Combinatorial bases for representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$

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Gelfand–Tsetlin bases for $\mathfrak{gl}_n$
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Finite-dimensional irreducible representations $L(\lambda)$ of $\mathfrak{gl}_n$ are in a one-to-one correspondence with $n$-tuples of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \quad \text{for} \quad i = 1, \ldots, n - 1.$$
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$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \quad \text{for} \quad i = 1, \ldots, n - 1.$$

$L(\lambda)$ contains a highest vector $\zeta \neq 0$ such that

$$E_{ii} \zeta = \lambda_i \zeta \quad \text{for} \quad i = 1, \ldots, n \quad \text{and}$$

$$E_{ij} \zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq n.$$
Suppose that $\lambda$ is a partition, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. 
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Depict it as a Young diagram.

Example. The diagram $\lambda = (5, 5, 3, 0, 0)$ is

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**Example.** The diagram $\lambda = (5, 5, 3, 0, 0)$ is

![Young diagram](image)

\[ \ell(\lambda) = 3 \]

The number of nonzero rows is the **length of $\lambda$**, denoted $\ell(\lambda)$. 
Given a diagram $\lambda$, a column-strict $\lambda$-tableau $T$ is obtained by filling in the boxes of $\lambda$ with the numbers $1, 2, \ldots, n$ in such a way that the entries weakly increase along the rows and strictly increase down the columns.
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Example. A column-strict $\lambda$-tableau for $\lambda = (5, 5, 3, 0, 0)$:

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 4 & 5 \\
4 & 5 & 5 & \\
\end{array}
\]
Theorem (Gelfand and Tsetlin, 1950). $L(\lambda)$ admits a basis $\zeta_T$ parameterized by all column-strict $\lambda$-tableaux $T$ such that the action of generators of $\mathfrak{gl}_n$ is given by the formulas

$$E_{ss} \zeta_T = \omega_s \zeta_T,$$

$$E_{s,s+1} \zeta_T = \sum_{T'} c_{TT'} \zeta_{T'},$$

$$E_{s+1,s} \zeta_T = \sum_{T'} d_{TT'} \zeta_{T'}.$$
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$$E_{s+1,s} \zeta_T = \sum_{T'} d_{TT'} \zeta_{T'}.$$

Here $\omega_s$ is the number of entries in $T$ equal to $s$, and the sums are taken over column-strict tableaux $T'$ obtained from $T$ respectively by replacing an entry $s + 1$ by $s$ and $s$ by $s + 1$. 
For any $1 \leq j \leq s \leq n$ denote by $\lambda_{sj}$ the number of entries in row $j$ which do not exceed $s$ and set

$$l_{sj} = \lambda_{sj} - j + 1.$$
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Then

$$c_{TT'} = -\frac{(l_{si} - l_{s+1,1}) \cdots (l_{si} - l_{s+1,1+s+1})}{(l_{si} - l_{s1}) \cdots \wedge \cdots (l_{si} - l_{ss})},$$

$$d_{TT'} = \frac{(l_{si} - l_{s-1,1}) \cdots (l_{si} - l_{s-1,1+s-1})}{(l_{si} - l_{s1}) \cdots \wedge \cdots (l_{si} - l_{ss})},$$

if the replacement occurs in row $i$. 
Equivalent parametrization of the basis vectors by the Gelfand–Tsetlin patterns:
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\[
\begin{array}{cccc}
\lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \\
\lambda_{n-1,1} & \cdots & \lambda_{n-1,n-1} \\
\lambda_{21} & \lambda_{22} \\
\lambda_{11} & \\
\end{array}
\]
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\begin{array}{cccc}
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\lambda_{n-1,1} & \cdots & \lambda_{n-1,n-1} \\
T & \cdots & \cdots & \cdots \\
\lambda_{21} & \lambda_{22} \\
\lambda_{11}
\end{array}
\]

The top row coincides with \( \lambda \) and the entries satisfy the betweenness conditions \( \lambda_{k,i} \geq \lambda_{k-1,i} \geq \lambda_{k,i+1} \).
Example. The column-strict tableau with \( \lambda = (5, 5, 3, 0, 0) \)

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 4 & 5 \\
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\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 4 & 5 \\
4 & 5 & 5 & \\
\end{array}
\]

corresponds to the pattern

\[
\begin{array}{cccc}
5 & 5 & 3 & 0 \\
5 & 3 & 1 & 0 \\
3 & 2 & 0 & \\
3 & 1 & \\
2 & \\
\end{array}
\]
Given $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_+^n$, consider the weight subspace

$$L(\lambda) \omega = \{ \eta \in L(\lambda) \mid E_{ss} \eta = \omega_s \eta \text{ for all } s \}.$$
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L(\lambda) \omega = \{ \eta \in L(\lambda) \mid E_{ss} \eta = \omega_s \eta \quad \text{for all} \quad s \}\.
\]

The character of \( L(\lambda) \) is the polynomial in variables \( x_1, \ldots, x_n \) defined by

\[
\text{ch} \ L(\lambda) = \sum_\omega \dim L(\lambda) \omega \ x_1^{\omega_1} \ldots x_n^{\omega_n}.
\]
Given $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_+^n$, consider the weight subspace

$$L(\lambda)_{\omega} = \{ \eta \in L(\lambda) \mid E_{ss}\eta = \omega_s\eta \text{ for all } s \}.$$ 

The character of $L(\lambda)$ is the polynomial in variables $x_1, \ldots, x_n$ defined by

$$\text{ch} \ L(\lambda) = \sum_{\omega} \dim L(\lambda)_{\omega} x_1^{\omega_1} \ldots x_n^{\omega_n}.$$ 

Corollary. $\text{ch} \ L(\lambda) = s_\lambda(x_1, \ldots, x_n)$, the Schur polynomial.
Lie superalgebra $\mathfrak{gl}_{m|n}$
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Basis elements of $\mathfrak{gl}_{m|n}$ are $E_{ij}$ with $1 \leq i, j \leq m + n$. 
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The $\mathbb{Z}_2$-degree (or parity) is given by

$$\deg(E_{ij}) = \bar{i} + \bar{j},$$

where $\bar{i} = 0$ for $1 \leq i \leq m$ and $\bar{i} = 1$ for $m + 1 \leq i \leq m + n$. 
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The commutation relations in $\mathfrak{gl}_{m|n}$ have the form

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}(-1)^{\bar{i}+\bar{j}}(\bar{k}+\bar{l)}),$$

where the square brackets denote the super-commutator.
The span of \( \{ E_{ij} \mid 1 \leq i, j \leq m \} \) is a Lie subalgebra isomorphic to \( \mathfrak{gl}_m \),
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the span of $\{E_{ij} \mid m + 1 \leq i, j \leq m + n\}$ is a Lie subalgebra of isomorphic to $\mathfrak{gl}_n$,

the Lie subalgebra of even elements of $\mathfrak{gl}_{m|n}$ is isomorphic to $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$. 
Finite-dimensional irreducible representations of $\mathfrak{g}_{m|n}$ are parameterized by their highest weights $\lambda$ of the form

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Finite-dimensional irreducible representations of $\mathfrak{gl}_{m|n}$ are parameterized by their highest weights $\lambda$ of the form $
abla \lambda = (\lambda_1, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{m+n})$, where

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where

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+, \quad \text{for} \quad i = 1, \ldots, m + n - 1, \quad i \neq m.$$ 

The corresponding representation $L(\lambda)$ contains a highest vector $\zeta \neq 0$ such that

$$E_{ii} \zeta = \lambda_i \zeta \quad \text{for} \quad i = 1, \ldots, m + n \quad \text{and}$$

$$E_{ij} \zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq m + n.$$
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These are the irreducible components of the representations

$$\mathbb{C}^{m|n} \otimes \ldots \otimes \mathbb{C}^{m|n}$$

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Covariant representations $L(\lambda)$

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They are distinguished by the conditions:

- all components $\lambda_1, \ldots, \lambda_{m+n}$ of $\lambda$ are nonnegative integers;
- the number $\ell$ of nonzero components among $\lambda_{m+1}, \ldots, \lambda_{m+n}$ is at most $\lambda_m$. 
To each highest weight $\lambda$ satisfying these conditions, associate the Young diagram $\Gamma_\lambda$ containing $\lambda_1 + \cdots + \lambda_{m+n}$ boxes.
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It is determined by the conditions that the first $m$ rows of $\Gamma_\lambda$ are $\lambda_1, \ldots, \lambda_m$ while the first $\ell$ columns are $\lambda_{m+1} + m, \ldots, \lambda_{m+\ell} + m$. 
To each highest weight $\lambda$ satisfying these conditions, associate the Young diagram $\Gamma_{\lambda}$ containing $\lambda_1 + \cdots + \lambda_{m+n}$ boxes.

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The condition $\ell \leq \lambda_m$ ensures that $\Gamma_{\lambda}$ is the diagram of a partition.
Example. The following is the diagram $\Gamma_\lambda$ associated with the highest weight $\lambda = (10, 7, 4, 3 | 3, 1, 0, 0, 0)$ of $\mathfrak{gl}_{4|5}$:
A supertableau $\Lambda$ of shape $\Gamma_\lambda$ is obtained by filling in the boxes of the diagram $\Gamma_\lambda$ with the numbers $1, \ldots, m+n$ in such a way that
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- the entries in $\{m+1, \ldots, m+n\}$ strictly increase from left to right along each row.
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$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 2 & 2 & 3 & 5 & 6 & 7 & 9 \\
2 & 2 & 3 & 3 & 4 & 4 & 5 \\
3 & 4 & 7 & 9 \\
3 & 4 & 7 & 9 \\
4 & 6 & 8 \\
5 & 6 \\
7 \\
7 \\
\end{array}
$$
Theorem. The covariant representation $L(\lambda)$ of $\mathfrak{gl}_{m|n}$ admits a basis $\zeta_{\Lambda}$ parameterized by all supertableaux $\Lambda$ of shape $\Gamma_\lambda$. 

The action of the generators of the Lie superalgebra $\mathfrak{gl}_{m|n}$ in this basis is given by the formulas:

$$E_{ss} \zeta_{\Lambda} = \omega_{ss} \zeta_{\Lambda},$$

$$E_{s, s+1} \zeta_{\Lambda} = \sum_{\Lambda'} c_{\Lambda \Lambda'} \zeta_{\Lambda'},$$

$$E_{s+1, s} \zeta_{\Lambda} = \sum_{\Lambda'} d_{\Lambda \Lambda'} \zeta_{\Lambda'}.$$

The sums are over supertableaux $\Lambda'$ obtained from $\Lambda$ by replacing an entry $s+1$ by $s$ and an entry $s$ by $s+1$, respectively.
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Here $\omega_s$ denotes the number of entries in $\Lambda$ equal to $s$. 
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**Corollary (Sergeev 1985, Berele and Regev 1987).** The character $\text{ch } L(\lambda)$ coincides with the supersymmetric Schur polynomial $s_{\Gamma_\lambda}(x_1, \ldots, x_m \mid x_{m+1}, \ldots, x_{m+n})$ associated with the Young diagram $\Gamma_\lambda$. 
Given such a supertableau $\Lambda$, for any $1 \leq i \leq s \leq m$ denote by $\lambda_{si}$ the number of entries in row $i$ which do not exceed $s$. 
Given such a supertableau $\Lambda$, for any $1 \leq i \leq s \leq m$ denote by $\lambda_{si}$ the number of entries in row $i$ which do not exceed $s$.

Set $r = \lambda_{m1}$ and for any $0 \leq p \leq n$ and $1 \leq j \leq r + p$ denote by $\lambda'_{r+p,j}$ the number of entries in column $j$ which do not exceed $m + p$. 
Example. The supertableau with \( \lambda = (7, 5, 2 \mid 2, 1) \)

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$$
\begin{array}{cccccc}
1 & 1 & 2 & 2 & 2 & 4 & 5 \\
2 & 3 & 3 & 4 & 5 &   &   \\
3 & 5 &   &   &   &   &   \\
4 & 5 &   &   &   &   &   \\
4 &   &   &   &   &   &   \\
\end{array}
$$

corresponds to the patterns $\mathcal{U}$ and $\mathcal{V}$:

$$
\begin{array}{ccccccccc}
5 & 3 & 1 &   &   &   &   &   &   \\
5 & 4 & 2 & 2 & 2 & 1 & 1 &   &   \\
5 & 1 &   &   &   &   &   &   &   \\
3 & 2 & 2 & 1 & 1 &   &   &   &   \\
2 &   &   &   &   &   &   &   &   \\
\end{array}
$$
Set \( l_i = \lambda_i - i + 1, \)

\[
l_{s_i} = \lambda_{s_i} - i + 1, \quad l'_{r+p,j} = \lambda'_{r+p,j} - j + 1.
\]
Set \( l_i = \lambda_i - i + 1, \)

\[ l_{si} = \lambda_{si} - i + 1, \quad l'_{r+p,j} = \lambda'_{r+p,j} - j + 1. \]

The coefficients in the expansions of \( E_{s,s+1} \zeta_\lambda \) and \( E_{s+1,s} \zeta_\lambda \) are given by
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\[
\begin{align*}
   l_{si} &= \lambda_{si} - i + 1, \\
   l'_{r+p, j} &= \lambda'_{r+p, j} - j + 1.
\end{align*}
\]

The coefficients in the expansions of \( E_{s, s+1} \zeta_{\Lambda} \) and \( E_{s+1, s} \zeta_{\Lambda} \) are given by

\[
\begin{align*}
   c_{\Lambda\Lambda'} &= -\frac{(l_{si} - l_{s+1, 1}) \cdots (l_{si} - l_{s+1, s+1})}{(l_{si} - l_{s1}) \cdots \wedge \cdots (l_{si} - l_{ss})}, \\
   d_{\Lambda\Lambda'} &= \frac{(l_{si} - l_{s-1, 1}) \cdots (l_{si} - l_{s-1, s-1})}{(l_{si} - l_{s1}) \cdots \wedge \cdots (l_{si} - l_{ss})},
\end{align*}
\]

if \( 1 \leq s \leq m - 1 \) and the replacement occurs in row \( i \),
and by

\[
\begin{align*}
c_{\wedge\wedge'} &= -\frac{(l'_{r+p,j} - l'_{r+p+1,1}) \cdots (l'_{r+p,j} - l'_{r+p+1,r+p+1})}{(l'_{r+p,j} - l'_{r+p,1}) \cdots \wedge \cdots (l'_{r+p,j} - l'_{r+p,r+p})}, \\
d_{\wedge\wedge'} &= \frac{(l'_{r+p,j} - l'_{r+p-1,1}) \cdots (l'_{r+p,j} - l'_{r+p-1,r+p-1})}{(l'_{r+p,j} - l'_{r+p,1}) \cdots \wedge \cdots (l'_{r+p,j} - l'_{r+p,r+p})},
\end{align*}
\]

if \( s = m + p \) for \( 1 \leq p \leq n - 1 \) and the replacement occurs in column \( j \).
Formulas for the expansions of $E_{m,m+1} \zeta_\Lambda$ and $E_{m+1,m} \zeta_\Lambda$ are also available.
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Example (Palev 1989). The basis $\zeta_{\Lambda}$ of the $\mathfrak{gl}_{m|1}$-module $L(\lambda_1, \ldots, \lambda_m | \lambda_{m+1})$ is parameterized by the patterns

\[ U = \begin{bmatrix} \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mm} \\ \lambda_{m-1,1} & \cdots & \lambda_{m-1,m-1} \\ \vdots & \ddots & \ddots & \ddots \\ \lambda_{21} & \lambda_{22} \\ \lambda_{11} \end{bmatrix} \]
The top row runs over partitions \((\lambda_{m_1}, \ldots, \lambda_{m_m})\) such that either \(\lambda_{mj} = \lambda_j\) or \(\lambda_{mj} = \lambda_j - 1\) for each \(j = 1, \ldots, m\).
The top row runs over partitions \((\lambda_{m1}, \ldots, \lambda_{mm})\) such that either \(\lambda_{mj} = \lambda_j\) or \(\lambda_{mj} = \lambda_j - 1\) for each \(j = 1, \ldots, m\).

\[
E_{m,m+1} \zeta_U = \sum_{i=1}^{m} (l_{mi} + \lambda_{m+1} + m) \\
\times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}} \frac{l_{mi} - l_j}{l_{mi} - l_{mj}} \prod_{j=i+1}^{m} \frac{l_{mi} - l_{mj} + 1}{l_{mi} - l_j + 1} \zeta_U^{\delta_{mi}},
\]

\[
E_{m+1,m} \zeta_U = \sum_{i=1}^{m} \frac{(l_{mi} - l_{m-1,1}) \cdots (l_{mi} - l_{m-1,m-1})}{(l_{mi} - l_{m1}) \cdots \wedge \cdots (l_{mi} - l_{mm})} \\
\times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}} \frac{l_{mi} - l_{mj} - 1}{l_{mi} - l_j - 1} \prod_{j=1}^{i-1} \frac{l_{mi} - l_{mj}}{l_{mi} - l_j} \zeta_U^{-\delta_{mi}}.
\]
Example. The basis $\zeta_{\Lambda}$ of the $\mathfrak{gl}_{1|n}$-module $L(\lambda_1 | \lambda_2, \ldots, \lambda_{n+1})$ is parameterized by the trapezium patterns

$$
\begin{array}{cccccc}
\lambda'_{r+n,1} & \lambda'_{r+n,2} & \cdots & \cdots & \lambda'_{r+n,r+n} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\lambda'_{r+1,1} & \lambda'_{r+1,2} & \cdots & \cdots & \lambda'_{r+1,r+1} \\
1 & 1 & \cdots & \cdots & 1 \\
\end{array}
$$

The number $r$ of $1'$s in the bottom row is nonnegative and varies between $\lambda_1 - n$ and $\lambda_1$. The top row coincides with $(\lambda'_1, \ldots, \lambda'_p, 0, \ldots, 0)$, where $p = \lambda_1$. 


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\vdots & \ddots & \cdots & \cdots & \vdots \\
\lambda'_{r+1,1} & \lambda'_{r+1,2} & \cdots & \lambda'_{r+1, r+1} \\
1 & 1 & \cdots & 1 \\
\end{array}
\]

The number $r$ of 1’s in the bottom row is nonnegative and varies between $\lambda_1 - n$ and $\lambda_1$. The top row coincides with $(\lambda'_1, \ldots, \lambda'_{p}, 0, \ldots, 0)$, where $p = \lambda_1$. 
Yangian $Y(gl_n)$
Yangian $\mathcal{Y}(\mathfrak{gl}_n)$

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The Yangian $\mathcal{Y}(\mathfrak{gl}_n)$ is a unital associative algebra with generators $t_{ij}^{(1)}$, $t_{ij}^{(2)}$, ..., where $i$ and $j$ run over the set $\{1, \ldots, n\}$. The defining relations are given by

\[ [t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \]

where $r, s \geq 0$ and $t_{ij}^{(0)} := \delta_{ij}$. 
Using the formal generating series

\[ t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \ldots \]
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A natural analogue of the Poincaré–Birkhoff–Witt theorem holds for the Yangian \( Y(\mathfrak{gl}_n) \).
Every finite-dimensional irreducible representation $L$ of $\mathfrak{gl}_n$ contains a highest vector $\zeta$ such that

$$t_{ij}(u)\zeta = 0 \quad \text{for } 1 \leq i < j \leq n,$$

and

$$t_{ii}(u)\zeta = \lambda_i(u)\zeta \quad \text{for } 1 \leq i \leq n,$$

for some formal series $\lambda_i(u) = 1 + \lambda_1(u)u - 1 + \lambda_2(u)u^2 - 1 + \ldots$, $\lambda_i(u) \in \mathbb{C}$. The $n$-tuple of formal series $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$ is the highest weight of $L$. 
Every finite-dimensional irreducible representation $L$ of $\mathfrak{gl}_n$ contains a highest vector $\zeta$ such that

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The $n$-tuple of formal series $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u))$ is the highest weight of $L$. 
Moreover, there exist monic polynomials $P_1(u), \ldots, P_{n-1}(u)$ in $u$ (the Drinfeld polynomials) such that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u + 1)}{P_i(u)}$$

for $i = 1, \ldots, n - 1$. 

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For an arbitrary representation $L(\lambda)$ of $\mathfrak{gl}_{m|n}$ consider the vector space isomorphism

$$L(\lambda) \cong \bigoplus_{\mu} L'(\mu) \otimes L(\lambda)^{+}_{\mu},$$
For an arbitrary representation \( L(\lambda) \) of \( \mathfrak{gl}_{m|n} \), consider the vector space isomorphism

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L(\lambda) \cong \bigoplus_{\mu} L'(\mu) \otimes L(\lambda)^{\perp},
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where \( L'(\mu) \) denotes the irreducible representation of the Lie algebra \( \mathfrak{gl}_m \) with the highest weight \( \mu = (\mu_1, \ldots, \mu_m) \),
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where $L'(\mu)$ denotes the irreducible representation of the Lie algebra $\mathfrak{gl}_m$ with the highest weight $\mu = (\mu_1, \ldots, \mu_m)$, and

$L(\lambda)^{\perp}_\mu$ is the multiplicity space spanned by the $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$ of weight $\mu$, 
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consider the vector space isomorphism

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where $L'(\mu)$ denotes the irreducible representation of the Lie algebra $\mathfrak{gl}_m$ with the highest weight $\mu = (\mu_1, \ldots, \mu_m)$, and

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$$L(\lambda)^+_\mu \cong \text{Hom}_{\mathfrak{gl}_m}(L'(\mu), L(\lambda)).$$
Olshanski homomorphism
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Set $E = [E_{ij}]_{i,j=1}^m$. The mapping $\psi : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{m|n})$ given by


t_{ij}^{(1)} \mapsto E_{m+i,m+j},

t_{ij}^{(r)} \mapsto \sum_{k,l=1}^m E_{m+i,k}(E^{r-2})_{k,l}E_{l,m+j}, \quad r \geq 2,

defines an algebra homomorphism.
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Set $E = [E_{ij}]_{i,j=1}^m$. The mapping $\psi : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{m|n})$ given by

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The image of $\psi$ is contained in the centralizer $U(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$. 
Theorem. The representation of \( Y(\mathfrak{gl}_n) \) in \( L(\lambda)^\mu_+ \) defined via the homomorphism \( \psi \) is irreducible.
Theorem. The representation of \( Y(\mathfrak{gl}_n) \) in \( L(\lambda)^{\pm}_\mu \) defined via the homomorphism \( \psi \) is irreducible.

Proof.

\( L(\lambda)^{\pm}_\mu \) is an irreducible representation of the centralizer \( U(\mathfrak{gl}_m|_n)^{\mathfrak{gl}_m} \).
Theorem. The representation of $Y(gl_n)$ in $L(\lambda)^+_{\mu}$ defined via the homomorphism $\psi$ is irreducible.

Proof.

- $L(\lambda)^+_{\mu}$ is an irreducible representation of the centralizer $U(gl_{m|n})_{gl_m}$.

- The centralizer $U(gl_{m|n})_{gl_m}$ is generated by the image of the homomorphism $Y(gl_n) \rightarrow U(gl_{m|n})_{gl_m}$ and the center of $U(gl_{m|n})$. 
Twist the Yangian action on $L(\lambda)_\mu^+$ by the automorphism

$$t_{ij}(u) \rightarrow t_{ij}(u + m).$$
Twist the Yangian action on $L(\lambda)^{\pm}\mu$ by the automorphism

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For each box $\alpha = (i, j)$ of a Young diagram define its content by $c(\alpha) = j - i$. 
Twist the Yangian action on $L(\lambda)^+_\mu$ by the automorphism $t_{ij}(u) \rightarrow t_{ij}(u + m)$.

For each box $\alpha = (i, j)$ of a Young diagram define its content by $c(\alpha) = j - i$.

**Theorem.** Suppose that $L(\lambda)$ is a covariant representation. The Drinfeld polynomials for the $\mathcal{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ are given by

$$P_k(u) = \prod_{\alpha} (u - c(\alpha)), \quad k = 1, \ldots, n - 1,$$

where $\alpha$ runs over the leftmost boxes of the rows of length $k$ in the diagram $\Gamma_{\lambda}/\mu$. 
Example. For $\lambda = (7, 5, 2 \mid 3, 1, 0, 0)$ and $\mu = (4, 2, 1)$ we have
Example. For \( \lambda = (7, 5, 2 \mid 3, 1, 0, 0) \) and \( \mu = (4, 2, 1) \) we have

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P_1(u) = (u + 1)(u + 4)(u + 5),
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\[
P_2(u) = u + 3,
\]

\[
P_3(u) = (u - 4)(u - 1).
\]
Introduce parameters of the diagram conjugate to $\Gamma_{\lambda/\mu}$. Set $r = \mu_1$ and let $\mu' = (\mu'_1, \ldots, \mu'_r)$ be the diagram conjugate to $\mu$ so that $\mu'_j$ equals the number of boxes in column $j$ of $\mu$. 

Corollary. The $\mathfrak{Y}(\mathfrak{g}l_n)$-module $L(\lambda) + \mu$ is isomorphic to $L(\lambda') + \mu'$, the skew representation associated with $\mathfrak{g}l_r + n$-module $L(\lambda')$ and the $\mathfrak{g}l_r$-highest weight $\mu'$. 

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Set $\lambda' = (\lambda'_1, \ldots, \lambda'_{r+n})$, where $\lambda'_j$ equals the number of boxes in column $j$ of the diagram $\Gamma_{\lambda}$. 
Introduce parameters of the diagram conjugate to $\Gamma_{\lambda}/\mu$. Set $r = \mu_1$ and let $\mu' = (\mu'_1, \ldots, \mu'_r)$ be the diagram conjugate to $\mu$ so that $\mu'_j$ equals the number of boxes in column $j$ of $\mu$.

Set $\lambda' = (\lambda'_1, \ldots, \lambda'_{r+n})$, where $\lambda'_j$ equals the number of boxes in column $j$ of the diagram $\Gamma_{\lambda}$.

**Corollary.** The $\mathbf{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^{\mu}_\mu$ is isomorphic to $\overline{L}(\lambda')^{\mu'}_{\mu'}$, the skew representation associated with $\mathfrak{gl}_{r+n}$-module $\overline{L}(\lambda')$ and the $\mathfrak{gl}_r$-highest weight $\mu'$. 
Construction of basis vectors

produce the highest vector of the \( Y(gl_n) \)-module \( L(\lambda + \mu) \), use the isomorphism \( L(\lambda) + \mu \sim L(\lambda') + \mu' \) to get the vectors of the trapezium Gelfand–Tsetlin basis of \( L'(\mu) \) in terms of the Yangian generators, combine with the Gelfand–Tsetlin basis of \( L'(\mu) \).
Construction of basis vectors

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- produce the highest vector of the $\mathcal{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$,

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Construction of basis vectors

- produce the highest vector of the Y(gl_n)-module $L(\lambda)_{\mu}^{+}$,

- use the isomorphism $L(\lambda)_{\mu}^{+} \cong \overline{L}(\lambda')_{\mu'}^{+}$ to get the vectors of the trapezium Gelfand–Tsetlin basis of $\overline{L}(\lambda')_{\mu'}^{+}$ in terms of the Yangian generators,

- combine with the Gelfand–Tsetlin basis of $L'(\mu)$. 
The extremal projector $p$ for $\mathfrak{gl}_m$ is given by

$$p = \prod_{i<j} \sum_{k=0}^{\infty} (E_{ji})^k (E_{ij})^k \frac{(-1)^k}{k! (h_i - h_j + 1) \ldots (h_i - h_j + k)},$$

where $h_i = E_{ii} - i + 1$. The product is taken in a normal order.
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The projector satisfies

$$E_{ij} p = p E_{ji} = 0 \quad \text{for} \quad 1 \leq i < j \leq m.$$
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For $i = 1, \ldots, m$ and $a = m + 1, \ldots, m + n$ set

\[ z_{i a} = p E_{i a}(h_i - h_1) \cdots (h_i - h_{i-1}), \]

\[ z_{a i} = p E_{a i}(h_i - h_{i+1}) \cdots (h_i - h_m). \]
For $i = 1, \ldots, m$ and $a = m + 1, \ldots, m + n$ set

$$z_{ia} = p E_{ia} (h_i - h_1) \cdots (h_i - h_{i-1}),$$

$$z_{ai} = p E_{ai} (h_i - h_{i+1}) \cdots (h_i - h_m).$$

$z_{ia}$ and $z_{ai}$ can be regarded as elements of $U(\mathfrak{gl}_{m|n})$ modulo the left ideal generated by $E_{ij}$ with $1 \leq i < j \leq m$. 
For \(i = 1, \ldots, m\) and \(a = m + 1, \ldots, m + n\) set

\[
    z_{i,a} = p E_{i,a}(h_i - h_1) \cdots (h_i - h_{i-1}),
\]
\[
    z_{a,i} = p E_{a,i}(h_i - h_{i+1}) \cdots (h_i - h_{m}).
\]

\(z_{i,a}\) and \(z_{a,i}\) can be regarded as elements of \(\mathcal{U}(\mathfrak{gl}_{m|n})\) modulo the left ideal generated by \(E_{ij}\) with \(1 \leq i < j \leq m\).

**Example.**

\[
    z_{1,a} = E_{1,a}, \quad z_{2,a} = E_{2,a}(h_2 - h_1) + E_{21}E_{1,a},
\]
\[
    z_{am} = E_{am}, \quad z_{a,m-1} = E_{a,m-1}(h_{m-1} - h_m) + E_{m,m-1}E_{am}.
\]
The elements $z_{ia}$ and $z_{ai}$ are odd; together with the even elements $E_{ab}$ with $a, b \in \{m + 1, \ldots, m + n\}$ they generate the Mickelsson–Zhelobenko superalgebra $Z(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ associated with the pair $\mathfrak{gl}_m \subseteq \mathfrak{gl}_{m|n}$. 
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The generators satisfy quadratic relations that can be written in an explicit form.
They preserve the subspace of $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$,

$$z_{ia} : L(\lambda)_{\mu}^+ \rightarrow L(\lambda)_{\mu+\delta_i}^+,$$

$$z_{ai} : L(\lambda)_{\mu}^+ \rightarrow L(\lambda)_{\mu-\delta_i}^+,$$

where $\mu \pm \delta_i$ is obtained from $\mu$ by replacing $\mu_i$ by $\mu_i \pm 1$. 

They preserve the subspace of $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$,

$$z_{ia} : L(\lambda)^+_{\mu} \rightarrow L(\lambda)^+_{\mu+\delta_i}, \quad z_{ai} : L(\lambda)^+_{\mu} \rightarrow L(\lambda)^+_{\mu-\delta_i},$$

where $\mu \pm \delta_i$ is obtained from $\mu$ by replacing $\mu_j$ by $\mu_j \pm 1$.

**Proposition.** The element

$$\zeta_\mu = \prod_{j=1}^{m} (z_{m+\lambda_j-\mu_j, j} \cdots z_{m+2,j} z_{m+1,j}) \zeta$$

with the product taken in the increasing order of $j$ is the highest vector of the $Y(\mathfrak{gl}_n)$-module $L(\lambda)^+_{\mu}$. 