Feigin–Frenkel center for classical types

Alexander Molev

University of Sydney
Affine Kac–Moody algebras
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Consider the standard invariant bilinear form on \( \mathfrak{g} \)

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where \( h^\vee \) is the dual Coxeter number.
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where $h^\vee$ is the dual Coxeter number.

For the classical types,

$$h^\vee = \begin{cases} 
  n & \text{for } \mathfrak{g} = \mathfrak{sl}_n, \\
  N - 2 & \text{for } \mathfrak{g} = \mathfrak{o}_N, \\
  n + 1 & \text{for } \mathfrak{g} = \mathfrak{sp}_{2n}. 
\end{cases}$$
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with the commutation relations

$$[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r,-s} \langle X, Y \rangle K,$$

where $X[r] = X t^r$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$. 
The affine Kac–Moody algebra $\hat{g}$ is the central extension

$$\hat{g} = g[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations

$$[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r,-s} \langle X, Y \rangle K,$$

where $X[r] = X t^r$ for any $X \in g$ and $r \in \mathbb{Z}$.

The vacuum module at the critical level $V(g)$ over $\hat{g}$ is the quotient of the universal enveloping algebra $U(\hat{g})$ by the left ideal generated by $g[t]$ and $K + h^\vee$. 
The Feigin–Frenkel center $\mathfrak{z}(\hat{g})$ is the algebra $\mathfrak{z}(\hat{g}) = \text{End}_{\hat{g}} V(g)$. Equivalently, $\mathfrak{z}(\hat{g}) = V(g) \{ t \} = \{ v \in V(g) | g[t] v = 0 \}$. The algebra $\mathfrak{z}(\hat{g})$ is commutative.
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The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative.
As a vector space, the vacuum module $V(\mathfrak{g})$ can be identified with the universal enveloping algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$. 

Define the translation operator $T: V(\mathfrak{g}) \to V(\mathfrak{g})$ as the derivation $T = -\partial_t$. 

The subspace $z(\hat{\mathfrak{g}})$ of $V(\mathfrak{g})$ is $T$-invariant. Any element of $z(\hat{\mathfrak{g}})$ is called a Segal–Sugawara vector.
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There exist Segal–Sugawara vectors $S_1, \ldots, S_n \in \mathbf{U}(t^{-1} g[t^{-1}])$ such that

$$\mathfrak{z}(\hat{g}) = \mathbb{C} [T^k S_l \mid l = 1, \ldots, n, \quad k \geq 0],$$

where $n = \text{rank } g$ and the symbols $S_1, \ldots, S_n$ coincide with the images of certain algebraically independent generators of the algebra of invariants $\mathbf{S}(g)$ under the embedding $\mathbf{S}(g) \hookrightarrow \mathbf{S}(t^{-1} g[t^{-1}])$ defined by $X \mapsto X[-1]$. We call $S_1, \ldots, S_n$ a complete set of Segal–Sugawara vectors.
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We call $S_1, \ldots, S_n$ a complete set of Segal–Sugawara vectors.
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Let \( P = P(Y_1, \ldots, Y_l) \) be a \( g \)-invariant in \( S(g) \).

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Then each \( P(r) \) is a \( g[t] \)-invariant in \( S(t^{-1}g[t^{-1}]) \).

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Moreover, \( k! P(-k-1) = T^k P(Y_1[-1], \ldots, Y_l[-1]) \) for \( k \geq 0 \).
Theorem (Beilinson–Drinfeld, 1997). If $P_1, \ldots, P_n$ are algebraically independent generators of $S(g)^g$, then the elements $P_{1,(r)}, \ldots, P_{n,(r)}$ with $r < 0$ are algebraically independent generators of $S(t^{-1}g[t^{-1}])^g[t]$. Earlier work: R. Goodman and N. Wallach, 1989, type $A$; T. Hayashi, 1988, types $A, B, C$; V. Kac and D. Kazhdan, 1979.
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Explicit formulas for Segal–Sugawara vectors

They will lead, in particular, to a simpler proof of the Feigin–Frenkel theorem for classical types.

We will need the extended Lie algebra \( \hat{g} \oplus \mathbb{C} \tau \), where for the element \( \tau = -\partial_t \) we have the relations

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\left[ \tau, X^r \right] = -r X^{r-1},
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Type A

Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$, the standard basis $\{E_{ij} | i, j = 1, \ldots, n\}$.

Consider the $n \times n$ matrix $\tau + E_{[-1]}$ given by

\[
\begin{pmatrix}
\tau + E_{11}[(-1)] & E_{12}[(-1)] & \cdots & E_{1n}[(-1)] \\
E_{21}[(-1)] & \tau + E_{22}[(-1)] & \cdots & E_{2n}[(-1)] \\
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Theorem (Chervov–Talalaev, 2006; also Chervov–M., 2009).

The coefficients $S_1, \ldots, S_n$ of the polynomial

$$\text{cdet}(\tau + E[-1]) = \tau^n + S_1 \tau^{n-1} + \cdots + S_{n-1} \tau + S_n$$

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Example. For $n = 2$

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cdet(\tau + E[-1]) = (\tau + E_{11}[-1])(\tau + E_{22}[-1]) - E_{21}[-1]E_{12}[-1]
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with

$$
S_1 = E_{11}[-1] + E_{22}[-1],
$$

$$
S_2 = E_{11}[-1]E_{22}[-1] - E_{21}[-1]E_{12}[-1] + E_{22}[-2].
$$
Corollary. For any $k \geq 0$ all coefficients $P_{kl}$ in the expansion

$$\text{tr}(\tau + E[-1])^k = P_{k0} \tau^k + P_{k1} \tau^{k-1} + \cdots + P_{kk}$$

are Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$. 

Remark. These results generalize to the Lie superalgebra $\mathfrak{gl}_m|_n$. The column-determinant is replaced by a noncommutative Berezinian (M.–Ragoucy, 2009).
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Moreover, the elements $P_{11}, \ldots, P_{nn}$ form a complete set of Segal–Sugawara vectors.
Corollary. For any \( k \geq 0 \) all coefficients \( P_{kl} \) in the expansion

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Moreover, the elements \( P_{11}, \ldots, P_{nn} \) form a complete set of Segal–Sugawara vectors.

Remark. These results generalize to the Lie superalgebra \( \mathfrak{gl}_{m|n} \).

The column-determinant is replaced by a noncommutative Berezinian (M.–Ragoucy, 2009).
Types $B$, $C$ and $D$

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The orthogonal Lie algebra $\mathfrak{o}_N$ of skew-symmetric matrices is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij} = E_{ij} - E_{ji}$.

Denote by $F$ the $N \times N$ matrix whose $(i,j)$ entry is $F_{ij}$. Regard $F$ as the element

$$F = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij} \in \text{End } \mathbb{C}^N \otimes \text{U}(\mathfrak{o}_N).$$
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Introduce elements of $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \cong \text{End } (\mathbb{C}^N \otimes \mathbb{C}^N)$ by

$$P = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}, \quad Q = \sum_{i,j=1}^N e_{ij} \otimes e_{ij}.$$
The defining relations of the algebra $U(\sigma_N)$ have the form

$$F_1 F_2 - F_2 F_1 = (P - Q) F_2 - F_2 (P - Q)$$

together with the relation $F + F^t = 0,$
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\[ F_1 F_2 - F_2 F_1 = (P - Q) F_2 - F_2 (P - Q) \]

together with the relation $F + F^t = 0$, where both sides are regarded as elements of the algebra $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes U(\mathfrak{o}_N)$ and

\[ F_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes F_{ij}, \quad F_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes F_{ij}. \]
In the affine Kac–Moody algebra $\hat{\mathfrak{o}}_N = \mathfrak{o}_N[t, t^{-1}] \oplus \mathbb{C} K$ set

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\]

The defining relations of the algebra \( U(\hat{\mathfrak{o}}_N) \) can be written as
\[
F[r_1] F[s_2] - F[s_2] F[r_1] = (P - Q) F[r + s_2] - F[r + s_2] (P - Q)
+ r \delta_{r,-s} (P - Q) K.
\]
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\end{bmatrix}.
$$

Note that

$$
F_{ij}[-1] + F_{ji}[-1] = 0.
$$
For each \( a \in \{1, \ldots, m\} \) define the element \( \Phi_a \) of the algebra

\[
\text{End } \mathbb{C}^N \otimes \ldots \otimes \text{End } \mathbb{C}^N \otimes U(\hat{\mathfrak{o}}_N \oplus \mathbb{C} \tau)
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\Phi_a = \sum_{i,j=1}^{N} 1^\otimes(a-1) \otimes e_{ij} \otimes 1^\otimes(m-a) \otimes \Phi_{ij},
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where \( \Phi_{ij} = \delta_{ij}\tau + F_{ij}[-1] \).
For each $a \in \{1, \ldots, m\}$ define the element $\Phi_a$ of the algebra

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where $\Phi_{ij} = \delta_{ij}\tau + F_{ij}[-1]$. 

The trace map $\text{tr} : \text{End } \mathbb{C}^N \rightarrow \mathbb{C}$ is defined by $\text{tr} : e_{ij} \mapsto \delta_{ij}$.
Introduce the element $S^{(m)}$ of the algebra

$$\text{End } \mathbb{C}^N \otimes \ldots \otimes \text{End } \mathbb{C}^N$$

by $S^{(m)} = 1_n$, the product is taken in the lexicographic order on the pairs $(a, b)$, and $P_{ab}$ and $Q_{ab}$ act as the respective operators $P$ and $Q$ in the $a$-th and $b$-th copies of $\mathbb{C}^N$ and as the identity operators in all the remaining copies.
Introduce the element $S^{(m)}$ of the algebra

$$\text{End } \mathbb{C}^N \otimes \ldots \otimes \text{End } \mathbb{C}^N$$

by

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leq a < b \leq m} \left( 1 + \frac{P_{ab}}{b - a} - \frac{Q_{ab}}{N/2 + b - a - 1} \right),$$

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Properties: for $1 \leq a < b \leq m$ we have

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Equivalent formula:

\[ S^{(m)} = \frac{1}{m!} \prod_{1 \leq a < b \leq m} \left( 1 - \frac{Q_{ab}}{N + a + b - 3} \right) \prod_{1 \leq a < b \leq m} \left( 1 + \frac{P_{ab}}{b - a} \right). \]

Remark. $S^{(m)}$ is the idempotent associated with the trivial representation of the Brauer algebra $B_m^\lambda$. In particular, $(S^{(m)})^2 = S^{(m)}$. 
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Remark. $S^{(m)}$ is the idempotent associated with the trivial representation of the Brauer algebra $\mathcal{B}_m(N)$. In particular, $(S^{(m)})^2 = S^{(m)}$. 
In a reduced form,

\[ S^{(m)} = H^{(m)} \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{(-1)^r}{2^r r!} \binom{N/2 + m - 2}{r}^{-1} \sum_{a_i < b_i} Q_{a_1 b_1} \cdots Q_{a_r b_r}, \]

where \( H^{(m)} \) is the symmetrizer in the group algebra \( \mathbb{C}[S_m] \).
In a reduced form,

\[ S^{(m)} = H^{(m)} \sum_{r=0}^{\lfloor m/2 \rfloor} \frac{(-1)^r}{2^r r!} \left( \frac{N}{2} + m - 2 \right)^{-1} \sum_{a_i < b_i} Q_{a_1 \ b_1} \cdots Q_{a_r \ b_r}, \]

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In terms of the Jucys–Murphy elements:

\[ S^{(m)} = \prod_{b=2}^{m} \frac{1}{b(N + 2b - 4)} \left( 1 + \sum_{a=1}^{b-1} (P_{ab} - Q_{ab}) \right) \times \left( N + b - 3 + \sum_{a=1}^{b-1} (P_{ab} - Q_{ab}) \right). \]
Theorem. The elements $\phi_{ma} \in U(t^{-1}o_N[t^{-1}])$ defined by

$$\text{tr} S^{(m)} \Phi_1 \ldots \Phi_m = \phi_{m0} \tau^m + \phi_{m1} \tau^{m-1} + \cdots + \phi_{mm}$$

are Segal–Sugawara vectors for $o_N$. 
Theorem. The elements $\phi_{ma} \in U(t^{-1}o_N[t^{-1}])$ defined by

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Moreover, $\phi_{22}, \phi_{44}, \ldots, \phi_{2n\cdot2n}$ is a complete set of Segal–Sugawara vectors for $o_{2n+1}$. 
Theorem. The elements $\phi_{ma} \in U(t^{-1} \mathfrak{o}_N[t^{-1}])$ defined by

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Moreover, $\phi_{22}, \phi_{44}, \ldots, \phi_{2n \cdot 2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$,

$\phi_{22}, \phi_{44}, \ldots, \phi_{2n - 2 \cdot 2n - 2}, \phi'_n$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$, where $\phi'_n = \text{Pf } F[-1]$ is the Pfaffian of the skew-symmetric matrix $F[-1]$. 
Example. For \( m = 2 \) we have

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S^{(2)} = \frac{1 + P}{2} - \frac{Q}{N}.
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Hence, $\phi_{22}$ is found from

$$\text{tr} S^{(2)} \Phi_1 \Phi_2 = \frac{1}{2} \left( \text{tr} (\tau + F[-1]) \right)^2 + \frac{1}{2} \text{tr} (\tau + F[-1])^2$$

$$- \frac{1}{N} \text{tr} (\tau - F[-1])(\tau + F[-1]).$$
Example. For $m = 2$ we have

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- \frac{1}{N} \text{tr} (\tau - F[-1])(\tau + F[-1])
= \frac{N + 2}{2N} \left( (N^2 - N) \tau^2 + \text{tr} F[-1]^2 \right).
\]
In the case of $\mathfrak{o}_{2n}$ the Pfaffian $\text{Pf} F[-1]$ is

$$\text{Pf} F[-1] = \sum_{\sigma} \text{sgn} \sigma \cdot F_{\sigma(1)\sigma(2)}[-1] \cdots F_{\sigma(2n-1)\sigma(2n)}[-1],$$

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summed over the permutations $\sigma \in \mathfrak{S}_{2n}$ such that

$$\sigma(1) < \sigma(2), \quad \sigma(3) < \sigma(4), \quad \ldots, \quad \sigma(2n−1) < \sigma(2n)$$

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$\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

**Example.** For $\sigma_4$ we have

For the proof of the theorem we show that

\[ F[0]_0 \, \text{tr} \, S^{(m)} \Phi_1 \ldots \Phi_m = 0 \quad \text{and} \quad F[1]_0 \, \text{tr} \, S^{(m)} \Phi_1 \ldots \Phi_m = 0 \]
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in the module

\[
\underbrace{\text{End } \mathbb{C}^N \otimes \ldots \otimes \text{End } \mathbb{C}^N}_{m+1} \otimes V(\mathfrak{o}_N)[\tau]
\]

with the copies of \( \text{End } \mathbb{C}^N \) labelled by \( 0, 1, \ldots, m \).
For the symbols of the Segal–Sugawara vectors $\phi_{2k2k}$ find

$$\text{tr } S^{(2k)} X_1 \ldots X_{2k}, \quad X \in \mathfrak{o}_N.$$
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Suppose that the eigenvalues of $X$ are

$$x_1, \ldots, x_n, -x_1, \ldots, -x_n, 0 \quad \text{if} \quad N = 2n + 1,$$

$$x_1, \ldots, x_n, -x_1, \ldots, -x_n \quad \text{if} \quad N = 2n.$$
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$$x_1, \ldots, x_n, -x_1, \ldots, -x_n \quad \text{if} \quad N = 2n.$$ 

Then

$$\text{tr } S^{(2k)} X_1 \ldots X_{2k} = \frac{N + 4k - 2}{N + 2k - 2} h_k(x_1^2, \ldots, x_n^2),$$

$h_k$ is the complete symmetric polynomial.
Vertex algebra structure

The vacuum module $V(g)$ is a vertex algebra with the vacuum vector $1$, the translation operator $T = -\partial_t$, and the state-field correspondence $Y$ which is a linear map $Y: V(g) \to \text{End} V(g)[[z, z^{-1}]]$.

It is determined by

$$Y(X[-1], z) = \sum_{r \in \mathbb{Z}} X[r] z^{-r-1} =: X(z).$$
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It is determined by

$$Y(X[-1], z) = \sum_{r \in \mathbb{Z}} X[r] z^{-r-1} =: X(z).$$
For any $r_i \geq 0$ we have

$$Y(X_1[-r_1 - 1] \ldots X_m[-r_m - 1], z) = \frac{1}{r_1! \ldots r_m!} : \partial_z^{r_1} X_1(z) \ldots \partial_z^{r_m} X_m(z) :,$$

with the convention that the normally ordered product is read from right to left;
For any $r_i \geq 0$ we have

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$$: a(z) b(w) : = a(z) + b(w) + b(w) a(z)_-, $$
For any $r_i \geq 0$ we have

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with the convention that the normally ordered product is read from right to left;

$$: a(z) b(w) : = a(z)_{+} b(w) + b(w) a(z)_{-},$$

where

$$a(z)_{+} = \sum_{r \geq 0} a_r z^r \quad \text{and} \quad a(z)_{-} = \sum_{r < 0} a_r z^r.$$
Suppose that $S_1, \ldots, S_n$ is a complete set of Segal–Sugawara vectors in $\hat{\mathfrak{g}}(\mathfrak{g})$. Apply the state-field correspondence map:

$$Y(S_l, z) = \sum_{r \in \mathbb{Z}} S_{l,r} z^{-r-1}.$$
Suppose that $S_1, \ldots, S_n$ is a complete set of Segal–Sugawara vectors in $\mathfrak{z}(\widehat{\mathfrak{g}})$. Apply the state-field correspondence map:

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The elements $S_{l,r}$ are Sugawara operators for $\widehat{\mathfrak{g}}$. They generate the center of the completed algebra $U(\widehat{\mathfrak{g}})$ at the critical level.
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Applications: Singular vectors in Verma modules and Weyl modules over $\hat{\mathfrak{g}}$ (E. Frenkel and D. Gaitsgory, 2006, 2007).
Example.

Apply $Y$ to the Segal–Sugawara vector $\text{tr} F[-1]^2$ for $\hat{o}_N$: 
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Example.

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= \sum_{i,j=1}^{N} \left( F_{ij}(z) + F_{ji}(z) + F_{ji}(z) F_{ij}(z)_{-} \right) = \sum_{p \in \mathbb{Z}} S_p z^{-p-2}.
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The $S_p$ are the Sugawara operators

$$S_p = \sum_{i,j=1}^{N} \left( \sum_{r < 0} F_{ij}[r] F_{ji}[p - r] + \sum_{r \geq 0} F_{ji}[p - r] F_{ij}[r] \right)$$

commuting with $\hat{o}_N$. 
Apply the state-field correspondence map $Y$: 

$$\text{cdet}(\tau + E[-1]) \mapsto \text{cdet}(\partial z + E(z)),$$

where $E_{ij}(z) = \sum_{r \in Z} E_{ij}[r]z^{r-1}$ and 

$$
\begin{bmatrix}
\partial z + E_{11}(z) & E_{12}(z) & \ldots & E_{1n}(z) \\
E_{21}(z) & \partial z + E_{22}(z) & \ldots & E_{2n}(z) \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1}(z) & E_{n2}(z) & \ldots & \partial z + E_{nn}(z)
\end{bmatrix}.
$$
Apply the state-field correspondence map

\[ Y : \text{cdet}(\tau + E[-1]) \leftrightarrow \text{cdet}(\partial_z + E(z)) : \]
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Type A

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and

\[ \partial_z + E(z) = \begin{bmatrix}
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E_{21}(z) & \partial_z + E_{22}(z) & \ldots & E_{2n}(z) \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1}(z) & E_{n2}(z) & \ldots & \partial_z + E_{nn}(z)
\end{bmatrix}. \]
Expand the normally ordered column-determinant

\[ : \text{cdet}(\partial_z + E(z)) : = \partial_z^n + S_1(z) \partial_z^{n-1} + \cdots + S_{n-1}(z) \partial_z + S_n(z) \].
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The coefficients $S_{l,r}$ of the $S_l(z)$ are Sugawara operators for $\hat{\mathfrak{gl}}_n$. 
Expand the normally ordered column-determinant

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The coefficients \( S_{l,r} \) of the \( S_l(z) \) are Sugawara operators for \( \hat{\mathfrak{gl}}_n \).

Using the vacuum axiom

\[ : \text{cdet}(\partial_z + E(z)) : 1 = \text{cdet}(\partial_z + E(z)_+), \]

we get explicit generators of \( \mathfrak{gl}_n \) and hence, generators of the commutative subalgebra of \( U(t^{-1} \mathfrak{gl}_n[t^{-1}]) \).
Types $B$, $C$ and $D$
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Apply the state-field correspondence map

$$Y : \text{tr} \ S^{(m)} \Phi_1 \ldots \Phi_m \mapsto \text{tr} \ S^{(m)} (\partial_z + F_1(z)) \ldots (\partial_z + F_m(z)) :$$
Types $B$, $C$ and $D$

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Types $B$, $C$ and $D$

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$$\partial_z + F(z) = \begin{bmatrix}
\partial_z & F_{12}(z) & \ldots & F_{1N}(z) \\
F_{21}(z) & \partial_z & \ldots & F_{2N}(z) \\
\vdots & \vdots & \ddots & \vdots \\
F_{N1}(z) & F_{N2}(z) & \ldots & \partial_z
\end{bmatrix}.$$
Expand into a polynomial in $\partial_z$:

$$: \text{tr} S^{(m)} \left( \partial_z + F_1(z) \right) \cdots \left( \partial_z + F_m(z) \right) :$$

$$= f_{m0}(z) \partial_z^m + f_{m1}(z) \partial_z^{m-1} + \cdots + f_{mm}(z).$$
Expand into a polynomial in $\partial_z$:

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All coefficients of the $f_{ma}(z)$ are Sugawara operators for $\hat{o}_N$. 
Expand into a polynomial in $\partial_z$:

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All coefficients of the $f_{ma}(z)$ are Sugawara operators for $\hat{o}_N$.

Applying them to the vacuum vector, we get explicit generators of the Feigin–Frenkel center $\mathfrak{z}(\hat{o}_N)$, and hence, generators of the commutative subalgebra of $U(t^{-1}o_N[t^{-1}])$. 
Introduce the matrix $F(z) = [F_{ij}(z)]$ and set $L(z) = \partial_z - F(z)$,

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**Corollary.** The coefficients of all series $l_{ma}(z)$ with $m = 1, 2, \ldots$

defined by the decompositions

$$\text{tr} S^{(m)} L_1(z) \ldots L_m(z) = l_{m0}(z) \partial_z^m + l_{m1}(z) \partial_z^{m-1} + \cdots + l_{mm}(z),$$

generate a commutative subalgebra of $U(o_N[t])$. 
Pfaffian-type Sugawara operators

In type D, $PfF[-1] \mapsto PfF(z)$ (no normal ordering).

Taking the coefficients of the powers of $z$ we get Sugawara operators $S_r, r \in \mathbb{Z}$, of the form

$$S_r = \sum_{r_1 + \cdots + r_n = r} \prod_{\sigma \in S_n} \text{sgn} \sigma \cdot F_{\sigma}(1)_{\sigma}(2)_{\sigma}(2n-1)_{\sigma}(2n).$$
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Harmonic polynomials

The operator $S(m)$ projects the vector space $(\mathbb{C}^N)^\otimes m$ to a subspace of the space of symmetric tensors, which carries an irreducible representation of the orthogonal group $O_N$.

Identify symmetric tensors with polynomials in variables $x_1, \ldots, x_N$. Then the subspace $S(m) (\mathbb{C}^N)^\otimes m$ is isomorphic to the space $H^N_m$ of harmonic polynomials of degree $m$.

These are polynomials annihilated by the Laplace operator $\partial^2_1 + \cdots + \partial^2_N$. 
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Harmonic polynomials

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Identify symmetric tensors with polynomials in variables $x_1, \ldots, x_N$. Then the subspace $S^{(m)}(\mathbb{C}^N)^\otimes m$ is isomorphic to the space $\mathcal{H}^m_N$ of harmonic polynomials of degree $m$.

These are polynomials annihilated by the Laplace operator $\partial_1^2 + \cdots + \partial_N^2$. 
The operator $S^{(m)}$ coincides with the restriction of the extremal projector $p : \mathbb{C}[x_1, \ldots, x_N] \rightarrow \mathcal{H}_N$ to the subspace of homogeneous polynomials of degree $m$, where
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$$\mathbb{C}[x_1, \ldots, x_N] = \mathcal{H}_N \oplus (x_1^2 + \cdots + x_N^2) \mathbb{C}[x_1, \ldots, x_N].$$
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**Remark.** The operator $p$ is associated with the action of $\mathfrak{sl}_2$ commuting with that of $O_N$ via the special case of Howe duality:
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**Remark.** The operator $p$ is associated with the action of $\mathfrak{sl}_2$ commuting with that of $O_N$ via the special case of Howe duality:

$$e \mapsto -\frac{1}{2} \sum_{i=1}^{N} \partial^2_i, \quad f \mapsto \frac{1}{2} \sum_{i=1}^{N} x_i^2, \quad h \mapsto -\frac{N}{2} - \sum_{i=1}^{N} x_i \partial_i,$$
The operator \( S^{(m)} \) coincides with the restriction of the extremal projector \( p : \mathbb{C}[x_1, \ldots, x_N] \rightarrow \mathcal{H}_N \) to the subspace of homogeneous polynomials of degree \( m \), where

\[
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\]

**Remark.** The operator \( p \) is associated with the action of \( \mathfrak{sl}_2 \) commuting with that of \( O_N \) via the special case of Howe duality:

\[
e \mapsto -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2, \quad f \mapsto \frac{1}{2} \sum_{i=1}^{N} x_i^2, \quad h \mapsto -\frac{N}{2} - \sum_{i=1}^{N} x_i \partial_i,
\]

and \( p \) satisfies \( e p = p f = 0 \).
Corollary. The Segal–Sugawara vectors $\phi_{mk}$ can be found from the expansion

$$\text{tr} p \Phi^{(m)}|_{\mathcal{H}^{m}_N} = \phi_{m0} \tau^m + \phi_{m1} \tau^{m-1} + \cdots + \phi_{mm}$$

with the trace taken over the subspace $\mathcal{H}^{m}_N$. 
Corollary. The Segal–Sugawara vectors $\phi_{mk}$ can be found from the expansion

$$\text{tr} \ p \Phi^{(m)}|_{\mathcal{H}_N^m} = \phi_{m0} \tau^m + \phi_{m1} \tau^{m-1} + \cdots + \phi_{mm}$$

with the trace taken over the subspace $\mathcal{H}_N^m$,

$$\Phi^{(m)} : x_{j_1} \cdots x_{j_m} \mapsto \sum_{i_1 \leq \cdots \leq i_m} x_{i_1} \cdots x_{i_m} \otimes \Phi_{i_1, \ldots, i_m}$$
Corollary. The Segal–Sugawara vectors $\phi_{mk}$ can be found from the expansion

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where

$$\Phi_{j_1, \ldots, j_m}^{i_1, \ldots, i_m} = \frac{1}{\alpha_1! \cdots \alpha_N! m!} \sum_{\sigma, \pi \in S_m} \Phi_{i_{\sigma(1)}j_{\pi(1)}} \cdots \Phi_{i_{\sigma(m)}j_{\pi(m)}}$$

and $\alpha_i$ is the multiplicity of $i$ in the multiset $\{i_1, \ldots, i_m\}$. 