Littlewood–Richardson polynomials

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A diagram (or partition) is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of integers \( \lambda_i \) such that \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \), depicted as an array of unit boxes.
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![Diagram](image)

The number of boxes is the **weight** of the diagram, denoted $|\lambda|$. The number of nonzero rows is its **length**, denoted $\ell(\lambda)$. 
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Let $\ell(\lambda) \leq n$ and let $V^\lambda$ denote the irreducible $\mathfrak{gl}_n$-module with the highest weight $\lambda$.

Then

$$V^\lambda \otimes V^\mu \cong \bigoplus_{\nu} c_{\lambda\mu}^\nu V^\nu.$$ 

Here $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$. 
Let $|\lambda| = k$ and let $\chi^\lambda$ denote the corresponding irreducible character of the symmetric group $S_k$. 

In particular, $c^\nu_{\lambda \mu} \neq 0 \Rightarrow |\nu| = |\lambda| + |\mu|$. 
Let $|\lambda| = k$ and let $\chi^\lambda$ denote the corresponding irreducible character of the symmetric group $\mathfrak{S}_k$.

Then

$$\text{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_l}^{\mathfrak{S}_{k+l}} (\chi^\lambda \times \chi^\mu) = \sum_{\nu} c_{\lambda \mu}^\nu \chi^\nu.$$

Here $|\lambda| = k$, $|\mu| = l$, $|\nu| = k + l$. 
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Here $|\lambda| = k$, $|\mu| = l$, $|\nu| = k + l$.

In particular,

$$c_{\lambda \mu}^\nu \neq 0 \quad \Rightarrow \quad |\nu| = |\lambda| + |\mu|.$$
Let $n$ and $N$ be nonnegative integers with $n \leq N$ and let $\text{Gr}_{n,N}$ denote the Grassmannian of the $n$-dimensional vector subspaces of $\mathbb{C}^N$. The cohomology ring $H^*(\text{Gr}_{n,N})$ has a basis of the Schubert classes $\sigma_\lambda$ parameterized by all diagrams $\lambda$ contained in the $n \times m$ rectangle, $m = N - n$. 

We have

$$\sigma_\lambda \sigma_\mu = \sum \nu c_{\nu \lambda \mu} \sigma_\nu.$$ 

Here $\lambda$, $\mu$, $\nu$ are contained in the $n \times m$ rectangle.
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The **Schur functions** \( s_\lambda(x) \) parameterized by all diagrams form a basis of the ring of symmetric functions \( \Lambda \).

We have

\[
s_\lambda(x) s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x).
\]

Here \( \lambda, \mu, \nu \) are any diagrams.
History:

D. E. Littlewood and A. R. Richardson (1934),
(general formulation, a proof in the case $\ell(\mu) \leq 2$),
G. de B. Robinson (1938), (proof contains gaps).
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Complete proofs:


Now:

A couple of dozens of versions of the LR rule, $c_{\chi\mu}^{\nu}$ counts
tableaux, trees, hives, honeycombs, puzzles, cartons, . . . .
A version of the LR rule

Let $R$ denote a sequence of diagrams

$$\mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

$\rho \rightarrow \sigma$ means $\sigma$ is obtained from $\rho$ by adding one box.
Let $R$ denote a sequence of diagrams

$$\mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

$\rho \rightarrow \sigma$ means $\sigma$ is obtained from $\rho$ by adding one box.

Let $r_i$ denote the row number of the box added to $\rho^{(i-1)}$.

The sequence $r_1 \ r_2 \ \ldots \ r_l$ is the Yamanouchi symbol of $R$. 
Example. Let

\[ R : (3, 1) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (3, 3, 1) \rightarrow (4, 3, 1) \]
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\]

or, with diagrams,

```
+----+    +----+    +----+    +----+    +----+
|    | →   |    | →   |    | →   | →   |    |
|    |     |    |     |    |     |     |    |
|    |     |    |     |    |     |     |    |
```

or

```
+----+    +----+    +----+    +----+    +----+
|    |    |    |    |    |    |    |    |
|    |    |    |    |    |    |    |    |
|    |    |    |    |    |    |    |    |
|    |    |    |    |    |    |    |    |
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or, with diagrams,

```
+---+---+   +---+---+   +---+---+   +---+---+   +---+---+
|   |   | → |   |   | → |   |   | → |   |   | → |   |   |
| * |   |   |   |   |   | * |   |   | * |   |   | * |   |   |
|   |   |   |   | * |   |   | * |   |   | * |   |   | * |   |   |
```

the Yamanouchi symbol is 2321.
A reverse $\lambda$-tableau $T$ is obtained by filling in the boxes of $\lambda$ with the numbers $1, 2, \ldots$ in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of $\lambda$ we let $T(\alpha) = T(i, j)$ denote the entry of $T$ in the box $\alpha$. 
A reverse $\lambda$-tableau $T$ is obtained by filling in the boxes of $\lambda$ with the numbers $1, 2, \ldots$ in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of $\lambda$ we let $T(\alpha) = T(i, j)$ denote the entry of $T$ in the box $\alpha$.

The column word of $T$ is the sequence of all entries of $T$ written in the column order: by reading the entries by columns from left to right and from bottom to top in each column.
Example. A reverse $\lambda$-tableau for $\lambda = (5, 5, 3)$:

\[
\begin{array}{cccccc}
5 & 5 & 4 & 2 & 2 \\
4 & 3 & 2 & 1 & 1 \\
2 & 1 & 1 & & & \\
\end{array}
\]

Its column word is 2 4 5 1 3 5 1 2 4 1 2 1 2.
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\begin{array}{cccc}
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5 & 5 & 4 & 2 \\
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\end{array}
$$

Its column word is $2 4 5 1 3 5 1 2 4 1 2 1 2$.

A reverse $\lambda$-tableau is $\nu$-bounded if

$$
T(1, j) \leq \nu_j' \quad \text{for all } j = 1, \ldots, \lambda_1,
$$

where $\nu_j'$ is the number of boxes in column $j$ of $\nu$. 
Theorem. The Littlewood–Richardson coefficient $c_{\lambda \mu}^{\nu}$ equals the number of $\nu$-bounded reverse $\lambda$-tableaux $T$ such that the column word of $T$ coincides with the Yamanouchi symbol of a certain sequence $R$ from $\mu$ to $\nu$. 

Remarks. $c_{\lambda \mu}^{\nu} \neq 0$ only if $|\nu| = |\lambda| + |\mu|$. 

$\Rightarrow c_{\lambda \mu}^{\nu} \neq 0$ only if $\lambda \subseteq \nu$ and $\mu \subseteq \nu$. 

$\Rightarrow c_{\lambda \mu}^{\nu} = c_{\mu \lambda}^{\nu}$. 

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- $c_{\lambda\mu}^\nu \neq 0$ only if $|\nu| = |\lambda| + |\mu|$.
- $c_{\lambda\mu}^\nu \neq 0$ only if $\lambda \subseteq \nu$ and $\mu \subseteq \nu$.
- $c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu$. 
Example.

Calculation of $c_{\lambda \mu}^\nu$, \( \lambda = (2, 1) \), \( \mu = (3, 1) \), \( \nu = (4, 2, 1) \).
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Calculation of $c_{\lambda\mu}^\nu$, $\lambda = (2, 1)$, $\mu = (3, 1)$, $\nu = (4, 2, 1)$.

Here $\nu'_1 = 3$, $\nu'_2 = 2$, $\nu'_3 = 1$, $\nu'_4 = 1$. The $\nu$-bounded $\lambda$-tableaux:

\[
\begin{array}{ccc}
3 & 2 & 3 \\
2 & 1 & 2 \\
\end{array}
\quad \begin{array}{ccc}
3 & 2 & 3 \\
1 & 2 & 1 \\
\end{array}
\quad \begin{array}{cc}
3 & 1 \\
1 & 2 \\
\end{array}
\quad \begin{array}{cc}
3 & 1 \\
2 & 1 \\
\end{array}
\quad \begin{array}{cc}
2 & 2 \\
1 & 1 \\
\end{array}
\quad \begin{array}{cc}
2 & 1 \\
1 & 1 \\
\end{array}
\]
Example.

Calculation of $c_{\lambda\mu}^{\nu}$, \( \lambda = (2, 1), \mu = (3, 1), \nu = (4, 2, 1) \).

Here $\nu'_1 = 3$, $\nu'_2 = 2$, $\nu'_3 = 1$, $\nu'_4 = 1$. The \( \nu \)-bounded \( \lambda \)-tableaux

\[
\begin{array}{cccc}
3 & 2 & & \\
2 & & & \\
\end{array}
\quad \begin{array}{cccc}
3 & 2 & & \\
1 & & & \\
\end{array}
\quad \begin{array}{cccc}
3 & 1 & & \\
2 & & & \\
1 & & & \\
\end{array}
\quad \begin{array}{cccc}
3 & 1 & & \\
2 & 2 & & \\
2 & 1 & & \\
1 & & & \\
\end{array}
\]

Two column words 1 3 2 and 2 3 1 are the Yamanouchi symbols of the respective sequences

\((3, 1) \rightarrow (4, 1) \rightarrow (4, 1, 1) \rightarrow (4, 2, 1) \) \quad \text{and} \quad \((3, 1) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (4, 2, 1) \).
Example.

Calculation of $c^{\nu}_{\lambda\mu}$, $\lambda = (2, 1)$, $\mu = (3, 1)$, $\nu = (4, 2, 1)$.

Here $\nu'_1 = 3$, $\nu'_2 = 2$, $\nu'_3 = 1$, $\nu'_4 = 1$. The $\nu$-bounded $\lambda$-tableaux

Two column words $1\ 3\ 2$ and $2\ 3\ 1$ are the Yamanouchi symbols of the respective sequences

$(3, 1) \rightarrow (4, 1) \rightarrow (4, 1, 1) \rightarrow (4, 2, 1)$ and

$(3, 1) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (4, 2, 1)$.

Hence $c^{\nu}_{\lambda\mu} = 2.$
Symmetric functions

Let \( a = (a_i), \ i \in \mathbb{Z} \), be a sequence of variables.

Denote by \( \Lambda_n \) the ring of symmetric polynomials in \( x_1, \ldots, x_n \) with coefficients in \( \mathbb{Z}[a] \).
Symmetric functions

Let \( a = (a_i), i \in \mathbb{Z}, \) be a sequence of variables.

Denote by \( \Lambda_n \) the ring of symmetric polynomials in \( x_1, \ldots, x_n \) with coefficients in \( \mathbb{Z}[a] \). Set

\[ \Lambda = \lim_{\leftarrow} \Lambda_n, \quad n \to \infty, \]

the inverse limit is taken with respect to the homomorphisms

\[ \varphi_n : \Lambda_n \to \Lambda_{n-1}, \quad P(x_1, \ldots, x_n) \mapsto P(x_1, \ldots, x_{n-1}, a_n) \]

in the category of filtered rings.
Examples. We have

$$\varphi_n : \sum_{i=1}^{n} (x_i^k - a_i^k) \mapsto \sum_{i=1}^{n-1} (x_i^k - a_i^k)$$

hence

$$p_k(x \| a) = \sum_{i=1}^{\infty} (x_i^k - a_i^k) \in \Lambda,$$

the double power sums symmetric function.
Examples. We have

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hence

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the double power sums symmetric function.

\( \Lambda \) is the ring of polynomials in

\[ p_1(x \| a), \quad p_2(x \| a), \quad \ldots. \]

with coefficients in \( \mathbb{Z}[a] \).
Double Schur functions

For any diagram $\lambda$ define the **double Schur function** by

$$s_\lambda(x \parallel a) = \sum_{T} \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha)} - c(\alpha)),$$

summed over the reverse $\lambda$-tableaux $T$,

$c(\alpha) = j - i$ is the content of the box $\alpha = (i, j)$. 
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summed over the reverse $\lambda$-tableaux $T$,

$c(\alpha) = j - i$ is the content of the box $\alpha = (i, j)$.

The double Schur functions form a basis of $\Lambda$ over $\mathbb{Z}[a]$. 
Example. For \( \lambda = (2, 1) \) the reverse tableaux are

\[
\begin{array}{c}
i \\ j \\ k
\end{array}
\]

with \( i \geq j \) and \( i > k \)
Example. For $\lambda = (2, 1)$ the reverse tableaux are

\[
\begin{array}{|c|c|}
\hline
i & j \\
\hline
k \\
\hline
\end{array}
\]

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Hence

\[
s_{(2,1)}(x \| a) = \sum_{i \geq j, \ i > k} (x_i - a_i)(x_j - a_{j-1})(x_k - a_{k+1}).
\]
Set \( h_k(x \parallel a) = s_{(k)}(x \parallel a), \quad e_k(x \parallel a) = s_{(1\,k)}(x \parallel a). \)
Set \( h_k(x \parallel a) = s_{(k)}(x \parallel a) \), \( e_k(x \parallel a) = s_{(1^k)}(x \parallel a) \).

Tableaux

\[
\begin{array}{cccc}
  i_1 & i_2 & \cdots & i_k \\
\end{array}
\]

\[
\begin{array}{c}
i_1 \\
i_2 \\
\vdots \\
i_k \\
\end{array}
\]
Set \( h_k(x \parallel a) = s_{(k)}(x \parallel a), \quad e_k(x \parallel a) = s_{(1k)}(x \parallel a) \).

Tableaux

\[
\begin{array}{cccc}
i_1 & i_2 & \cdots & i_k \\
\end{array}
\]

Other generators of \( \Lambda \):

\[
h_k(x \parallel a) = \sum_{i_1 \geq \cdots \geq i_k} (x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_{k-k+1}}),
\]

\[
e_k(x \parallel a) = \sum_{i_1 > \cdots > i_k} (x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_{k+k-1}}).
\]
Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^\nu(a) \in \mathbb{Z}[a]$ by

$$s_\lambda(x \parallel a) \cdot s_\mu(x \parallel a) = \sum_{\nu} c_{\lambda\mu}^\nu(a) \cdot s_\nu(x \parallel a).$$
Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^\nu(a) \in \mathbb{Z}[a]$ by

$$s_\lambda(x \| a) \ s_\mu(x \| a) = \sum_\nu c_{\lambda\mu}^\nu(a) \ s_\nu(x \| a).$$

**Properties.**

- $c_{\lambda\mu}^\nu(a) \neq 0$ only if $|\nu| \leq |\lambda| + |\mu|$. 

- $c_{\lambda\mu}^\nu(a) = c_{\nu\mu\lambda}(a)$.
Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a) \in \mathbb{Z}[a]$ by

$$s_\lambda(x \parallel a) s_\mu(x \parallel a) = \sum_{\nu} c_{\lambda\mu}^{\nu}(a) s_\nu(x \parallel a).$$

Properties.

- $c_{\lambda\mu}^{\nu}(a) \neq 0$ only if $|\nu| \leq |\lambda| + |\mu|$.

- $c_{\lambda\mu}^{\nu}(a)$ is homogeneous of degree $|\lambda| + |\mu| - |\nu|$.
Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^\nu(a) \in \mathbb{Z}[a]$ by

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- $c_{\lambda\mu}^\nu(a)$ is homogeneous of degree $|\lambda| + |\mu| - |\nu|$.
- $c_{\lambda\mu}^\nu(a) = c_{\lambda\mu}^\nu$ if $|\lambda| + |\mu| = |\nu|$ or $a = (0)$. 
Define the Littlewood–Richardson polynomials $c_{\lambda \mu}^\nu (a) \in \mathbb{Z}[a]$ by

$$s_\lambda(x \| a) s_\mu(x \| a) = \sum_{\nu} c_{\lambda \mu}^\nu (a) s_\nu(x \| a).$$

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- $c_{\lambda\mu}^\nu(a) = c_{\mu\lambda}^\nu(a)$.
- $c_{\lambda\mu}^\nu(a) \neq 0$ only if $\lambda \subseteq \nu$ and $\mu \subseteq \nu$. 
Calculation of $c_{\lambda \mu}^\nu(a)$

Given a sequence $R$ from $\mu$ to $\nu$ with the Yamanouchi symbol $r_1 \, r_2 \ldots \, r_l$, introduce the set $T(\lambda, R)$ of barred reverse $\lambda$-tableaux $T$ with entries from $\{1, 2, \ldots\}$ such that $T$ contains entries $r_1, \, r_2, \ldots, \, r_l$ listed in the column order.
Calculation of $c_{\lambda\mu}^\nu (a)$

Given a sequence $R$ from $\mu$ to $\nu$ with the Yamanouchi symbol $r_1 \ r_2 \ldots \ r_l$, introduce the set $T(\lambda, R)$ of barred reverse $\lambda$-tableaux $T$ with entries from $\{1, 2, \ldots \}$ such that $T$ contains entries $r_1, r_2, \ldots, r_l$ listed in the column order.

We will distinguish these entries by barring each of them.
Calculation of $c^\nu_{\lambda\mu}(a)$

Given a sequence $R$ from $\mu$ to $\nu$ with the Yamanouchi symbol $r_1 r_2 \ldots r_l$, introduce the set $\mathcal{T}(\lambda, R)$ of barred reverse $\lambda$-tableaux $T$ with entries from $\{1, 2, \ldots\}$ such that $T$ contains entries $r_1, r_2, \ldots, r_l$ listed in the column order.

We will distinguish these entries by barring each of them.

An element $T \in \mathcal{T}(\lambda, R)$ is a pair consisting of a reverse $\lambda$-tableau and a sequence of barred entries compatible with $R$. 
Example. Let $R$ be the sequence

$$(3, 1) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (3, 3, 1) \rightarrow (4, 3, 1)$$

so that the Yamanouchi symbol is $2 \ 3 \ 2 \ 1$. 
Example. Let $R$ be the sequence

$$(3, 1) \rightarrow (3, 2) \rightarrow (3, 2, 1) \rightarrow (3, 3, 1) \rightarrow (4, 3, 1)$$

so that the Yamanouchi symbol is $2321$.

Let $\lambda = (5, 5, 3)$. The barred $\lambda$-tableau

\[
\begin{array}{cccc}
7 & 7 & 4 & \bar{2} \\
4 & \bar{3} & 2 & 1 \\
2 & 1 & 1 & 1
\end{array}
\]

belongs to $T(\lambda, R)$. 
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$$
\begin{array}{cccc}
7 & 7 & 4 & \bar{2} \\
4 & \bar{3} & 2 & 1 \\
\bar{2} & 1 & 1 \\
\end{array}
$$

belongs to $\mathcal{I}(\lambda, R)$. 


Given a sequence of diagrams

\[ R : \quad \mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu, \]

set \( \rho(\alpha) = \rho^{(i)} \) for any box \( \alpha \) occupied by an unbarred entry of \( T \), between \( \overline{r}_i \) and \( \overline{r}_{i+1} \) in column order.
Given a sequence of diagrams

\[ R : \quad \mu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \nu, \]

set \( \rho(\alpha) = \rho^{(i)} \) for any box \( \alpha \) occupied by an unbarred entry of \( T \), between \( r_i \) and \( r_{i+1} \) in column order.

The barred entries \( \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_l \) of \( T \) divide the tableau into regions marked by the elements of the sequence \( R \):

\[
\begin{array}{c|c|c|c}
\rho^{(0)} & \bar{r}_1 & \rho^{(1)} & \bar{r}_2 \\
& & \cdots & \\
& \rho^{(l)} & & \bar{r}_l
\end{array}
\]
Theorem. We have

\[ c_{\lambda\mu}^{\nu}(a) = \sum_{R} \sum_{T} \prod_{\alpha \in \lambda} \left( a_{T(\alpha) - \rho(\alpha)_{T(\alpha)}} - a_{T(\alpha) - c(\alpha)} \right) , \]

 summed over all sequences \( R \) from \( \mu \) to \( \nu \) and all \( \nu \)-bounded reverse \( \lambda \)-tableaux \( T \in \mathcal{T}(\lambda, R) \).
Theorem. We have

\[ c_\nu^{\lambda\mu}(a) = \sum_{R} \sum_{T} \prod_{\alpha \in \lambda} \left( a_{T(\alpha)} - \rho(\alpha)_{T(\alpha)} - a_{T(\alpha)} - c(\alpha) \right), \]

summed over all sequences \( R \) from \( \mu \) to \( \nu \) and all \( \nu \)-bounded reverse \( \lambda \)-tableaux \( T \in T(\lambda, R) \).

Remarks.

- If \(|\nu| = |\lambda| + |\mu|\) then this is a version of the LR rule.
**Theorem.** We have

\[
c^\nu_{\lambda\mu}(a) = \sum_{R} \sum_{T} \prod_{\alpha \in \lambda} \left( a_{T(\alpha) - \rho(\alpha)_{T(\alpha)}} - a_{T(\alpha) - c(\alpha)} \right),
\]

summed over all sequences \( R \) from \( \mu \) to \( \nu \) and all \( \nu \)-bounded reverse \( \lambda \)-tableaux \( T \in \mathcal{T}(\lambda, R) \).

**Remarks.**

- If \( |\nu| = |\lambda| + |\mu| \) then this is a version of the LR rule.
- \( c^\nu_{\lambda\mu}(a) \) is a polynomial in the differences \( a_i - a_j, i < j \), with positive integer coefficients.
Example. Calculation of $c^\nu_{\lambda\mu}(a)$,

$\lambda = (2, 1), \quad \mu = (3, 1), \quad \nu = (4, 1, 1)$. 
Example. Calculation of $c^{\nu}_{\lambda \mu}(a)$,

$\lambda = (2, 1), \mu = (3, 1), \nu = (4, 1, 1)$. 

Here $\nu'_1 = 3, \nu'_2 = 1, \nu'_3 = 1, \nu'_4 = 1$. The $\nu$-bounded $\lambda$-tableaux

\[
\begin{array}{cc}
3 & 1 \\
2 & \\
\end{array}
\quad
\begin{array}{cc}
3 & 1 \\
1 & \\
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\begin{array}{cc}
2 & 1 \\
1 & \\
\end{array}
\]
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\]

There are two sequences

$R_1 : \quad (3, 1) \rightarrow (4, 1) \rightarrow (4, 1, 1)$ \quad \text{and} \quad \quad

$R_2 : \quad (3, 1) \rightarrow (3, 1, 1) \rightarrow (4, 1, 1)$

with the respective Yamanouchi symbols $1 \ 3$ and $3 \ 1$. 
\( \mathcal{T}(\lambda, R_1) \) contains one barred tableau

\[
\begin{array}{c|c}
3 & 1 \\
\hline
1 & 1 \\
\end{array}
\]

with \( T(\alpha) = 1, \quad \rho(\alpha) = (4, 1, 1), \quad c(\alpha) = 1, \)

contributing

\[ a_{T(\alpha) - \rho(\alpha)}T(\alpha) - a_{T(\alpha) - c(\alpha)} = a_3 - a_0. \]
\( T(\lambda, R_1) \) contains one barred tableau

\[
\begin{array}{c|c|c}
3 & 1 \\
1 & & \\
\end{array}
\]

with \( T(\alpha) = 1 \), \( \rho(\alpha) = (4, 1, 1) \), \( c(\alpha) = 1 \),

contributing

\[
a_{T(\alpha) - \rho(\alpha)T(\alpha)} - a_{T(\alpha) - c(\alpha)} = a_{-3} - a_0.
\]

\( T(\lambda, R_2) \) contains two barred tableaux with contributions

\[
\begin{array}{c|c|c}
3 & 1 \,
1 & & \\
\end{array}
\]

\( a_{-2} - a_2 \),

\[
\begin{array}{c|c|c}
3 & 1 \,
2 & & \\
\end{array}
\]

\( a_1 - a_3 \).
$\mathcal{T}(\lambda, R_1)$ contains one barred tableau

\[
\begin{array}{c|c}
\overline{3} & 1 \\
\hline
\overline{1} & 1
\end{array}
\]

with $T(\alpha) = 1$, $\rho(\alpha) = (4, 1, 1)$, $c(\alpha) = 1$,

contributing $a_{T(\alpha)-\rho(\alpha)} - a_{T(\alpha)-c(\alpha)} = a_3 - a_0$.

$\mathcal{T}(\lambda, R_2)$ contains two barred tableaux with contributions

\[
\begin{array}{c|c}
\overline{3} & \overline{1} \\
\hline
1 & 1
\end{array} \quad a_{-2} - a_2,
\quad \begin{array}{c|c}
\overline{3} & \overline{1} \\
\hline
2 & 1
\end{array} \quad a_1 - a_3.
\]

Hence $c_{\lambda\mu}^{\nu}(a) = a_3 - a_0 + a_{-2} - a_2 + a_1 - a_3$. 
Example. For the product of the double Schur functions \( s_{(2)}(x || a) \) and \( s_{(2,1)}(x || a) \) we have

\[
s_{(2)}(x || a) s_{(2,1)}(x || a) = s_{(4,1)}(x || a) + s_{(3,2)}(x || a) + s_{(3,1,1)}(x || a) + s_{(2,2,1)}(x || a)
+ (a_{-1} - a_0) s_{(2,1,1)}(x || a) + (a_{-1} - a_2) s_{(2,2)}(x || a)
+ (a_{-1} - a_2 + a_{-2} - a_0) s_{(3,1)}(x || a)
+ (a_{-1} - a_2) (a_{-1} - a_0) s_{(2,1)}(x || a).
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Example. For the product of the double Schur functions $s_{(2)}(x \| a)$ and $s_{(2,1)}(x \| a)$ we have

$$s_{(2)}(x \| a) s_{(2,1)}(x \| a)$$

$$= s_{(4,1)}(x \| a) + s_{(3,2)}(x \| a) + s_{(3,1,1)}(x \| a) + s_{(2,2,1)}(x \| a) + (a_{-1} - a_0) s_{(2,1,1)}(x \| a) + (a_{-1} - a_2) s_{(2,2)}(x \| a) + (a_{-1} - a_2 + a_{-2} - a_0) s_{(3,1)}(x \| a) + (a_{-1} - a_2) (a_{-1} - a_0) s_{(2,1)}(x \| a).$$
Example. For the product of the double Schur functions

\( s_{(2)}(x \parallel a) \) and \( s_{(2,1)}(x \parallel a) \) we have

\[
s_{(2)}(x \parallel a) s_{(2,1)}(x \parallel a) = s_{(4,1)}(x \parallel a) + s_{(3,2)}(x \parallel a) + s_{(3,1,1)}(x \parallel a) + s_{(2,2,1)}(x \parallel a) \\
+ (a_{-1} - a_0) s_{(2,1,1)}(x \parallel a) + (a_{-1} - a_2) s_{(2,2)}(x \parallel a) \\
+ (a_{-1} - a_2 + a_{-2} - a_0) s_{(3,1)}(x \parallel a) \\
+ (a_{-1} - a_2) (a_{-1} - a_0) s_{(2,1)}(x \parallel a).
\]
Example. For any diagram $\lambda$, 

$$c_{\lambda\lambda}(a) = \prod_{(i,j) \in \lambda} (a_i - \lambda_i - a_{\lambda'_j - j + 1}).$$
Example. For any diagram $\lambda$,

$$c_{\lambda\lambda}^\lambda(a) = \prod_{(i,j) \in \lambda} (a_{i-\lambda_i} - a_{\lambda'_j - j + 1}).$$

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Example. For any diagram $\lambda$,

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More generally,

$$c_{\lambda\mu}^{\mu}(a) = s_\lambda(a_\mu \parallel a), \quad a_\mu = (a_{1-\mu_1}, a_{2-\mu_2}, \ldots).$$
Example. For any diagram $\lambda$,

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More generally,

$$c_{\lambda\mu}^\mu(a) = s_\lambda(a_\mu \parallel a), \quad a_\mu = (a_{1-\mu_1}, a_{2-\mu_2}, \ldots).$$

The evaluation of $s_\lambda(x \parallel a)$ at $x = a_\mu$ is well-defined since $(a_\mu)_n = a_n$ for all sufficiently large $n$. 
Proof of the theorem. Calculate $c_{\lambda \mu}^\nu(a)$ by induction on $|\nu| - |\mu|$. 

Starting point: the Vanishing Theorem (Okounkov, 96):

$s_{\lambda}(a_{\rho}) = 0$ unless $\lambda \subseteq \rho$, implying the formula for $c_{\nu \lambda \mu}(a)$. 

Then use the recurrence:

$c_{\nu \lambda \mu}(a) = 1$ if $|a_{\nu}| - |a_{\mu}| = \sum_{\nu \rightarrow \mu} c_{\nu \lambda \mu} + (a) - \sum_{\nu \rightarrow \lambda} c_{\nu \lambda \mu}(a)$,

where $|a_{\nu}| - |a_{\mu}| = \sum_{i \geq 1} (a_{\nu})_i - (a_{\mu})_i$ (M. and Sagan, 99).
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implying the formula for $c_{\lambda\mu}^\nu(a)$. Then use the recurrence

$$c_{\lambda\mu}^\nu(a) = \frac{1}{|a_\nu| - |a_\mu|} \left( \sum_{\mu \rightarrow \mu^+} c_{\lambda\mu}^\nu(a) - \sum_{\nu^- \rightarrow \nu} c_{\lambda\mu}^{\nu^-}(a) \right),$$

where $|a_\nu| - |a_\mu| = \sum_{i \geq 1} \left( (a_\nu)_i - (a_\mu)_i \right)$ (M. and Sagan, 99).
Equivariant Schubert calculus
on the Grassmannian

The torus $T = (\mathbb{C}^*)^N$ acts naturally on $\text{Gr}_{n,N}$. The equivariant cohomology ring $H_T^*(\text{Gr}_{n,N})$ is a module over

$\mathbb{Z}[t_1, \ldots, t_N] = H_T^*(\{pt\})$. 
Equivariant Schubert calculus on the Grassmannian

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$$\mathbb{Z}[t_1, \ldots, t_N] = H^*_T(\{pt\}).$$

It has a basis of the equivariant Schubert classes $\sigma_\lambda$ parameterized by all diagrams $\lambda$ contained in the $n \times m$ rectangle, $m = N - n$. 
Let $x_1, \ldots, x_n$ denote the Chern roots of the dual $S^\vee$ of the tautological subbundle $S$ of the trivial bundle $\mathcal{C}^N_{\text{Gr}_{n,N}}$ so that for the total equivariant Chern class of $S$ we have

$$c^T(S) = \prod_{i=1}^{n} (1 - x_i).$$
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$$c^T(S) = \prod_{i=1}^{n} (1 - x_i).$$

Then $\sigma_\lambda = s_\lambda(x \parallel a)$ with $x = (x_1, \ldots, x_n, 0, \ldots)$,

$$a_{-m+1} = -t_1, \ldots, a_n = -t_N,$$

and $a_i = 0$ otherwise (Knutson and Tao, 03; Fulton, 07).
Corollary. We have

\[ \sigma_{\lambda} \sigma_{\mu} = \sum_{\nu} d_{\lambda\mu}^{\nu} \sigma_{\nu}, \]

where \( d_{\lambda\mu}^{\nu} = c_{\lambda\mu}^{\nu}(a) \) with the sequence \( a \) specialized as above.
Corollary. We have

\[ \sigma_\lambda \sigma_\mu = \sum_{\nu} d_{\lambda\mu}^{\nu} \sigma_\nu, \]

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The \( d_{\lambda\mu}^{\nu} \) are polynomials in the \( t_i - t_j, \ i > j \) with positive integer coefficients. (Positivity property, Graham 01).
Corollary. We have

\[ \sigma_\lambda \sigma_\mu = \sum_{\nu} d_{\lambda \mu}^\nu \sigma_\nu, \]

where \( d_{\lambda \mu}^\nu = c_{\lambda \mu}^\nu(a) \) with the sequence \( a \) specialized as above.

The \( d_{\lambda \mu}^\nu \) are polynomials in the \( t_i - t_j, i > j \) with positive integer coefficients. (Positivity property, Graham 01).

The coefficients \( d_{\lambda \mu}^\nu \), regarded as polynomials in the \( a_i \), are independent of \( n \) and \( m \), as soon as the inequalities

\[ n \geq \lambda'_1 + \mu'_1 \text{ and } m \geq \lambda_1 + \mu_1 \]

hold. (Stability property.)
Example. For any \( n \geq 3 \) and \( m \geq 4 \) we have

\[
\sigma(2) \sigma(2,1) = \sigma(4,1) + \sigma(3,2) + \sigma(3,1,1) + \sigma(2,2,1) \\
+ (t_m - t_{m-1}) \sigma(2,1,1) + (t_{m+2} - t_{m-1}) \sigma(2,2) \\
+ (t_{m+2} - t_{m-1} + t_m - t_{m-2}) \sigma(3,1) \\
+ (t_{m+2} - t_{m-1})(t_m - t_{m-1}) \sigma(2,1) .
\]
Example. For any $n \geq 3$ and $m \geq 4$ we have

\[ \sigma(2) \sigma(2,1) = \sigma(4,1) + \sigma(3,2) + \sigma(3,1,1) + \sigma(2,2,1) \]

\[ + (t_m - t_{m-1}) \sigma(2,1,1) + (t_{m+2} - t_{m-1}) \sigma(2,2) \]

\[ + (t_{m+2} - t_{m-1} + t_m - t_{m-2}) \sigma(3,1) \]

\[ + (t_{m+2} - t_{m-1})(t_m - t_{m-1}) \sigma(2,1). \]

Remark. The puzzle rule of Knutson and Tao (2003) was the first manifestly positive formula for the $d_{\lambda \mu}^\nu$. The stability property was not observed by Knutson and Tao, although it can be derived from their rule.
Quantum immanants

The symmetric group $\mathcal{S}_k$ acts in a natural way in the tensor space $(\mathbb{C}^n)^\otimes k$. We identify elements of $\mathcal{S}_k$ with the corresponding operators.
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For a diagram $\lambda$ with $\ell(\lambda) \leq n$ and $|\lambda| = k$ denote by $T_0$ the $\lambda$-tableau obtained by filling in the boxes by the numbers $1, \ldots, k$ from left to right in successive rows.
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For a diagram $\lambda$ with $\ell(\lambda) \leq n$ and $|\lambda| = k$ denote by $T_0$ the $\lambda$-tableau obtained by filling in the boxes by the numbers $1, \ldots, k$ from left to right in successive rows.

Let $R_\lambda$ and $C_\lambda$ denote the row symmetrizer and column antisymmetrizer of $T_0$ respectively.
By $c_{\lambda}(r)$ denote the content of the cell of $T_0$ occupied by $r$. 

*Introduce the matrix $E = \sum_{i,j=1}^{n} E_{ij} \otimes e_{ij} \in \mathbb{U}(\mathfrak{gl}_n) \otimes \text{End}(C_n)$ and define the quantum immanant $S_{\lambda}$ by $S_{\lambda} = \frac{1}{h(\lambda)} \text{tr}(E - c_{\lambda}(1)) \otimes \cdots \otimes (E - c_{\lambda}(k)) \cdot R_{\lambda}C_{\lambda}$, where $h(\lambda)$ is the product of the hooks of $\lambda$ (Okounkov, 96).*
By \( c_\lambda(r) \) denote the content of the cell of \( T_0 \) occupied by \( r \).

Introduce the matrix

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\]

and define the quantum immanant \( S_\lambda \) by

\[
S_\lambda = \frac{1}{h(\lambda)} \text{tr} (E - c_\lambda(1)) \otimes \cdots \otimes (E - c_\lambda(k)) \cdot R_\lambda C_\lambda,
\]

where \( h(\lambda) \) is the product of the hooks of \( \lambda \) (Okounkov, 96).
Examples. Capelli elements (quantum minors)

\[ S_{(1^k)} = \sum_{a_1 < \cdots < a_k} \sum_{p \in \mathcal{S}_k} \text{sgn } p \cdot E_{a_1, a_{p(1)}} \cdots (E + k - 1)_{a_k, a_{p(k)}}. \]
Examples. Capelli elements (quantum minors)

\[ S_{(1^k)} = \sum_{a_1 < \cdots < a_k} \sum_{p \in S_k} \text{sgn } p \cdot E_{a_1,a_p(1)} \cdots (E + k - 1)_{a_k,a_p(k)}. \]

Quantum permanents

\[ S_{(k)} = \sum_{a_1 \leq \cdots \leq a_k} \frac{1}{\alpha_1! \cdots \alpha_n!} \sum_{p \in S_k} E_{a_1,a_p(1)} \cdots (E - k + 1)_{a_k,a_p(k)}, \]

where \( \alpha_i \) is the multiplicity of \( i \) in \( a_1, \ldots, a_k \), each 
\( a_r \in \{1, \ldots, n\} \).
The quantum immanants $S_\lambda$ with $\ell(\lambda) \leq n$ form a basis of the center of the universal enveloping algebra $U(\mathfrak{gl}_n)$. 

Corollary. $f_{\nu}^{\lambda \mu} = c_{\nu}^{\lambda \mu}(a)$ for the specialization $a_i = -i$ for $i \in \mathbb{Z}$. 

The quantum immanants \( S_\lambda \) with \( \ell(\lambda) \leq n \) form a basis of the center of the universal enveloping algebra \( U(\mathfrak{gl}_n) \).

Define the coefficients \( f_{\nu}^{\lambda\mu} \) by the expansion

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The quantum immanants $S_\lambda$ with $\ell(\lambda) \leq n$ form a basis of the center of the universal enveloping algebra $U(\mathfrak{gl}_n)$.

Define the coefficients $f_{\lambda \mu}^\nu$ by the expansion

$$S_\lambda S_\mu = \sum_\nu f_{\lambda \mu}^\nu S_\nu.$$

Corollary. $f_{\lambda \mu}^\nu = c_{\lambda \mu}^\nu (a)$ for the specialization $a_i = -i$ for $i \in \mathbb{Z}$. 
The coefficient $f_{\lambda\mu}^\nu$ is zero unless $\lambda, \mu \subseteq \nu$. If $\lambda, \mu \subseteq \nu$ then

$$f_{\lambda\mu}^\nu = \sum_{R} \sum_{T} \prod_{\alpha \in \lambda} \left( \rho(\alpha)_{T(\alpha)} - c(\alpha) \right),$$

summed over all sequences $R$ from $\mu$ to $\nu$ and all $\nu$-bounded reverse $\lambda$-tableaux $T \in T(\lambda, R)$. In particular, the $f_{\lambda\mu}^\nu$ are nonnegative integers.
Example. For any $n \geq 3$ we have

$$S(2) S(2, 1) = S(4, 1) + S(3, 2) + S(3, 1, 1) + S(2, 2, 1)$$

$$+ S(2, 1, 1) + 5 S(3, 1) + 3 S(2, 2) + 3 S(2, 1).$$
Example. For any $n \geq 3$ we have

$$S(2) S(2,1) = S(4,1) + S(3,2) + S(3,1,1) + S(2,2,1)$$

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If $n = 2$ then

$$S(2) S(2,1) = S(4,1) + S(3,2) + 5 S(3,1) + 3 S(2,2) + 3 S(2,1).$$