Classical Lie algebras and Yangians

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Recall \( E = [E_{ij}] \) with \( i, j \in \{1, \ldots, N\} \). We have

\[
[E_{ij}, (E^s)_{kl}] = \delta_{kj}(E^s)_{il} - \delta_{il}(E^s)_{kj}.
\]

This implies that \( \text{tr} \ E^s \) are Casimir elements for \( \mathfrak{gl}_N \) (the Gelfand invariants).
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This implies that $\text{tr} E^s$ are Casimir elements for $\mathfrak{gl}_N$ (the Gelfand invariants).

More generally, we have

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj}(E^s)_{il} - (E^s)_{kj}(E^r)_{il},$$

where $r, s \geq 0$ and $E^0 = 1$ is the identity matrix.
Yangian for $\mathfrak{gl}_N$

Definition

The Yangian for $\mathfrak{gl}_N$ is the associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $i, j = 1, \ldots, N$, and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)},$$

where $r, s = 0, 1, \ldots$ and $t_{ij}^{(0)} = \delta_{ij}$. 
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where $r, s = 0, 1, \ldots$ and $t_{ij}^{(0)} = \delta_{ij}$.

This algebra is denoted by $Y(\mathfrak{gl}_N)$. 
Introduce the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y(gl_N)[[u^{-1}]].$$

The defining relations take the form

$$(u - v) [t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u).$$
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The defining relations are equivalent to

\[ [t^{(r)}_{ij}, t^{(s)}_{kl}] = \sum_{a=1}^{\min\{r,s\}} \left( t^{(a-1)}_{kj} t^{(r+s-a)}_{il} - t^{(r+s-a)}_{kj} t^{(a-1)}_{il} \right). \]
Proposition

The assignment

$$\pi_N : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}$$

defines a surjective homomorphism $Y(gl_N) \to U(gl_N)$. Moreover, the assignment

$$E_{ij} \mapsto t_{ij}^{(1)}$$

defines an embedding $U(gl_N) \hookrightarrow Y(gl_N)$. 
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We may regard \( U(\mathfrak{gl}_N) \) as a subalgebra of \( Y(\mathfrak{gl}_N) \).
Matrix form of the defining relations

Introduce the $N \times N$ matrix $T(u)$ whose $ij$-th entry is the series $t_{ij}(u)$. We regard $T(u)$ as an element of the algebra $\text{End } \mathbb{C}^N \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$. Then

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u),$$

where $e_{ij} \in \text{End } \mathbb{C}^N$ are the standard matrix units.
For any positive integer $m$ consider the algebra

$$(\text{End } \mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{gl}_N).$$

For any $a \in \{1, \ldots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the $a$-th copy of the algebra $\text{End } \mathbb{C}^N$ in the tensor product algebra.
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$$(\text{End} \mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{gl}_N).$$

For any $a \in \{1, \ldots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the $a$-th copy of the algebra $\text{End} \mathbb{C}^N$ in the tensor product algebra. That is, $T_a(u)$ is a formal power series in $u^{-1}$ given by

$$T_a(u) = \sum_{i,j=1}^{N} 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (m-a)} \otimes t_{ij}(u),$$

where $1$ is the identity matrix.
If
\[ C = \sum_{i,j,k,l=1}^{N} c_{ijkl} e_{ij} \otimes e_{kl} \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N, \]

then for any two indices \( a, b \in \{1, \ldots, m\} \) such that \( a < b \), define the element \( C_{ab} \) of the algebra \((\text{End } \mathbb{C}^N)^{\otimes m}\) by

\[ C_{ab} = \sum_{i,j,k,l=1}^{N} c_{ijkl} 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (b-a-1)} \otimes e_{kl} \otimes 1^{\otimes (m-b)}. \]

The tensor factors \( e_{ij} \) and \( e_{kl} \) belong to the \( a \)-th and \( b \)-th copies of \( \text{End } \mathbb{C}^N \), respectively.
Consider now the permutation operator

\[ P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N. \]
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The rational function

\[ R(u) = 1 - Pu^{-1} \]

with values in \( \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \) is called the Yang \( R \)-matrix.
Proposition

In the algebra \( (\text{End} \mathbb{C}^N)^{\otimes 3} (u, v) \) we have the identity

\[
R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u).
\]
Proposition

In the algebra \((\text{End } \mathbb{C}^N)^\otimes 3(u, \nu)\) we have the identity

\[ R_{12}(u) R_{13}(u + \nu) R_{23}(\nu) = R_{23}(\nu) R_{13}(u + \nu) R_{12}(u). \]

This relation is known as the Yang–Baxter equation. The Yang \(R\)-matrix is its simplest nontrivial solution.
Proposition

The defining relations of the algebra $\mathcal{Y}(\mathfrak{gl}_N)$ can be written in the equivalent form

$$ R(u - v) \, T_1(u) \, T_2(v) = T_2(v) \, T_1(u) \, R(u - v). $$
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$$ R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v). $$

Here $T_1(u)$ and $T_2(v)$ as formal power series with the coefficients in the algebra

$$ \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes Y(\mathfrak{gl}_N). $$
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The matrix relation is called the \textit{RTT} relation (or ternary relation).
Symmetries of $Y(\mathfrak{gl}_N)$

Let $f(u)$ be a formal power series in $u^{-1}$ of the form

\[ f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]. \]

Let $c \in \mathbb{C}$ and let $B$ be any nonsingular complex $N \times N$ matrix.
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Let $c \in \mathbb{C}$ and let $B$ be any nonsingular complex $N \times N$ matrix.

**Proposition.** Each of the mappings

1. $T(u) \mapsto f(u) T(u)$,
2. $T(u) \mapsto T(u - c)$,
3. $T(u) \mapsto B T(u) B^{-1}$

defines an automorphism of $Y(gl_N)$. 
Proposition. Each of the mappings

\[ \sigma_N : T(u) \mapsto T(-u), \]
\[ t : T(u) \mapsto T^t(u), \]
\[ S : T(u) \mapsto T^{-1}(u) \]

defines an anti-automorphism of \( Y(gl_N) \).
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Corollary. The mapping

\[ \omega_N : T(u) \mapsto T^{-1}(-u) \]

defines an involutive automorphism of \( Y(\mathfrak{gl}_N) \).
Poincaré–Birkhoff–Witt theorem

**Theorem**

*Given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$, any element of the algebra $Y(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators.*
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*Given an arbitrary linear order on the set of generators* $t_{ij}^{(r)}$, *any element of the algebra* $\mathcal{Y}(\mathfrak{gl}_N)$ *can be uniquely written as a linear combination of ordered monomials in these generators.*

**Corollary.** Consider the ascending filtration on $\mathcal{Y}(\mathfrak{gl}_N)$ defined by

$$\deg t_{ij}^{(r)} = r.$$ 

The graded algebra $\text{gr} \ \mathcal{Y}(\mathfrak{gl}_N)$ is an algebra of polynomials.
A coalgebra (over the field \( \mathbb{C} \)) is a vector space \( A \) equipped with linear maps \( \Delta : A \rightarrow A \otimes A \), the comultiplication, and \( \varepsilon : A \rightarrow \mathbb{C} \), the counit, satisfying some axioms; e.g.,

\[
A \otimes A \otimes A \xrightarrow{\Delta \otimes \text{id}} A \otimes A
\]

\[
\text{id} \otimes \Delta
\]

\[
A \otimes A \xrightarrow{\Delta} A
\]

the coassociativity of \( \Delta \).
A bialgebra is an associative unital algebra $A$ equipped with a coalgebra structure, such that $\Delta$ and $\varepsilon$ are algebra homomorphisms. In particular, then we have $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. 
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A bialgebra $A$ is called a Hopf algebra, if it is also equipped with an anti-automorphism $S : A \to A$, the antipode, such that the following two diagrams commute:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{S \otimes \text{id}} & A \otimes A \\
\Delta & \uparrow & \downarrow \mu \\
A & \xrightarrow{\delta \circ \varepsilon} & A
\end{array}
\quad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\text{id} \otimes S} & A \otimes A \\
\Delta & \uparrow & \downarrow \mu \\
A & \xrightarrow{\delta \circ \varepsilon} & A
\end{array}
\]
Theorem

The Yangian $\mathcal{Y}(\mathfrak{gl}_N)$ is a Hopf algebra with comultiplication

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u),$$

the antipode

$$S : T(u) \mapsto T^{-1}(u),$$

and the counit $\varepsilon : T(u) \mapsto 1.$
Quantum determinant

For any $m \geq 2$ introduce the rational function $R(u_1, \ldots, u_m)$ with values in the tensor product algebra $(\text{End } \mathbb{C}^N)^\otimes m$ by

$$R(u_1, \ldots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \cdots (R_{1m} \cdots R_{12}),$$

where $u_1, \ldots, u_m$ are independent complex variables and we abbreviate $R_{ij} = R_{ij}(u_i - u_j) = 1 - P_{ij}(u_i - u_j)^{-1}$. 
Quantum determinant

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where $u_1, \ldots, u_m$ are independent complex variables and we abbreviate $R_{ij} = R_{ij}(u_i - u_j) = 1 - P_{ij}(u_i - u_j)^{-1}$.

Using the Yang–Baxter equation, we get

$$R(u_1, \ldots, u_m) = (R_{12} \cdots R_{1m}) \cdots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}).$$
Applying the \( RTT \) relation repeatedly, we come to the fundamental relation

\[
R(u_1, \ldots, u_m) \ T_1(u_1) \ldots \ T_m(u_m) = T_m(u_m) \ldots \ T_1(u_1) \ R(u_1, \ldots, u_m).
\]
Applying the $RTT$ relation repeatedly, we come to the fundamental relation

$$R(u_1, \ldots, u_m) T_1(u_1) \ldots T_m(u_m) = T_m(u_m) \ldots T_1(u_1) R(u_1, \ldots, u_m).$$

**Lemma**

If $u_i - u_{i+1} = 1$ for all $i = 1, \ldots, m - 1$ then

$$R(u_1, \ldots, u_m) = A_m,$$

the image of the anti-symmetrizer $\sum_{p \in S_m} \text{sgn } p \cdot p \in \mathbb{C}[S_m]$ in the algebra $\text{End } (\mathbb{C}^N)^{\otimes m}$. 
Hence, we have

$$A_m \ T_1(u) \ldots T_m(u - m + 1) = T_m(u - m + 1) \ldots T_1(u) \ A_m.$$
Hence, we have

\[ A_m \ T_1(u) \ldots T_m(u - m + 1) = T_m(u - m + 1) \ldots T_1(u) A_m. \]

If \( m = N \) then the operator \( A_N \) on \( (C^N)^\otimes N \) is one-dimensional.

**Definition**

The quantum determinant of the matrix \( T(u) \) with the coefficients in \( Y(gl_N) \) is the formal series

\[ \text{qdet} \ T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \ldots \]

such that both sides of the above relation with \( m = N \), are equal to \( A_N \text{qdet} \ T(u) \).
Proposition

For any permutation $q \in \mathcal{S}_N$ we have

$$q \det T(u) = \text{sgn } q \sum_{p \in \mathcal{S}_N} \text{sgn } p \cdot t_{p(1),q(1)}(u) \cdots t_{p(N),q(N)}(u - N + 1)$$

$$= \text{sgn } q \sum_{p \in \mathcal{S}_N} \text{sgn } p \cdot t_{q(1),p(1)}(u - N + 1) \cdots t_{q(N),p(N)}(u).$$
Proposition

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$$= \sgn q \sum_{p \in \mathfrak{S}_N} \sgn p \cdot t_{q(1), p(1)}(u - N + 1) \ldots t_{q(N), p(N)}(u).$$

In particular,

$$q\det T(u) = \sum_{p \in \mathfrak{S}_N} \sgn p \cdot t_{p(1), 1}(u) \ldots t_{p(N), N}(u - N + 1)$$

$$= \sum_{p \in \mathfrak{S}_N} \sgn p \cdot t_{1, p(1)}(u - N + 1) \ldots t_{N, p(N)}(u).$$
**Example**

For $N = 2$ we have

$$\text{qdet } T(u) = t_{11}(u) t_{22}(u - 1) - t_{21}(u) t_{12}(u - 1)$$

$$= t_{22}(u) t_{11}(u - 1) - t_{12}(u) t_{21}(u - 1)$$

$$= t_{11}(u - 1) t_{22}(u) - t_{12}(u - 1) t_{21}(u)$$

$$= t_{22}(u - 1) t_{11}(u) - t_{21}(u - 1) t_{12}(u).$$
Assuming that \( m \leq N \) is arbitrary, define the \( m \times m \) quantum minors \( t_{a_1 \ldots a_m}^{b_1 \ldots b_m}(u) \) so that each side of

\[
A_m \ T_1(u) \ldots T_m(u - m + 1) = T_m(u - m + 1) \ldots T_1(u) \ A_m
\]
equals

\[
\sum e_{a_1 b_1} \otimes \ldots \otimes e_{a_m b_m} \otimes t_{b_1 \ldots b_m}^{a_1 \ldots a_m}(u),
\]
summed over the indices \( a_i, b_i \in \{1, \ldots, N\} \).
Proposition

The images of quantum minors under the comultiplication are given by

$$\Delta(t^{a_1 \ldots a_m}(u)) = \sum_{c_1 < \cdots < c_m} t^{a_1 \ldots a_m}(u) \otimes t^{c_1 \ldots c_m}(u),$$

summed over all subsets of indices \(\{c_1, \ldots, c_m\}\) from \(\{1, \ldots, N\}\).
Proposition

The images of quantum minors under the comultiplication are given by

\[ \Delta(t_{b_1\ldots b_m}(u)) = \sum_{c_1<\cdots<c_m} t_{c_1\ldots c_m}(u) \otimes t_{b_1\ldots b_m}(u), \]

summed over all subsets of indices \( \{c_1, \ldots, c_m\} \) from \( \{1, \ldots, N\} \).

In particular, as \( \text{qdet } T(u) = t_{1\ldots N}(u) \),

\[ \Delta : \text{qdet } T(u) \mapsto \text{qdet } T(u) \otimes \text{qdet } T(u). \]
Center of $\mathcal{Y}(\mathfrak{gl}_N)$

**Proposition**

We have the relations

\[
(u - v) \left[ t_{kl}(u), t_{b_1 \ldots b_m}(v) \right] = \sum_{i=1}^{m} t_{a_i l}(u) t_{b_1 \ldots k \ldots b_m}(v) - \sum_{i=1}^{m} t_{b_1 \ldots l \ldots b_m}(v) t_{k b_i}(u)
\]

where the indices $k$ and $l$ in the quantum minors replace $a_i$ and $b_i$, respectively.
Theorem

The coefficients \(d_1, d_2, \ldots\) of the series \(q \det T(u)\) belong to the center \(Z \mathcal{Y}(\mathfrak{gl}_N)\) of the algebra \(\mathcal{Y}(\mathfrak{gl}_N)\). Moreover, these elements are algebraically independent and generate \(Z \mathcal{Y}(\mathfrak{gl}_N)\).

Proof.

The first part follows from the Proposition. For the second part introduce another filtration on \(\mathcal{Y}(\mathfrak{gl}_N)\) by setting

\[
\deg' t_{ij}^{(r)} = r - 1
\]

for every \(r \geq 1\). Then the corresponding graded algebra \(\text{gr}' \mathcal{Y}(\mathfrak{gl}_N)\) is isomorphic to the universal enveloping algebra \(U(\mathfrak{gl}_N[z])\). \(\square\)
Yangian for $\mathfrak{sl}_N$

For any series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ consider the automorphism $\mu_f : T(u) \mapsto f(u) T(u)$ of $Y(\mathfrak{gl}_N)$.
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For any series $f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]$ consider the automorphism $\mu_f : T(u) \mapsto f(u) T(u)$ of $Y(\mathfrak{gl}_N)$.

The Yangian for $\mathfrak{sl}_N$ is the subalgebra $Y(\mathfrak{sl}_N)$ of $Y(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms $\mu_f$. 
**Yangian for $\mathfrak{sl}_N$**

For any series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ consider the automorphism $\mu_f : T(u) \mapsto f(u) T(u)$ of $Y(\mathfrak{gl}_N)$.

The **Yangian** for $\mathfrak{sl}_N$ is the subalgebra $Y(\mathfrak{sl}_N)$ of $Y(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms $\mu_f$.

**Theorem**

*We have the isomorphism*

$$Y(\mathfrak{gl}_N) = ZY(\mathfrak{gl}_N) \otimes Y(\mathfrak{sl}_N).$$

*In particular, the center of $Y(\mathfrak{sl}_N)$ is trivial.*
Corollary

The algebra $Y(\mathfrak{sl}_N)$ is isomorphic to the quotient of $Y(\mathfrak{gl}_N)$ by the ideal generated by the elements $d_1, d_2, \ldots$, i.e.,

$$Y(\mathfrak{sl}_N) \cong Y(\mathfrak{gl}_N)/(\text{qdet } T(u) = 1).$$
Corollary

The algebra $\mathcal{Y}(\mathfrak{sl}_N)$ is isomorphic to the quotient of $\mathcal{Y}(\mathfrak{gl}_N)$ by the ideal generated by the elements $d_1, d_2, \ldots$, i.e.,

$$\mathcal{Y}(\mathfrak{sl}_N) \cong \mathcal{Y}(\mathfrak{gl}_N)/(\text{qdet } T(u) = 1).$$

Proposition

The subalgebra $\mathcal{Y}(\mathfrak{sl}_N)$ of $\mathcal{Y}(\mathfrak{gl}_N)$ is a Hopf algebra whose comultiplication, antipode and counit are obtained by restricting those from $\mathcal{Y}(\mathfrak{gl}_N)$.
Quantum Liouville formula

The quantum comatrix $\hat{T}(u)$ is defined by

$$\hat{T}(u) \ T(u - N + 1) = \text{qdet} \ T(u).$$
Quantum Liouville formula

The quantum comatrix $\hat{T}(u)$ is defined by

$$\hat{T}(u) \ T(u - N + 1) = \text{qdet} \ T(u).$$

Proposition

The entries $\hat{t}_{ij}(u)$ of the matrix $\hat{T}(u)$ are given by

$$\hat{t}_{ij}(u) = (-1)^{i+j} t_{\hat{1}\ldots\hat{j}\ldots\hat{i}\ldots\hat{N}}(u),$$

where the hats on the right hand side indicate the indices to be omitted. Moreover, we have the relation

$$\hat{T}^t(u - 1) \ T^t(u) = \text{qdet} \ T(u).$$
Consider the series $z(u)$ with coefficients from $Y(gl_N)$ given by the formula

$$z(u)^{-1} = \frac{1}{N} \text{tr} \left( T(u) T^{-1}(u - N) \right),$$

so that

$$z(u) = 1 + z_2 u^{-2} + z_3 u^{-3} + \ldots \quad \text{where} \quad z_i \in Y(gl_N).$$
Consider the series $z(u)$ with coefficients from $\mathcal{Y}(\mathfrak{gl}_N)$ given by the formula

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so that

$$z(u) = 1 + z_2 u^{-2} + z_3 u^{-3} + \ldots \quad \text{where} \quad z_i \in \mathcal{Y}(\mathfrak{gl}_N).$$

**Theorem**

*We have the relation*

$$z(u) = \frac{\text{qdet} \ T(u - 1)}{\text{qdet} \ T(u)}.$$
Proof.

We have

\[ z(u)^{-1} = \frac{1}{N} \text{tr} (T(u) \hat{T}(u - 1) (\text{qdet } T(u - 1))^{-1}) \, . \]

Using the centrality of \( \text{qdet } T(u) \) we get

\[ T^t(u) \hat{T}^t(u - 1) = \text{qdet } T(u) \]

and so

\[ \text{tr} (T(u) \hat{T}(u - 1)) = N \text{qdet } T(u), \]

implying the formula.
Theorem

The square of the antipode $S$ is the automorphism of $\mathcal{Y}(\mathfrak{gl}_N)$ given by

$$S^2 : T(u) \mapsto z(u + N) T(u + N).$$

In particular, $\text{qdet } T(u)$ is stable under $S^2$. 
Application to $\mathfrak{gl}_N$

Recall the evaluation homomorphism $\pi_N : T(u) \mapsto 1 + E \, u^{-1}$:

$$\pi_N : z(-u + N)^{-1} \mapsto \frac{1}{N} \, \text{tr} \left( (1 - E \, (u - N)^{-1}) (1 - E \, u^{-1})^{-1} \right)$$

$$= 1 - \frac{1}{u - N} \sum_{k=1}^{\infty} \text{tr} \, E^k \, u^{-k}.$$
Application to $\mathfrak{gl}_N$

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$$= 1 - \frac{1}{u - N} \sum_{k=1}^{\infty} \text{tr} E^k u^{-k}.$$  

The quantum Liouville formula gives

$$z(u + 1)^{-1} = \frac{\text{qdet} \ T(u + 1)}{\text{qdet} \ T(u)}.$$

Applying the evaluation homomorphism to both sides of this relation, we get Newton’s formulas (see Lecture 1).
Factorization of the quantum determinant

Let $A = [a_{ij}]$ be an $N \times N$ matrix over a ring with 1. The $ij$-th quasideterminant of $A$ is defined by

$$|A|_{ij} = ((A^{-1})_{ji})^{-1}.$$  

Example

For a $2 \times 2$ matrix $A$ the four quasideterminants are

$$|A|_{11} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \quad |A|_{12} = a_{12} - a_{11} a_{22}^{-1} a_{21},$$

$$|A|_{21} = a_{21} - a_{22} a_{12}^{-1} a_{11}, \quad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}.$$
For $m = 1, \ldots, N$ denote by $T^{(m)}(u)$ the submatrix of $T(u)$ corresponding to the first $m$ rows and columns.
For \( m = 1, \ldots, N \) denote by \( T^{(m)}(u) \) the submatrix of \( T(u) \) corresponding to the first \( m \) rows and columns.

**Theorem**

The quantum determinant \( \text{qdet} \ T(u) \) admits the factorization in the algebra \( \mathcal{Y}(\mathfrak{gl}_N)[[u^{-1}]] \)

\[
\text{qdet} \ T(u) = t_{11}(u) \left| T^{(2)}(u - 1) \right|_{22} \cdots \left| T^{(N)}(u - N + 1) \right|_{NN}.
\]

Moreover, the \( N \) factors on the right hand side of this equality pairwise commute.
Set

\[ \tilde{C}(q) = \sum_{p \in \mathcal{S}_N} \text{sgn} \; p \cdot (1 + q \; E)_{p(1),1} \cdots (1 + q \; (E - N + 1))_{p(N),N}. \]

Then \( \tilde{C}(q) = q^N C(q^{-1}), \) where \( C(u) \) is the Capelli determinant.
Set

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Then \( \tilde{C}(q) = q^N C(q^{-1}) \), where \( C(u) \) is the Capelli determinant.

Apply the evaluation homomorphism to the decomposition of the Theorem to get

\[ \tilde{C}(q) = \left| 1 + q E^{(1)} \right|_{11} \cdots \left| 1 + q (E^{(N)} - N + 1) \right|_{NN}, \]

where \( E^{(m)} \) is the submatrix of \( E \) corresponding to the first \( m \) rows and columns.
For the Harish-Chandra image of $\tilde{C}(q)$ we have

$$\chi(\tilde{C}(q)) = (1 + q l_1) \ldots (1 + q l_N), \quad l_i = \lambda_i - i + 1.$$
For the Harish-Chandra image of \( \widetilde{C}(q) \) we have

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\chi(\widetilde{C}(q)) = (1 + q l_1) \cdots (1 + q l_N), \quad l_i = \lambda_i - i + 1.
\]

Hence, if we define the Casimir elements \( \Phi_k \) by

\[
\sum_{k=1}^{\infty} \Phi_k q^{k-1} = -\frac{d}{dq} \log \widetilde{C}(-q),
\]

then

\[
\chi(\Phi_k) = l_1^k + \cdots + l_N^k.
\]
On the other hand, by the quasideterminant decomposition,

\[
\sum_{k=1}^{\infty} \Phi_k q^{k-1} = - \sum_{m=1}^{N} \frac{d}{dq} \log |1 - q (E^{(m)} - m + 1)|_{mm}.
\]
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\sum_{k=1}^{\infty} \Phi_k q^{k-1} = - \sum_{m=1}^{N} \frac{d}{dq} \log |1 - q (E^{(m)} - m + 1)|_{mm}.
$$

Therefore,

$$
\Phi_k = \Phi_k^{(1)} + \cdots + \Phi_k^{(N)},
$$

where

$$
\sum_{k=1}^{\infty} \Phi_k^{(m)} q^{k-1} = - \frac{d}{dq} \log |1 - q (E^{(m)} - m + 1)|_{mm}.
$$
Quantum Sylvester theorem

Suppose that $A = [a_{ij}]$ is a numerical $(M + N) \times (M + N)$ matrix. For any indices $i, j = 1, \ldots, N$ introduce the minors $c_{ij}$ of $A$ corresponding to the rows $1, \ldots, M, M+i$ and columns $1, \ldots, M, M+j$ so that

$$c_{ij} = a_{1\ldots M,M+i}^{1\ldots M,M+j}.$$

Let $A^{(M)}$ be the submatrix of $A$ determined by the first $M$ rows and columns. The classical Sylvester theorem provides a formula for the determinant of the matrix $C = [c_{ij}]$:

$$\det C = \det A \cdot \left( \det A^{(M)} \right)^{N-1}.$$
Introduce the series with coefficients in $\mathcal{Y}(gl_{M+N})$ by

$$t_{ij}^\#(u) = t_{1\ldots M,M+i}^{1\ldots M,M+j}(u)$$

and set $T^\#(u) = [t_{ij}^\#(u)]$. 
Introduce the series with coefficients in $Y(gl_{M+N})$ by

$$t_{ij}^\#(u) = t^{1\ldots M,M+i}_{1\ldots M,M+j}(u)$$

and set $T^\#(u) = [t_{ij}^\#(u)]$.

**Theorem**

The mapping

$$t_{ij}(u) \mapsto t_{ij}^\#(u), \quad 1 \leq i, j \leq N,$$

defines a homomorphism $Y(gl_N) \to Y(gl_{M+N})$. Moreover,

$$\text{qdet } T^\#(u) = \text{qdet } T(u) \cdot \text{qdet } T^{(M)}(u-1) \ldots \text{qdet } T^{(M)}(u-N+1).$$
Consider the orthogonal Lie algebra $\mathfrak{o}_N$ as the subalgebra of $\mathfrak{gl}_N$ spanned by the skew-symmetric matrices. The elements $F_{ij} = E_{ij} - E_{ji}$ with $i < j$ form a basis of $\mathfrak{o}_N$. Introduce the $N \times N$ matrix $F$ whose $ij$-th entry is $F_{ij}$.
Twisted Yangians

Consider the orthogonal Lie algebra $\mathfrak{o}_N$ as the subalgebra of $\mathfrak{gl}_N$ spanned by the skew-symmetric matrices. The elements $F_{ij} = E_{ij} - E_{ji}$ with $i < j$ form a basis of $\mathfrak{o}_N$. Introduce the $N \times N$ matrix $F$ whose $ij$-th entry is $F_{ij}$.

The matrix elements of the powers of the matrix $F$ are known to satisfy the relations

$$[F_{ij}, (F^s)_{kl}] = \delta_{kj}(F^s)_{il} - \delta_{il}(F^s)_{kj} - \delta_{ik}(F^s)_{jl} + \delta_{lj}(F^s)_{ki}. $$
Introduce the generating series

$$f_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} (F^r)_{ij} \left( u + \frac{N - 1}{2} \right)^{-r}.$$
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Then we have the relations
\[
(u^2 - v^2) [f_{ij}(u), f_{kl}(v)] = (u + v) \left( f_{kj}(u) f_{il}(v) - f_{kj}(v) f_{il}(u) \right) \\
- (u - v) \left( f_{ik}(u) f_{jl}(v) - f_{ki}(v) f_{lj}(u) \right) \\
+ f_{ki}(u) f_{jl}(v) - f_{ki}(v) f_{jl}(u).
\]
More generally, equip $\mathbb{C}^N$ with a nonsingular bilinear form which may be either symmetric or alternating. The alternating case can only occur if $N$ is even. Let $G = [g_{ij}]$ be the corresponding matrix so that $G$ is nonsingular with $G^t = \pm G$. 
More generally, equip $\mathbb{C}^N$ with a nonsingular bilinear form which may be either symmetric or alternating. The alternating case can only occur if $N$ is even. Let $G = [g_{ij}]$ be the corresponding matrix so that $G$ is nonsingular with $G^t = \pm G$.

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the symmetric case and the lower sign to the alternating case. Introduce the elements $F_{ij}$ of the Lie algebra $\mathfrak{gl}_N$ by the formulas

$$F_{ij} = \sum_{k=1}^{N} (E_{ik} g_{kj} \mp E_{jk} g_{ki}).$$
Obviously,

$$F_{ji} = \mp F_{ij}$$

and the elements $F_{ij}$ satisfy the commutation relations

$$[F_{ij}, F_{kl}] = g_{kj} F_{il} - g_{il} F_{kj} - g_{ik} F_{jl} + g_{lj} F_{ki}.$$
Obviously,

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and the elements \( F_{ij} \) satisfy the commutation relations

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The Lie subalgebra of \( \mathfrak{gl}_N \) spanned by the elements \( F_{ij} \) is isomorphic to the orthogonal Lie algebra \( \mathfrak{o}_N \) in the symmetric case and to the symplectic Lie algebra \( \mathfrak{sp}_N \) in the alternating case. This Lie algebra will be denoted by \( \mathfrak{g}_N \).
The twisted Yangian $Y_G(g_N)$ is an associative algebra with generators $s_{ij}^{(1)}$, $s_{ij}^{(2)}$, \ldots where $1 \leq i, j \leq N$, and the defining relations written in terms of the generating series

$$s_{ij}(u) = g_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \ldots$$
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$$s_{ij}(u) = g_{ij} + s^{(1)}_{ij}u^{-1} + s^{(2)}_{ij}u^{-2} + \ldots$$

as follows

$$(u^2 - v^2) [s_{ij}(u), s_{kl}(v)] = (u + v) (s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u))$$

$$- (u - v) (s_{ik}(u)s_{jl}(v) - s_{ki}(v)s_{jl}(u))$$

$$+ s_{ki}(u)s_{jl}(v) - s_{ki}(v)s_{jl}(u)$$

and

$$s_{ji}(-u) = \pm s_{ij}(u) + \frac{s_{ij}(u) - s_{ij}(-u)}{2u}.$$
If $G$ and $G'$ are two nonsingular symmetric (respectively, skew-symmetric) $N \times N$-matrices then the algebras $Y_G(g_N)$ and $Y_{G'}(g_N)$ are isomorphic to each other.
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**Proposition**

The assignment

\[ s_{ij}(u) \mapsto g_{ij} + F_{ij} \left( u \pm \frac{1}{2} \right)^{-1} \]

defines an algebra epimorphism $\varrho_N : Y(g_N) \to U(g_N)$. Moreover, the assignment

\[ F_{ij} \mapsto s_{ij}^{(1)} \]

defines an embedding $U(g_N) \hookrightarrow Y(g_N)$.
Matrix form of the defining relations

Introduce the $N \times N$ matrix $S(u)$ by

$$S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes s_{ij}(u) \in \text{End} \mathbb{C}^N \otimes Y(g_N)[[u^{-1}]]$$
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Proposition

The defining relations of $Y(g_N)$ have the form

$$R(u-v) S_1(u) R^t(-u-v) S_2(v) = S_2(v) R^t(-u-v) S_1(u) R(u-v)$$

and

$$S^t(-u) = \pm S(u) + \frac{S(u) - S(-u)}{2u}.$$
Here

\[ R(u) = 1 - Pu^{-1} \]

is the Yang \( R \)-matrix, while

\[ R^t(u) = 1 - Q u^{-1}, \quad Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}. \]
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**Theorem**

*The mapping*

$$S(u) \mapsto T(u) G T^t(-u)$$

*defines an embedding $Y(gl_N) \hookrightarrow Y(gl_{1N})$.***
Sklyanin determinant

The Sklyanin determinant is a series in $u^{-1}$ defined by

$$s\text{det } S(u) = \gamma_{n,G}(u) q\text{det } T(u) q\text{det } T(-u + N - 1),$$

where

$$\gamma_{n,G}(u) = \begin{cases} 
\det G & \text{if } g_N = o_N, \\
\frac{2u + 1}{2u - 2n + 1} \det G & \text{if } g_N = sp_{2n}.
\end{cases}$$
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\frac{2u + 1}{2u - 2n + 1} \det G & \text{if } g_N = sp_{2n}.
\end{cases}$$

All coefficients of $s\text{det} \ S(u)$ are contained in $Y(g_N)$ and belong to the center of $Y(g_N)$. 
Introduce the scalar $\gamma_n(u)$ by

\[
\gamma_n(u) = \begin{cases} 
1 & \text{if } g_N = o_N, \\
(-1)^n \frac{2u + 1}{2u - 2n + 1} & \text{if } g_N = sp_{2n}.
\end{cases}
\]

**Theorem**

We have

\[
\text{sdet } S(u) = \gamma_n(u) \sum_{p \in G_N} \text{sgn } pp' \cdot s_{p(1),p'(1)}^t(-u) \cdots s_{p(n),p'(n)}^t(-u + n - 1) \\
\times s_{p(n+1),p'(n+1)}(u - n) \cdots s_{p(N),p'(N)}(u - N + 1).
\]
Here we denote the matrix elements of the transposed matrix $S^t(u)$ by $s^t_{ij}(u)$, and for any permutation $p \in \mathfrak{S}_N$ we denote by $p'$ its image under the map $\varphi_N : \mathfrak{S}_N \rightarrow \mathfrak{S}_N$ (Lecture 1).
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**Example**

For $N = 2$ we have

$$\text{sdet } S(u) = \frac{1 + 2u}{1 - 2u} \left( s_{11}^t(-u) s_{22}(u - 1) - s_{21}^t(-u) s_{12}(u - 1) \right).$$
Here we denote the matrix elements of the transposed matrix \( S^t(u) \) by \( s_{ij}^t(u) \), and for any permutation \( p \in \mathfrak{S}_N \) we denote by \( p' \) its image under the map \( \varphi_N : \mathfrak{S}_N \to \mathfrak{S}_N \) (Lecture 1).

**Example**

For \( N = 2 \) we have

\[
\text{sdet } S(u) = \frac{1 \mp 2u}{1 - 2u} (s_{11}^t(-u) s_{22}(u - 1) - s_{21}^t(-u) s_{12}(u - 1)).
\]

If \( N = 3 \) then \( \text{sdet } S(u) = \)

\[
s_{22}^t(-u) s_{11}(u - 1) s_{33}(u - 2) + s_{12}^t(-u) s_{31}(u - 1) s_{23}(u - 2) \\
+ s_{21}^t(-u) s_{32}(u - 1) s_{13}(u - 2) - s_{12}^t(-u) s_{21}(u - 1) s_{33}(u - 2) \\
- s_{32}^t(-u) s_{11}(u - 1) s_{23}(u - 2) - s_{31}^t(-u) s_{22}(u - 1) s_{13}(u - 2).
\]
The center of the twisted Yangian

**Theorem**

*All coefficients of the series*

\[
\text{sdet } S(u) = c_0 + c_1 u^{-1} + c_2 u^{-2} + \ldots
\]

*belong to the center of the algebra } \mathcal{Y}(\mathfrak{g}_N). Moreover, the even coefficients } c_2, c_4, \ldots \text{ are algebraically independent and generate the center of } \mathcal{Y}(\mathfrak{g}_N).
Coideal property

Theorem

The subalgebra $Y(g_N)$ is a left coideal of the Hopf algebra $Y(gl_N)$, i.e.,

\[ \Delta(Y(g_N)) \subset Y(gl_N) \otimes Y(g_N). \]

Moreover,

\[ \Delta : s_{ij}(u) \mapsto \sum_{a,b=1}^{N} t_{ia}(u) t_{jb}(-u) \otimes s_{ab}(u). \]
Twisted analogues of some Yangian theorems

- Quantum Liouville formula
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- Quantum Liouville formula
- Quasideterminant factorization of $s\det S(u)$
Twisted analogues of some Yangian theorems

- Quantum Liouville formula
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Applications to classical Lie algebras $\mathfrak{g}_N$

- Constructions of Casimir elements
Twisted analogues of some Yangian theorems

- Quantum Liouville formula
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Applications to classical Lie algebras $\mathfrak{g}_N$

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- Cayley–Hamilton theorem
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- Quantum Liouville formula
- Quasideterminant factorization of $\text{sdet } S(u)$
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Applications to classical Lie algebras $g_N$

- Constructions of Casimir elements
- Cayley–Hamilton theorem
- Characteristic identities