:

$$E_{ij}(u):=\delta_{ij}u^m+(E_{ij}^{(0)}-m(j-1)\delta_{ij})u^{m-1}+E_{ij}^{(1)}u^{m-2}+\dots+E_{ij}^{(m-1)},$$

where $i,j=1,\dots,n$, and define the "determinant" of the noncommutative matrix $E(u)=(E_{ij}(u))_{i,j=1}^n$ by the formula

$$\det E(u):=\sum_{p\in\mathfrak{S}_n}\mathrm{sgn}(p)E_{p(1),1}(u)\cdots E_{p(n),n}(u), \tag{1.1}$$

where $\mathfrak{S}_n$ denotes the symmetric group. This is a polynomial in u with coefficients in $\mathrm{U}(\mathfrak{g}_m)$:

$$\det E(u)=u^{mn}+\sum_{k=1}^{mn}\widetilde{\zeta}_k u^{mn-k},\qquad \widetilde{\zeta}_k\in\mathrm{U}(\mathfrak{g}_m). \tag{1.2}$$

Note that setting $\deg E_{ij}^{(p)}=p$ defines a grading on $\mathrm{U}(\mathfrak{g}_m)$. We let ζ_k denote the component of the highest degree of the element $\widetilde{\zeta}_k$, $k\in\{1,\dots,mn\}$. To write explicit formulas for the ζ_k, denote

$$\begin{aligned} F_{ij}^{(r)}&=E_{ij}^{(r-1)}\quad\text{for}\quad 1<r\le m,\quad\text{and}\\ F_{ij}^{(1)}&=E_{ij}^{(0)}-m(j-1)\delta_{ij}.\end{aligned}$$

Define the numbers $r\in\{1,\dots,m\}$ and $s\in\{1,\dots,n\}$ by the formula $k=m(s-1)+r$. Then

$$\zeta_k=\sum_{\substack{i_1<\dots<i_s\\ j_1+\dots+j_s=k}}\ \sum_{\sigma\in\mathfrak{S}_s}\mathrm{sgn}(\sigma)F_{i_{\sigma(1)}i_1}^{(j_1)}\cdots F_{i_{\sigma(s)}i_s}^{(j_s)}. \tag{1.3}$$

Note that in the case $m=1$ the ζ_k coincide with the well-known *Capelli elements* in $\mathrm{Z}(\mathfrak{gl}(n))$ (see, e.g., [HU]).

Let us now describe the second construction. For any $0\le p\le m-1$ introduce the matrix $E^{(p)}=(E_{ij}^{(p)})_{i,j=1}^n$. Represent $k\ge 1$ in the form $k=m(s-1)+r$ with $s\ge 1$ and $1\le r\le m$ and define the element θ_k by the formula

$$\theta_k=\sum_{r_1+\dots+r_s=k}r_s\cdot\mathrm{tr}\,E^{(r_1-1)}\cdots E^{(r_s-1)}, \tag{1.4}$$

where each r_i runs over the set $\{1,\dots,m\}$.

In particular, in the case $m=1$ we have $\theta_k=\operatorname{tr} E^k$, where $E=E^{(0)}$. These are the well-known central elements of $\mathrm{U}(\mathfrak{gl}(n))$ (see, e.g., [PP]).

The following is our main result.

Theorem 1.1. *All the elements ζ_k with $k=1,\dots,mn$ and θ_k with $k=1,2,\dots$ belong to the center $\mathrm{Z}(\mathfrak{g}_m)$ of the algebra $\mathrm{U}(\mathfrak{g}_m)$. Moreover, each family $\{\zeta_1,\dots,\zeta_{mn}\}$ and $\{\theta_1,\dots,\theta_{mn}\}$ is algebraically independent and generates $\mathrm{Z}(\mathfrak{g}_m)$.*

The theorem will be proved in Section 3.

Example 1.2. Let $m=n=2$. We have

$$\det E(u)=u^4+\Big(E^{(0)}_{11}+E^{(0)}_{22}-2\Big)\,u^3+\Big(E^{(1)}_{11}+E^{(1)}_{22}+E^{(0)}_{11}(E^{(0)}_{22}-2)-E^{(0)}_{21}E^{(0)}_{12}\Big)\,u^2$$
$$+\Big(E^{(0)}_{11}E^{(1)}_{22}+E^{(1)}_{11}(E^{(0)}_{22}-2)-E^{(0)}_{21}E^{(1)}_{12}-E^{(1)}_{21}E^{(0)}_{12}\Big)\,u+E^{(1)}_{11}E^{(1)}_{22}-E^{(1)}_{21}E^{(1)}_{12}.$$

Hence, the first family looks as follows:

$$\begin{aligned}
\zeta_1&=E^{(0)}_{11}+E^{(0)}_{22}-2,\\
\zeta_2&=E^{(1)}_{11}+E^{(1)}_{22},\\
\zeta_3&=E^{(0)}_{11}E^{(1)}_{22}+E^{(1)}_{11}(E^{(0)}_{22}-2)-E^{(0)}_{21}E^{(1)}_{12}-E^{(1)}_{21}E^{(0)}_{12},\\
\zeta_4&=E^{(1)}_{11}E^{(1)}_{22}-E^{(1)}_{21}E^{(1)}_{12}.
\end{aligned}$$

For the second family we have:

$$\begin{aligned}
\theta_1&=E^{(0)}_{11}+E^{(0)}_{22},\\
\theta_2&=2\left(E^{(1)}_{11}+E^{(1)}_{22}\right),\\
\theta_3&=2\left(E^{(0)}_{11}E^{(1)}_{11}+E^{(0)}_{22}E^{(1)}_{22}+E^{(0)}_{21}E^{(1)}_{12}+E^{(0)}_{12}E^{(1)}_{21}\right)\\
&\qquad+E^{(1)}_{11}E^{(0)}_{11}+E^{(1)}_{22}E^{(0)}_{22}+E^{(1)}_{21}E^{(0)}_{12}+E^{(1)}_{12}E^{(0)}_{21},\\
\theta_4&=2\left(E^{(1)}_{11}E^{(1)}_{11}+E^{(1)}_{22}E^{(1)}_{22}+E^{(1)}_{21}E^{(1)}_{12}+E^{(1)}_{12}E^{(1)}_{21}\right).
\end{aligned}$$

2. Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_m(n)$

First we define the *Yangian* $\mathrm{Y}(n)=\mathrm{Y}(\mathfrak{gl}(n))$ for the Lie algebra $\mathfrak{gl}(n)$ (see, e.g., [Dr], [MNO]). It is the associative algebra with countably many generators $t^{(1)}_{ij}$, $t^{(2)}_{ij},\dots$ where $1\le i,j\le n$, and defining relations

$$[t^{(r+1)}_{ij},t^{(s)}_{kl}]-[t^{(r)}_{ij},t^{(s+1)}_{kl}]=t^{(r)}_{kj}t^{(s)}_{il}-t^{(s)}_{kj}t^{(r)}_{il}, \tag{2.1}$$

where $r,s=0,1,2,\dots$ and $t^{(0)}_{ij}:=\delta_{ij}$.

The relations (2.1) can be rewritten in the following equivalent form:

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{p=0}^{\min(r,s)-1} \Big(t_{kj}^{(p)} t_{il}^{(r+s-1-p)} - t_{kj}^{(r+s-1-p)} t_{il}^{(p)} \Big), \tag{2.2}$$

where $r, s = 1, 2, \dots$ and $1 \le i, j, k, l \le n$.

The algebra $\mathrm{Y}_m(n)$ is defined as the factor-algebra of $\mathrm{Y}(n)$ by the ideal J_m generated by all the elements $t_{ij}^{(r)}$ with $1 \le i, j \le n$ and $r > m$. Following [C], we call it the *Yangian of level* m for the Lie algebra $\mathfrak{gl}(n)$.

Let us denote by $\tau_{ij}^{(r)}$, $(1 \le r \le m)$ the image of the generator $t_{ij}^{(r)} \in \mathrm{Y}(n)$ in $\mathrm{Y}_m(n)$. Then the $\tau_{ij}^{(r)}$ satisfy the same relations (2.1) and (2.2), where one replaces $t_{ij}^{(r)}$ with $\tau_{ij}^{(r)}$ or 0, depending on whether $1 \le r \le m$ or $r > m$.

Equivalently, $\mathrm{Y}_m(n)$ can be defined as the algebra with the generators $\tau_{ij}^{(1)}, \dots, \tau_{ij}^{(m)}$, $1 \le i, j \le n$, subject to the relations

$$[\tau_{ij}(u), \tau_{kl}(v)] = \frac{1}{u-v} (\tau_{kj}(u)\tau_{il}(v) - \tau_{kj}(v)\tau_{il}(u)),$$

where u is a formal variable and

$$\tau_{ij}(u) := \delta_{ij} u^m + \sum_{r=1}^{m} \tau_{ij}^{(r)} u^{m-r} \in \mathrm{Y}_m(n)[u]. \tag{2.3}$$

We shall need the following analogue of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_m(n)$.

Theorem 2.1. *Given an arbitrary linear ordering on the set of the generators* $\tau_{ij}^{(r)}$, *any element of the algebra* $\mathrm{Y}_m(n)$ *is uniquely written as a linear combination of ordered monomials in the generators.*

Proof. By definition, $\mathrm{Y}_m(n) \simeq \mathrm{Y}(n)/J_m$. Let us first consider a particular linear ordering defined on the generators $t_{ij}^{(r)}$ by setting $t_{ij}^{(r)} \le t_{kl}^{(s)}$ if $(r, i, j) \le (s, k, l)$ in the lexicographical ordering. By the Poincaré–Birkhoff–Witt theorem for the Yangian $\mathrm{Y}(n)$ (see, e.g., [MNO], Corollary 1.23), the ordered monomials

$$t_{i_1 j_1}^{(r_1)} \cdots t_{i_q j_q}^{(r_q)}, \qquad t_{i_1 j_1}^{(r_1)} \le \cdots \le t_{i_q j_q}^{(r_q)}, \tag{2.4}$$

form a basis in $\mathrm{Y}(n)$. We have to show that these monomials with the additional condition $r_q \le m$ form a basis of $\mathrm{Y}(n)$ modulo the ideal J_m (note that $r_1 \le \cdots \le r_q$ by definition of the ordering). Since any element of J_m is uniquely represented as a linear combination of monomials (2.4), it suffices to prove that for each monomial occuring in this representation the inequality $r_q > m$ holds.

For each pair of generators $t_{ij}^{(r)} \ge t_{kl}^{(s)}$ with $r > m$ one has

$$\begin{aligned} t_{ij}^{(r)} t_{kl}^{(s)} = t_{kl}^{(s)} t_{ij}^{(r)} &+ \text{a linear combination of } t_{ab}^{(p)} t_{cd}^{(q)} \\ &\text{with } t_{ab}^{(p)} < t_{cd}^{(q)}, \quad p + q < r + s, \quad \text{and} \quad q > m. \end{aligned} \tag{2.5}$$

Indeed, by (2.2),

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{p=0}^{s-1} \Big(t_{kj}^{(p)} t_{il}^{(r+s-1-p)} - t_{kj}^{(r+s-1-p)} t_{il}^{(p)} \Big),$$

and (2.5) follows by induction on $r+s$.

The ideal J_m is spanned by monomials of the form

$$t_{k_1 l_1}^{(s_1)} \cdots t_{k_p l_p}^{(s_p)}, \qquad p \geq 1, \tag{2.6}$$

such that at least one of the indices $s_1, \dots, s_p$ is greater than m. Choose a maximal generator $t_{k_i l_i}^{(s_i)}$ in the monomial (2.6) (the inequality $s_i > m$ should then hold) and move it to the extreme right position, permuting with the other generators using (2.5). Repeating this procedure and applying an obvious induction on $s_1 + \dots + s_p$ we get a representation of the monomial (2.6) as a linear combination of the ordered monomials (2.4) with $r_q > m$, which proves the claim for the chosen ordering.

Finally, by (2.2), any two generators $\tau_{ij}^{(r)}$ and $\tau_{kl}^{(s)}$ commute modulo products $\tau_{ab}^{(p)} \tau_{cd}^{(q)}$ with $p+q < r+s$. This reduces the proof in the general case (with an arbitrary ordering on the generators $\tau_{ij}^{(r)}$) to the particular case considered above.

A filtration on the algebra $\mathrm{Y}_m(n)$ can be defined by setting

$$\deg \tau_{ij}^{(r)} = r - 1. \tag{2.7}$$

Denote by $\operatorname{gr} \mathrm{Y}_m(n)$ the corresponding graded algebra.

Corollary 2.2 (cf. [MNO], Theorem 1.26)**.** *The algebra* $\operatorname{gr} \mathrm{Y}_m(n)$ *is isomorphic to the universal enveloping algebra* $\mathrm{U}(\mathfrak{g}_m)$.

Proof. Denote by $\overline{\tau}_{ij}^{(r)}$ the image of the generator $\tau_{ij}^{(r)}$ in the $(r-1)$th component of $\operatorname{gr} \mathrm{Y}_m(n)$. Then, comparing the degrees of the elements on the left and right hand sides of (2.2) we obtain that the elements $\overline{\tau}_{ij}^{(r)}$ satisfy the relations

$$[\overline{\tau}_{ij}^{(r)}, \overline{\tau}_{kl}^{(s)}] = \delta_{kj} \overline{\tau}_{il}^{(r+s-1)} - \delta_{il} \overline{\tau}_{kj}^{(r+s-1)},$$

where $\overline{\tau}_{ij}^{(r)} := 0$ for $r > m$. These are exactly the commutation relations for the Lie algebra $\mathfrak{g}_m$ in the basis $E_{ij}^{(r-1)}$. So, the mapping $E_{ij}^{(r-1)} \mapsto \overline{\tau}_{ij}^{(r)}$ defines an algebra homomorphism

$$\mathrm{U}(\mathfrak{g}_m) \to \operatorname{gr} \mathrm{Y}_m(n). \tag{2.8}$$

By Theorem 2.1 and the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{U}(\mathfrak{g}_m)$, its kernel is trivial.

3. Proof of Theorem 1.1

Let us combine the polynomials $\tau_{ij}(u) \in \mathrm{Y}_m(n)[u]$ defined by (2.3) into the matrix $\mathcal{T}(u) = (\tau_{ij}(u))_{i,j=1}^n$. The polynomial

$$\operatorname{qdet} \mathcal{T}(u) := \sum_{p\in\mathfrak{S}_n} \operatorname{sgn}(p)\tau_{p(1),1}(u)\cdots\tau_{p(n),n}(u-n+1)$$

is called the *quantum determinant* of the matrix $\mathcal{T}(u)$ (see [C], [Dr]). Write

$$\operatorname{qdet} \mathcal{T}(u) = u^{mn} + \sum_{k=1}^{mn} d_k u^{mn-k}, \qquad d_k \in \mathrm{Y}_m(n). \tag{3.1}$$

All the coefficients d_k belong to the center of the algebra $\mathrm{Y}_m(n)$. This follows immediately from the corresponding property of the quantum determinant for the Yangian $\mathrm{Y}(n)$ (see, e.g., [MNO], Theorem 2.10). Therefore, the images $\overline{d}_k$ of the elements d_k in the graded algebra $\operatorname{gr}\mathrm{Y}_m(n)$ also belong to its center.

To prove the centrality of the elements ζ_k it suffices to show that ζ_k corresponds to $\overline{d}_k$ under the isomorphism (2.8). This follows from the fact that the image of the polynomial

$$\tau_{ij}(u-j+1) = \delta_{ij}(u-j+1)^m + \sum_{k=1}^{m} \tau_{ij}^{(k)}(u-j+1)^{m-k}$$

in $\operatorname{gr}\mathrm{Y}_m(n)[u]$ is

$$\delta_{ij}u^m + (\overline{\tau}_{ij}^{(1)} - m(j-1)\delta_{ij})u^{m-1} + \overline{\tau}_{ij}^{(2)}u^{m-2} + \cdots + \overline{\tau}_{ij}^{(m)}$$

which corresponds to $E_{ij}(u)$ under the isomorphism (2.8).

Our next step is the proof of the algebraic independence of the elements ζ_k.

Let $\mathfrak{h}$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ denote the subalgebras of $\mathfrak{gl}(n)$ consisting of all diagonal, upper triangular and lower triangular matrices, respectively. For any subspace $\mathfrak{a}$ of $\mathfrak{gl}(n)$ we put

$$\mathfrak{a}_m = \mathfrak{a} \oplus \mathfrak{a}x \oplus \cdots \oplus \mathfrak{a}x^{m-1}.$$

Let $\mathrm{U}(\mathfrak{g}_m)_0$ denote the centralizer of $\mathfrak{h}$ in $\mathrm{U}(\mathfrak{g}_m)$. Set

$$\mathrm{L} = \mathrm{U}(\mathfrak{g}_m)\mathfrak{n}_m^+ \cap \mathrm{U}(\mathfrak{g}_m)_0.$$

One can derive the following facts from the Poincaré–Birkhoff–Witt theorem for $\mathrm{U}(\mathfrak{g}_m)$ (see [G], Proposition 4.2):

(i) $\mathrm{L} = \mathfrak{n}_m^- \,\mathrm{U}(\mathfrak{g}_m) \cap \mathrm{U}(\mathfrak{g}_m)_0$;

(ii) L is a two-sided ideal in $\mathrm{U}(\mathfrak{g}_m)_0$;

(iii) $\mathrm{U}(\mathfrak{g}_m)_0 = \mathrm{U}(\mathfrak{h}_m) \oplus \mathrm{L}$.

Hence, the projection $\chi : \mathrm{U}(\mathfrak{g}_m)_0 \to \mathrm{U}(\mathfrak{h}_m)$ with the kernel L is an algebra homomorphism which is an analogue of the Harish-Chandra homomorphism; cf. [Di], Ch. 7.4.

Note that the elements $\zeta_1, \dots, \zeta_{mn}$ belong to $\mathrm{U}(\mathfrak{g}_m)_0$. Their algebraic independence will follow from the algebraic independence of their images in $\mathrm{U}(\mathfrak{h}_m)$ under the homomorphism χ. Set

$$\lambda_i^{(r)} = E_{ii}^{(r-1)} \quad \text{for} \quad 1 < r \le m, \quad \text{and}$$
$$\lambda_i^{(1)} = E_{ii}^{(0)} - m(i-1),$$

where $i = 1, \dots, n$. Let us parametrize the indices k by the pairs (r, s) where $r \in \{1, \dots, m\}$, $s \in \{1, \dots, n\}$ and $k = m(s-1) + r$. Then we get from (1.3) that

$$\chi(\zeta_k) = \sum_{\substack{i_1 < \dots < i_s \\ j_1 + \dots + j_s = k}} \lambda_{i_1}^{(j_1)} \cdots \lambda_{i_s}^{(j_s)}. \tag{3.2}$$

Let us denote the polynomial (3.2) by $\Lambda_s^{(r)}$ and prove that the differentials $d\Lambda_s^{(r)}$ are linearly independent.

Note that $\Lambda_s^{(m)}$ is the elementary symmetric polynomial of degree s in the variables $\lambda_1^{(m)}, \dots, \lambda_n^{(m)}$. Therefore, the matrix $A = (a_{st})_{s,t=1}^n$ defined by

$$d\Lambda_s^{(m)} = a_{s1}\, d\lambda_1^{(m)} + \dots + a_{sn}\, d\lambda_n^{(m)} \tag{3.3}$$

is non-degenerated. Further, for $1 \le r < m$ we have

$$\begin{aligned} d\Lambda_s^{(r)} &= a_{s1}\, d\lambda_1^{(r)} + \dots + a_{sn}\, d\lambda_n^{(r)} \\ &+ \text{a linear combination of} \quad d\lambda_q^{(p)} \quad \text{with} \quad p > r. \end{aligned} \tag{3.4}$$

Let us combine the coefficients of $d\lambda_s^{(r)}$ in the expressions for the differentials $d\Lambda_s^{(r)}$ into a matrix and arrange its rows and columns in accordance with the lexicographical ordering on the pairs (r, s). Then (3.3) and (3.4) imply that this matrix is block-triangular with m identical diagonal $n \times n$-blocks equal to the matrix A. This proves the linear independence of the differentials $d\Lambda_s^{(r)}$ and hence, the algebraic independence of the polynomials $\Lambda_s^{(r)}$.

To complete the proof of Theorem 1.1 for the family $\{\zeta_k\}$ we need to show that any central element in the algebra $\mathrm{U}(\mathfrak{g}_m)$ can be expressed as a polynomial in the elements ζ_k.

It follows from [RT], Théorème 4.5, that the algebra $\mathrm{I}(\mathfrak{g}_m)$ admits a system of algebraically independent homogeneous generators $P_s^{(r)}$ with $1 \le r \le m$ and $1 \le s \le n$ such that $P_s^{(r)}$ has the degree s. Thus, due to the isomorphism $\operatorname{gr} \mathrm{Z}(\mathfrak{g}_m) \simeq \mathrm{I}(\mathfrak{g}_m)$, it suffices to verify that the elements ζ_k have the same degrees as the generators $P_s^{(r)}$ (in the sense of the canonical filtration of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_m)$). However, it is clear from (1.3) that ζ_k with $k = m(s-1) + r$ is an element of degree s, which completes the proof.

The above argument implies the following corollaries.

Corollary 3.1. *The coefficients d_k, $k = 1, \dots, mn$ of the quantum determinant* $\operatorname{qdet} \mathcal{T}(u)$, *defined by* (3.1) *form a family of algebraically independent generators of the center of the algebra* $\mathrm{Y}_m(n)$.

Corollary 3.2. *The center* $\mathrm{Z}(\mathfrak{g}_m)$ *of the algebra* $\mathrm{U}(\mathfrak{g}_m)$ *is isomorphic to the subalgebra in* $\mathbb{C}[\lambda_s^{(r)}]$, $(1 \le r \le m,\ 1 \le s \le n)$, *generated by the polynomials* $\Lambda_s^{(r)}$.

This isomorphism can be regarded as an analogue of the Harish-Chandra isomorphism for the algebra $\mathrm{U}(\mathfrak{g}_m)$ (cf. [Di], Ch. 7.4).

Let us turn now to the second family $\{\theta_k\}$.

Define the *quantum contraction* $z(u)$ for $\mathrm{Y}_m(n)$ by the formula (cf. [MNO])

$$z(u) = \frac{1}{n}\,\mathrm{tr}\,\mathcal{T}^{-1}(u-n+1)\mathcal{T}(u+1). \tag{3.5}$$

We regard it as a formal series in u^{-1} with coefficients from $\mathrm{Y}_m(n)$:

$$z(u) = 1 + z_1 u^{-1} + z_2 u^{-2} + \cdots, \qquad z_p \in \mathrm{Y}_m(n).$$

The polynomial $\mathrm{qdet}\,\mathcal{T}(u)$ and the series $z(u)$ are related by means of the *quantum Liouville formula* (see [MNO], Theorem 5.7):

$$z(u) = \frac{\mathrm{qdet}\,\mathcal{T}(u+1)}{\mathrm{qdet}\,\mathcal{T}(u)}. \tag{3.6}$$

So, the elements z_p are central in $\mathrm{Y}_m(n)$. The identity $z(u)\,\mathrm{qdet}\,\mathcal{T}(u) = \mathrm{qdet}\,\mathcal{T}(u+1)$ implies that for $k \ge 1$

$$z_k + z_{k-1}d_1 + \cdots + z_1 d_{k-1} + d_k = \sum_{i=0}^{k} \binom{mn-k+i}{i} d_{k-i}, \tag{3.7}$$

where $d_0 := 1$ and $d_k := 0$ for $k > mn$. Using induction on k we deduce from (3.7) that for $k = 1, \dots, mn$

$$z_{k+1} = -k\, d_k + \text{ a polynomial in } d_1, \dots, d_{k-1}. \tag{3.8}$$

Moreover, examining the degrees of the elements in (3.7) we obtain that the image of (3.8) in the graded algebra $\mathrm{gr}\,\mathrm{Y}_m(n)$ has the form

$$\overline{z}_{k+1} = -k\, \overline{d}_k + \text{ a polynomial in } \overline{d}_1, \dots, \overline{d}_{k-1}. \tag{3.9}$$

Now, Corollary 3.1 and (3.8) imply that $z_2, \dots, z_{mn+1}$ are algebraically independent generators of the center of the algebra $\mathrm{Y}_m(n)$, while the proof of the first part of Theorem 1.1 and (3.9) imply that $\overline{z}_2, \dots, \overline{z}_{mn+1}$ are those of the center of $\mathrm{gr}\,\mathrm{Y}_m(n)$.

To complete the proof of Theorem 1.1 let us compute the images of $\overline{z}_2, \dots, \overline{z}_{mn+1}$ under the isomorphism $\mathrm{gr}\,\mathrm{Y}_m(n) \simeq \mathrm{U}(\mathfrak{g}_m)$.

Denote

$$\mathcal{T}_0(u) = u^{-m}\mathcal{T}(u) = 1 + \mathcal{T}^{(1)}u^{-1} + \cdots + \mathcal{T}^{(m)}u^{-m},$$

where $\mathcal{T}^{(k)}$ is the matrix $(\tau_{ij}^{(k)})_{i,j=1}^n$, and consider the series

$$\frac{1}{n}\,\mathrm{tr}\,\mathcal{T}_0^{-1}(u)\mathcal{T}_0(u+n) =: y(u) = 1 + y_1 u^{-1} + y_2 u^{-2} + \cdots .$$

We have

$$\mathcal{T}_0^{-1}(u) = 1 + \sum_{k=1}^{\infty} \Theta_k u^{-k}$$

with

$$\Theta_k = \sum_{l\geq 1} \sum_{i_1+\cdots+i_l=k} (-1)^l \mathcal{T}^{(i_1)} \cdots \mathcal{T}^{(i_l)},$$

and

$$\mathcal{T}_0(u+n) = 1 + \sum_{k=1}^{\infty} u^{-k} \sum_{p=1}^{k} (-n)^{k-p} \binom{k-1}{k-p} \mathcal{T}^{(p)},$$

where $\mathcal{T}^{(k)} := 0$ for $k > m$. The identity $\mathcal{T}_0^{-1}(u)\mathcal{T}_0(u) = 1$ implies the relations

$$\Theta_k + \Theta_{k-1}\mathcal{T}^{(1)} + \cdots + \mathcal{T}^{(k)} = 0, \qquad k \geq 1. \tag{3.10}$$

Let $\overline{\mathcal{T}}^{(k)} = (\overline{\tau}_{ij}^{(k)})_{i,j=1}^n$ denote the image of $\mathcal{T}^{(k)}$ in $\operatorname{gr} \mathrm{Y}_m(n)$. We find with the use of (3.10) that for $k = m(s-1)+r$ with $2 \leq r \leq m$ the image of y_{k+1} in $\operatorname{gr} \mathrm{Y}_m(n)$ is given by the formula

$$\overline{y}_{k+1} = (-1)^s \sum_{r_1+\cdots+r_s=k} r_s \cdot \operatorname{tr} \overline{\mathcal{T}}^{(r_1)} \cdots \overline{\mathcal{T}}^{(r_s)} \tag{3.11}$$

and for $k = m(s-1)+1$ by the formula

$$\overline{y}_{k+1} = (-1)^s \sum_{r_1+\cdots+r_s=k} r_s \cdot \operatorname{tr} \overline{\mathcal{T}}^{(r_1)} \cdots \overline{\mathcal{T}}^{(r_s)} + \mathrm{const} \cdot \operatorname{tr} \left(\overline{\mathcal{T}}^{(m)}\right)^{s-1}. \tag{3.12}$$

Finally, since

$$z(u) = \left(\frac{u+1}{u-n+1}\right)^m y(u-n+1),$$

the same formulas (3.11) and (3.12) hold for the $\overline{z}_{k+1}$ with a different value of 'const' in (3.12).

This means that the image of $\overline{z}_{k+1}$ in the algebra $\mathrm{U}(\mathfrak{g}_m)$ coincides with $(-1)^s\theta_k$ or $(-1)^s\theta_k + \mathrm{const} \cdot \theta_{k-1}$ for $k = m(s-1)+r$ with $2 \leq r \leq m$ or $r = 1$, respectively. This completes the proof.

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