Recall that the Yangian $Y(gl_N)$ is an associative algebra with generators $t_{ij}^{(r)}$ and the defining relations

$$(u - v) [t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u),$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y(gl_N)[[u^{-1}]].$$
Definition. A representation $L$ of the Yangian $\mathcal{Y}(\mathfrak{gl}_N)$ is called a highest weight representation if there exists a nonzero vector $\zeta \in L$ such that $L$ is generated by $\zeta$ and the following relations hold

\[ t_{ij}(u) \zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq N, \quad \text{and} \]
\[ t_{ii}(u) \zeta = \lambda_i(u) \zeta \quad \text{for} \quad 1 \leq i \leq N \]
Definition. A representation $L$ of the Yangian $\mathcal{Y}(\mathfrak{gl}_N)$ is called a highest weight representation if there exists a nonzero vector $\zeta \in L$ such that $L$ is generated by $\zeta$ and the following relations hold

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and

$$t_{ii}(u)\zeta = \lambda_i(u)\zeta \quad \text{for} \quad 1 \leq i \leq N$$

for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \ldots, \quad \lambda_i^{(r)} \in \mathbb{C}.$$ 

The vector $\zeta$ is called the highest vector of $L$, and the $N$-tuple of formal series $\lambda(u) = (\lambda_1(u), \ldots, \lambda_N(u))$ is the highest weight of $L$. 
Verma module

Definition

Let $\lambda(u) = (\lambda_1(u), \ldots, \lambda_N(u))$ be an arbitrary tuple of formal series. The Verma module $M(\lambda(u))$ is the quotient of $\mathcal{Y}(\mathfrak{gl}_N)$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ for $1 \leq i < j \leq N$ and $t_{ii}(u) - \lambda_i(u)$ for $1 \leq i \leq N$. 

Proposition. For any given order on the set of generators $t_{ji}$ with $1 \leq i < j \leq N$ and $r \geq 1$, the elements $t_{r1}^{j1} t_{r2}^{j2} \ldots t_{rm}^{jm} \lambda(u)$, $m \geq 0$, with ordered products of the generators, form a basis of $M(\lambda(u))$. 

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Proposition. For any given order on the set of generators $t_{ji}^{(r)}$ with $1 \leq i < j \leq N$ and $r \geq 1$, the elements

$$t_{j_1 i_1}^{(r_1)} \cdots t_{j_m i_m}^{(r_m)} 1\lambda(u), \quad m \geq 0,$$

with ordered products of the generators, form a basis of $M(\lambda(u))$. 
The irreducible highest weight representation $L(\lambda(u))$ of $\mathcal{Y}(gl_N)$ with the highest weight $\lambda(u)$ is defined as the quotient of the Verma module $M(\lambda(u))$ by the unique maximal proper submodule.
The irreducible highest weight representation $L(\lambda(u))$ of $Y(\mathfrak{gl}_N)$ with the highest weight $\lambda(u)$ is defined as the quotient of the Verma module $M(\lambda(u))$ by the unique maximal proper submodule.

**Theorem**

*Every finite-dimensional irreducible representation of $Y(\mathfrak{gl}_N)$ is isomorphic to $L(\lambda(u))$ for some $\lambda(u)$.*

**Proof.**

Regard the representation of $Y(\mathfrak{gl}_N)$ as a $\mathfrak{gl}_N$-module using the embedding $E_{ij} \mapsto t_{ij}^{(1)}$. 
Given an $N$-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$ denote by $L(\lambda)$ the irreducible representation of the Lie algebra $\mathfrak{gl}_N$ with the highest weight $\lambda$. So, $L(\lambda)$ is generated by a nonzero vector $\zeta$ such that

$$E_{ij} \zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq N,$$

and

$$E_{ii} \zeta = \lambda_i \zeta \quad \text{for} \quad 1 \leq i \leq N.$$
Given an \( N \)-tuple of complex numbers \( \lambda = (\lambda_1, \ldots, \lambda_N) \) denote by \( L(\lambda) \) the irreducible representation of the Lie algebra \( \mathfrak{gl}_N \) with the highest weight \( \lambda \). So, \( L(\lambda) \) is generated by a nonzero vector \( \zeta \) such that

\[
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\]

\[
E_{ii} \zeta = \lambda_i \zeta \quad \text{for} \quad 1 \leq i \leq N.
\]

Equip \( L(\lambda) \) with a structure of \( Y(\mathfrak{gl}_N) \)-module via the evaluation homomorphism

\[
t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}.
\]
$L(\lambda)$ is the evaluation module over $Y(gl_N)$. 
$L(\lambda)$ is the evaluation module over $Y(gl_N)$.

$L(\lambda)$ is a highest weight representation of the Yangian with the highest vector $\zeta$, and the components of the highest weight are given by

$$\lambda_i(u) = 1 + \lambda_i u^{-1}, \quad i = 1, \ldots, N.$$
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If $L$ and $M$ are any two $Y(gl_N)$-modules, then the tensor product space $L \otimes M$ can be equipped with a $Y(gl_N)$-action with the use of the comultiplication $\Delta$ on $Y(gl_N)$.
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By the coassociativity of $\Delta$, we may unambiguously define multiple tensor product modules of the form

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \ldots \otimes L(\lambda^{(k)}).$$
Consider the irreducible highest weight representation $L(\lambda(u))$ of $Y(gl_2)$ with an arbitrary highest weight $\lambda(u) = (\lambda_1(u), \lambda_2(u))$. 
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**Proposition**

If $\dim L(\lambda(u)) < \infty$ then there exists a formal series

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \ldots, \quad f_r \in \mathbb{C},$$

such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in $u^{-1}$. 


let $\lambda_1(u)$ and $\lambda_2(u)$ be polynomials in $u^{-1}$ of degree not more than $k$. Write the decompositions

$$\lambda_1(u) = (1 + \alpha_1 u^{-1}) \ldots (1 + \alpha_k u^{-1}),$$

$$\lambda_2(u) = (1 + \beta_1 u^{-1}) \ldots (1 + \beta_k u^{-1}).$$
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**Proposition**

Suppose that for every $i = 1, \ldots, k - 1$ the following condition holds: if the multiset \{ $\alpha_p - \beta_q$ $|$ $i \leq p, q \leq k$ \} contains nonnegative integers, then $\alpha_i - \beta_i$ is minimal amongst them. Then the representation $L(\lambda_1(u), \lambda_2(u))$ of $Y(gl_2)$ is isomorphic to the tensor product module

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k).$$
Theorem

The irreducible highest weight representation $L(\lambda_1(u), \lambda_2(u))$ of $\mathcal{Y}(\mathfrak{gl}_2)$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in $u$ such that

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u + 1)}{P(u)}.$$

In this case $P(u)$ is unique.
Theorem

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In this case $P(u)$ is unique.

The polynomial $P(u)$ is called the Drinfeld polynomial of the finite-dimensional representation $L(\lambda_1(u), \lambda_2(u))$. 
Proof.

\[ \dim L(\alpha, \beta) < \infty \text{ if and only if } \alpha - \beta \in \mathbb{Z}_+. \]
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\[ \dim L(\alpha, \beta) < \infty \text{ if and only if } \alpha - \beta \in \mathbb{Z}_+. \]

The highest weight of the \( Y(\mathfrak{gl}_2) \)-evaluation module is

\[ \lambda_1(u) = 1 + \alpha u^{-1}, \quad \lambda_2(u) = 1 + \beta u^{-1}. \]

Hence, if \( \alpha - \beta \in \mathbb{Z}_+ \) then

\[ \frac{\lambda_1(u)}{\lambda_2(u)} = \frac{u + \alpha}{u + \beta} = \frac{P(u + 1)}{P(u)} \]

for

\[ P(u) = (u + \beta)(u + \beta + 1) \ldots (u + \alpha - 1). \]
Recall that the Yangian $Y(\mathfrak{sl}_2)$ is the subalgebra of $Y(\mathfrak{gl}_2)$ which consists of the elements stable under all automorphisms of the form $T(u) \mapsto f(u) \ T(u)$. 

Corollary

The isomorphism classes of finite-dimensional irreducible representations of the Yangian $Y(\mathfrak{sl}_2)$ are parameterized by monic polynomials in $u$. Every such representation is isomorphic to the restriction of a $Y(\mathfrak{gl}_2)$-module of the form $L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k)$, where each difference $\alpha_i - \beta_i$ is a positive integer.
Recall that the Yangian $\mathcal{Y}(\mathfrak{sl}_2)$ is the subalgebra of $\mathcal{Y}(\mathfrak{gl}_2)$ which consists of the elements stable under all automorphisms of the form $T(u) \mapsto f(u)T(u)$.

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where each difference $\alpha_i - \beta_i$ is a positive integer.
Irreducibility criterion

Define the string corresponding to a pair of complex numbers \((\alpha, \beta)\) with \(\alpha - \beta \in \mathbb{Z}_+\) as the set

\[ S(\alpha, \beta) = \{\beta, \beta + 1, \ldots, \alpha - 1\}. \]

If \(\alpha = \beta\) then the set \(S(\alpha, \beta)\) is regarded to be empty.
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If \(\alpha = \beta\) then the set \(S(\alpha, \beta)\) is regarded to be empty.

**Definition**

Two strings \(S_1\) and \(S_2\) are in general position if either

(i) \(S_1 \cup S_2\) is not a string, or

(ii) \(S_1 \subset S_2\), or \(S_2 \subset S_1\).
Suppose that all differences $\alpha_i - \beta_i$ are nonnegative integers.
Suppose that all differences $\alpha_i - \beta_i$ are nonnegative integers.

**Corollary**

*The representation*

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$$

*of* $Y(\mathfrak{gl}_2)$ (or $Y(\mathfrak{sl}_2)$) *is irreducible if and only if the strings* $S(\alpha_1, \beta_1), \ldots, S(\alpha_k, \beta_k)$ *are pairwise in general position.*
Example. The representation $L(7, 1) \otimes L(6, 4)$ of $\mathfrak{Y}(\mathfrak{gl}_2)$ is irreducible:

![Diagram]

1 2 3 4 5 6
Example. The representation $L(7, 1) \otimes L(6, 4)$ of $\mathfrak{gl}_2$ is irreducible:
Example. The representation $L(7, 1) \otimes L(6, 4)$ of $\mathcal{Y}(\mathfrak{gl}_2)$ is irreducible:

$$\begin{array}{cccc}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

while $L(6, 1) \otimes L(7, 4)$ is reducible:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
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Example. The representation $L(7, 1) \otimes L(6, 4)$ of $Y(\mathfrak{gl}_2)$ is irreducible:

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet & \circ & \circ & \bullet \\
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\end{array}
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Representations of $\mathcal{Y}(\mathfrak{gl}_N)$

Let $\lambda(u)$ be an $N$-tuple of formal series in $u^{-1}$,

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_N(u)).$$
Representations of $\mathcal{Y}(\mathfrak{gl}_N)$

Let $\lambda(u)$ be an $N$-tuple of formal series in $u^{-1}$,

$$\lambda(u) = (\lambda_1(u), \ldots, \lambda_N(u)).$$

**Theorem**

*The irreducible highest weight representation $L(\lambda(u))$ of the Yangian $\mathcal{Y}(\mathfrak{gl}_N)$ is finite-dimensional, if and only if there exist monic polynomials $P_1(u), \ldots, P_{N-1}(u)$ in $u$ such that*

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u + 1)}{P_i(u)}, \quad i = 1, \ldots, N - 1.$$
Definition

The polynomials $P_i(u)$ with $i = 1, \ldots, N - 1$ are called the Drinfeld polynomials of $L(\lambda(u))$. 
Definition

The polynomials \( P_i(u) \) with \( i = 1, \ldots, N - 1 \) are called the Drinfeld polynomials of \( L(\lambda(u)) \).

Lemma. Suppose that \( L \) and \( M \) are finite-dimensional irreducible representations of \( \mathcal{Y}(\mathfrak{gl}_N) \) with the respective sets of Drinfeld polynomials

\[
(P_1(u), \ldots, P_{N-1}(u)) \quad \text{and} \quad (Q_1(u), \ldots, Q_{N-1}(u)).
\]

Then the irreducible quotient of the cyclic \( \mathcal{Y}(\mathfrak{gl}_N) \)-span of the tensor product of the highest vectors of \( L \) and \( M \) corresponds to

\[
(P_1(u)Q_1(u), \ldots, P_{N-1}(u)Q_{N-1}(u)).
\]
The evaluation $Y(\mathfrak{gl}_N)$-module $L(\alpha + 1, \ldots, \alpha + 1, \alpha, \ldots, \alpha)$ with $i$ copies of $\alpha + 1$ is a fundamental representation; its Drinfeld polynomials are given by

$$P_i(u) = u + \alpha \quad \text{and} \quad P_j(u) = 1 \quad \text{if} \quad j \neq i.$$
The evaluation $\mathcal{Y}(\mathfrak{gl}_N)$-module $L(\alpha + 1, \ldots, \alpha + 1, \alpha, \ldots, \alpha)$ with $i$ copies of $\alpha + 1$ is a fundamental representation; its Drinfeld polynomials are given by

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**Corollary**

Every finite-dimensional irreducible representation of the Yangian $\mathcal{Y}(\mathfrak{gl}_N)$ is isomorphic to a subquotient of a tensor product of fundamental representations.
Remark

Contrary to the case $N = 2$, it is not true for $N \geq 3$ that every finite-dimensional irreducible representation of $Y(sl_N)$ is isomorphic to a tensor product of evaluation modules. For example, the $Y(sl_3)$-module $L(\lambda(u))$ with

$$\lambda_1(u) = (1 + 3u^{-1})(1 + u^{-1}),$$

$$\lambda_2(u) = 1 + 3u^{-1}, \quad \lambda_3(u) = 1 + 2u^{-1}$$

is 8-dimensional. On the other hand, the possible dimensions of the evaluation modules are $1, 3, 6, 8, \ldots$ so that $L(\lambda(u))$ cannot be isomorphic to a tensor product of such modules.
Irreducibility criterion for tensor products
of evaluation modules

Let the $\lambda^{(i)}$ be $\mathfrak{gl}_N$-highest weights.
Irreducibility criterion for tensor products of evaluation modules

Let the $\lambda^{(i)}$ be $\mathfrak{gl}_N$-highest weights.

Theorem (Binary property). The $\mathcal{Y}(\mathfrak{gl}_N)$-module

$$L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \ldots \otimes L(\lambda^{(l)})$$

is irreducible if and only if the modules $L(\lambda^{(i)}) \otimes L(\lambda^{(j)})$ are irreducible for all $1 \leq i < j \leq l$. 
Let
\[ \lambda = (\lambda_1, \ldots, \lambda_N), \quad \mu = (\mu_1, \ldots, \mu_N) \]
with \( \lambda_i, \mu_i \in \mathbb{Z} \) and
\[ \lambda_1 \geq \cdots \geq \lambda_N, \quad \mu_1 \geq \cdots \geq \mu_N. \]
Let

$$\lambda = (\lambda_1, \ldots, \lambda_N), \quad \mu = (\mu_1, \ldots, \mu_N)$$

with $\lambda_i, \mu_i \in \mathbb{Z}$ and

$$\lambda_1 \geq \cdots \geq \lambda_N, \quad \mu_1 \geq \cdots \geq \mu_N.$$

We will call two disjoint finite subsets $A$ and $B$ of $\mathbb{Z}$ crossing if there exist elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that

$$a_1 < b_1 < a_2 < b_2 \quad \text{or} \quad b_1 < a_1 < b_2 < a_2.$$  

Otherwise, $A$ and $B$ are called non-crossing.
For any $\mathfrak{gl}_N$-highest weight $\lambda$ with integer components introduce the subset $\mathcal{A}_\lambda \subset \mathbb{Z}$ by

$$\mathcal{A}_\lambda = \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_N - N + 1\}.$$
For any \( \mathfrak{gl}_N \)-highest weight \( \lambda \) with integer components introduce the subset \( A_\lambda \subset \mathbb{Z} \) by

\[
A_\lambda = \{ \lambda_1, \lambda_2 - 1, \ldots, \lambda_N - N + 1 \}.
\]

**Theorem**

The \( Y(\mathfrak{gl}_N) \)-module \( L(\lambda) \otimes L(\mu) \) is irreducible if and only if the sets \( A_\lambda \setminus A_\mu \) and \( A_\mu \setminus A_\lambda \) are non-crossing.
Example. The $Y(gl_4)$-module $L(7, 5, 5, 4) \otimes L(9, 8, 8, 6)$ is irreducible:
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\[
A_\lambda \quad 1 \quad 3 \quad 4 \quad 7
\]

\[
A_\mu \quad 3 \quad 6 \quad 7 \quad 9
\]

The $\mathcal{Y}(\mathfrak{gl}_4)$-module $L(7, 6, 6, 4) \otimes L(9, 8, 8, 6)$ is reducible:

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Example. The $\mathcal{Y}(\mathfrak{gl}_4)$-module $L(7,5,5,4) \otimes L(9,8,8,6)$ is irreducible:

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3 & 6 & 7 & 9 & \mathcal{A}_\mu \\
\bullet & \bullet & \bullet & \circ & \circ & \circ \\
\mathcal{A}_\lambda & 1 & 3 & 4 & 7 \\
\end{array}\]

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3 & 6 & 7 & 9 & \mathcal{A}_\mu \\
\bullet & \bullet & \bullet & \circ & \circ & \circ \\
\mathcal{A}_\lambda & 1 & 4 & 5 & 7 \\
\end{array}\]
The irreducible representations of $\mathfrak{S}_k$ over $\mathbb{C}$ are parameterized by partitions of $k$. Given a partition $\lambda$ of $k$ denote the corresponding irreducible representation of $\mathfrak{S}_k$ by $V_\lambda$. The vector space $V_\lambda$ is equipped with an $\mathfrak{S}_k$-invariant inner product $(\ , \ )$. The orthonormal Young basis $\{v_{\mathcal{U}}\}$ of $V_\lambda$ is parameterized by the set of standard $\lambda$-tableaux $\mathcal{U}$. 
Set $s_i = (i, i + 1)$ for $i \in \{1, \ldots, k - 1\}$. We have
\[ s_i \cdot v_{\mathcal{U}} = d v_{\mathcal{U}} + \sqrt{1 - d^2} v_{s_i \mathcal{U}}, \]
where $d = (c_{i+1} - c_i)^{-1}$ and $c_i = c_i(\mathcal{U})$ the content of the cell occupied by the number $i$ in a standard $\lambda$-tableau $\mathcal{U}$. The tableau $s_i \mathcal{U}$ is obtained from $\mathcal{U}$ by swapping the entries $i$ and $i + 1$. 
The group algebra $\mathbb{C}[S_k]$ is isomorphic to the direct sum of matrix algebras

$$\mathbb{C}[S_k] \cong \bigoplus_{\lambda \vdash k} \text{Mat}_{f_\lambda}(\mathbb{C}),$$

where $f_\lambda = \dim V_\lambda$. The matrix units $e_{UU'} \in \text{Mat}_{f_\lambda}(\mathbb{C})$ are parameterized by pairs of standard $\lambda$-tableaux $U$ and $U'$. 
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where $f_\lambda = \dim V_\lambda$. The matrix units $e_{UU'} \in \text{Mat}_{f_\lambda}(\mathbb{C})$ are parameterized by pairs of standard $\lambda$-tableaux $U$ and $U'$. Identify $\mathbb{C}[S_k]$ with the direct sum of matrix algebras by

$$e_{UU'} = \frac{f_\lambda}{k!} \phi_{UU'},$$

where $\phi_{UU'}$ is the matrix element corresponding to the basis vectors $v_U$ and $v_{U'}$ of the representation $V_\lambda$,

$$\phi_{UU'} = \sum_{s \in S_k} (s \cdot v_U, v_{U'}) \cdot s^{-1} \in \mathbb{C}[S_k].$$
For the diagonal elements we will simply write $e_u = e_{uu}$ and $\phi_u = \phi_{uu}$.
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The Jucys–Murphy elements of $\mathbb{C}[\mathfrak{S}_k]$ are defined by

$$x_1 = 0, \quad x_i = (1 \ i) + (2 \ i) + \cdots + (i - 1 \ i), \quad i = 2, \ldots, k.$$ 

They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, $x_k$ commutes with all elements of $\mathfrak{S}_{k-1}$. 
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They generate a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_k]$. Moreover, $x_k$ commutes with all elements of $\mathfrak{S}_{k-1}$.

The vectors of the Young basis are eigenvectors for the action of $x_i$ on $V_\lambda$. For any standard $\lambda$-tableau $U$ we have

$$x_i \cdot v_U = c_i(U) v_U, \quad i = 1, \ldots, k.$$
Fix a standard $\lambda$-tableau $\mathcal{U}$ and denote by $\mathcal{V}$ the standard tableau obtained from $\mathcal{U}$ by removing the cell $\alpha$ occupied by $k$. Denote the shape of $\mathcal{V}$ by $\mu$.  

Proposition (Murphy's formula). We have the relation in $C[S_k]$, 

$$e_\mathcal{U} = e_\mathcal{V} \left( x_k - a_1 \right) \cdots \left( x_k - a_l \right) \left( c - a_1 \right) \cdots \left( c - a_l \right),$$

where $a_1, \ldots, a_l$ are the contents of all addable cells of $\mu$ except for $\alpha$, while $c$ is the content of the latter. Equivalently, 

$$e_\mathcal{U} = e_\mathcal{V} u - c u - x_k \Big|_{u = c}.$$
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**Proposition (Murphy’s formula).** We have the relation in $\mathbb{C}[\mathfrak{S}_k]$,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_k - a_1) \ldots (x_k - a_l)}{(c - a_1) \ldots (c - a_l)},$$

where $a_1, \ldots, a_l$ are the contents of all addable cells of $\mu$ except for $\alpha$, while $c$ is the content of the latter.
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**Proposition (Murphy’s formula).** We have the relation in $\mathbb{C}[S_k]$, 

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_k - a_1) \ldots (x_k - a_l)}{(c - a_1) \ldots (c - a_l)},$$

where $a_1, \ldots, a_l$ are the contents of all addable cells of $\mu$ except for $\alpha$, while $c$ is the content of the latter.

Equivalently,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \left. \frac{u - c}{u - x_k} \right|_{u=c}.$$
For any distinct indices $i, j \in \{1, \ldots, k\}$ introduce the rational function in two variables $u, v$ with values in the group algebra $\mathbb{C}[S_k]$ by

$$\rho_{ij}(u, v) = 1 - \frac{(ij)}{u - v}. $$
For any distinct indices \( i, j \in \{1, \ldots, k\} \) introduce the rational function in two variables \( u, v \) with values in the group algebra \( \mathbb{C}[\mathfrak{S}_k] \) by

\[
\rho_{ij}(u, v) = 1 - \frac{(ij)}{u - v}.
\]

**Proposition**

*Let \( r \) be a fixed index, \( r \geq k + 1 \). We have the equalities of rational functions in \( u \) valued in \( \mathbb{C}[\mathfrak{S}_r] \),*

\[
\phi_U \rho_{k,r}(-c_k, u) \cdots \rho_{1,r}(-c_1, u) = \rho_{1,r}(-c_1, u) \cdots \rho_{k,r}(-c_k, u) \phi_U
\]

\[
= \phi_U \left( 1 + \frac{(1r) + (2r) + \cdots + (kr)}{u} \right).
\]
Take $k$ complex variables $u_1, \ldots, u_k$ and set

$$\phi(u_1, \ldots, u_k) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \times \ldots \rho_{1k}(u_1, u_k) \rho_{2k}(u_2, u_k) \ldots \rho_{k-1,k}(u_{k-1}, u_k).$$
Take $k$ complex variables $u_1, \ldots, u_k$ and set

$$
\phi(u_1, \ldots, u_k) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \\
\times \ldots \rho_{1k}(u_1, u_k) \rho_{2k}(u_2, u_k) \ldots \rho_{k-1,k}(u_{k-1}, u_k).
$$

**Theorem**

Suppose that $\lambda$ is a partition of $k$ and let $\mathcal{U}$ be a standard $\lambda$-tableau. Set $c_i = c_i(\mathcal{U})$ for $i = 1, \ldots, k$. 
Take $k$ complex variables $u_1, \ldots, u_k$ and set

$$
\phi(u_1, \ldots, u_k) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \\
\times \ldots \rho_{1k}(u_1, u_k) \rho_{2k}(u_2, u_k) \ldots \rho_{k-1,k}(u_{k-1}, u_k).
$$

**Theorem**

Suppose that $\lambda$ is a partition of $k$ and let $\mathcal{U}$ be a standard $\lambda$-tableau. Set $c_i = c_i(\mathcal{U})$ for $i = 1, \ldots, k$.

Then the consecutive evaluations

$$
\phi(u_1, \ldots, u_k) \bigg|_{u_1 = c_1} \bigg|_{u_2 = c_2} \ldots \bigg|_{u_k = c_k}
$$

of the rational function $\phi(u_1, \ldots, u_k)$ are well-defined. The corresponding value coincides with the matrix element $\phi_{\mathcal{U}}$. 
Example: \( \lambda = (k) \). Then

\[
\mathcal{U} = \begin{array}{cccc}
1 & 2 & \cdots & k \\
\end{array}
\]

\( c_i = i - 1 \),
Example: $\lambda = (k)$. Then

$$U = \begin{bmatrix} 1 & 2 & \cdots & k \end{bmatrix} \quad c_i = i - 1,$$

and

$$\phi_U = \sum_{\sigma \in \mathcal{S}_k} \sigma,$$

is the symmetrizer in $\mathbb{C}[\mathcal{S}_k]$. 
Example: \( \lambda = (k) \). Then

\[
\mathcal{U} = \begin{bmatrix} 1 & 2 & \cdots & k \end{bmatrix} \quad c_i = i - 1,
\]

and

\[
\phi_{\mathcal{U}} = \sum_{\sigma \in \mathfrak{S}_k} \sigma,
\]

is the symmetrizer in \( \mathbb{C}[\mathfrak{S}_k] \). By the theorem,

\[
\phi_{\mathcal{U}} = \left(1 + \frac{(12)}{1}\right) \left(1 + \frac{(13)}{2}\right) \left(1 + \frac{(23)}{1}\right) \times \cdots \left(1 + \frac{(1k)}{k-1}\right) \left(1 + \frac{(2k)}{k-2}\right) \cdots \left(1 + \frac{(k-1k)}{1}\right).
\]
Example: $\lambda = (1^k)$. Then

$$U = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix}$$

$c_i = -i + 1,$
Example: \( \lambda = (1^k) \). Then

\[
U = \begin{bmatrix}
1 \\
2 \\
\vdots \\
k
\end{bmatrix}
\]

\( c_i = -i + 1 \),

and \( \phi_U = \sum_{\sigma \in \mathcal{S}_k} \text{sgn } \sigma \cdot \sigma \) is the anti-symmetrizer in \( \mathbb{C}[\mathcal{S}_k] \).
Example: \( \lambda = (1^k) \). Then

\[
U = \begin{array}{c}
1 \\
2 \\
\vdots \\
k
\end{array}
\]

and \( \phi_U = \sum_{\sigma \in \mathcal{S}_k} \text{sgn} \sigma \cdot \sigma \) is the anti-symmetrizer in \( \mathbb{C}[\mathcal{S}_k] \),

\[
\phi_U = \left(1 - \frac{(12)}{1}\right) \left(1 - \frac{(13)}{2}\right) \left(1 - \frac{(23)}{1}\right) \\
\times \ldots \left(1 - \frac{(1k)}{k-1}\right) \left(1 - \frac{(2k)}{k-2}\right) \ldots \left(1 - \frac{(k-1k)}{1}\right).
\]
Example: \( \lambda = (2, 1) \),

\[
U = \begin{pmatrix} 1 & 2 \\ 3 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 3 \\ 2 \end{pmatrix}
\]
Example: $\lambda = (2, 1)$,

\[
\begin{bmatrix}
1 & 2 \\
3 & & \end{bmatrix} \quad \begin{bmatrix}
1 & 3 \\
2 & & \end{bmatrix}
\]

Then $c_1 = 0, \ c_2 = 1, \ c_3 = -1$ for $\mathcal{U}$, and

\[
\phi_\mathcal{U} = \left(1 + (12)\right) \left(1 - (13)\right) \left(1 - \frac{(23)}{2}\right),
\]
Example: $\lambda = (2, 1)$,

$$
\begin{bmatrix}
1 & 2 \\
3 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
2 & 2
\end{bmatrix}

Then $c_1 = 0, \ c_2 = 1, \ c_3 = -1$ for $U$, and

$$
\phi_U = \left(1 + (12)\right) \left(1 - (13)\right) \left(1 - \frac{(23)}{2}\right),
$$

while $c_1 = 0, \ c_2 = -1, \ c_3 = 1$ for $V$, and

$$
\phi_V = \left(1 - (12)\right) \left(1 + (13)\right) \left(1 + \frac{(23)}{2}\right).
$$
Example: \( \lambda = (2^2) \),

\[
\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \\
\times \rho_{14}(u_1, u_4) \rho_{24}(u_2, u_4) \rho_{34}(u_3, u_4).
\]
Example: \( \lambda = (2^2) \),

\[
\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \\
\times \rho_{14}(u_1, u_4) \rho_{24}(u_2, u_4) \rho_{34}(u_3, u_4).
\]

Take the standard \( \lambda \)-tableau

\[
\mathcal{U} = \begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\]
Example: \( \lambda = (2^2), \)

\[
\phi(u_1, u_2, u_3, u_4) = \rho_{12}(u_1, u_2) \rho_{13}(u_1, u_3) \rho_{23}(u_2, u_3) \\
\times \rho_{14}(u_1, u_4) \rho_{24}(u_2, u_4) \rho_{34}(u_3, u_4).
\]

Take the standard \( \lambda \)-tableau

\[
\mathcal{U} = \begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\]

The contents are \( c_1 = 0, \quad c_2 = 1, \quad c_3 = -1, \quad c_4 = 0. \)
Taking $u_1 = 0, \ u_2 = 1, \ u_3 = -1, \ u_4 = u$ we get

$$\phi(0, 1, -1, u) = \left(1 + (1\ 2)\right) \left(1 - (1\ 3)\right) \left(1 - \frac{2\ 3}{2}\right)$$

$$\times \left(1 + \frac{1\ 4}{u}\right) \left(1 + \frac{2\ 4}{u - 1}\right) \left(1 + \frac{3\ 4}{u + 1}\right).$$
Taking $u_1 = 0$, $u_2 = 1$, $u_3 = -1$, $u_4 = u$ we get

$$\phi(0, 1, -1, u) = \left(1 + (1\ 2)\right)\left(1 - (1\ 3)\right)\left(1 - \frac{(2\ 3)}{2}\right)$$

$$\times \left(1 + \frac{(1\ 4)}{u}\right)\left(1 + \frac{(2\ 4)}{u - 1}\right)\left(1 + \frac{(3\ 4)}{u + 1}\right).$$

By the theorem, this rational function is regular at $u = 0$ and the corresponding value coincides with $\phi_U$. 
We have

\[
\phi(0, 1, -1, u) = \phi_V \left( 1 + \frac{(14)}{u} \right) \left( 1 + \frac{(24)}{u - 1} \right) \left( 1 + \frac{(34)}{u + 1} \right),
\]
We have

\[ \phi(0, 1, -1, u) = \phi_V \left( 1 + \left( \frac{14}{u} \right) \right) \left( 1 + \left( \frac{24}{u - 1} \right) \right) \left( 1 + \left( \frac{34}{u + 1} \right) \right), \]

where

\[ V = \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} \]
Next step:

\[ \phi_\mathcal{V} \left( 1 + \frac{(14)}{u} \right) \left( 1 + \frac{(24)}{u - 1} \right) \left( 1 + \frac{(34)}{u + 1} \right) = \prod_{i=1}^{3} \left( 1 - \frac{1}{(u - c_i)^2} \right) \frac{u}{u - c_4} \cdot \phi_\mathcal{V} \frac{u - c_4}{u - x_4}, \]
Next step:

\[ \phi_\nu \left( 1 + \frac{(14)}{u} \right) \left( 1 + \frac{(24)}{u - 1} \right) \left( 1 + \frac{(34)}{u + 1} \right) \]

\[ = \prod_{i=1}^{3} \left( 1 - \frac{1}{(u - c_i)^2} \right) \frac{u}{u - c_4} \cdot \phi_\nu \frac{u - c_4}{u - x_4}, \]

where \( c_1 = 0, \ c_2 = 1, \ c_3 = -1, \ c_4 = 0 \) and \( x_4 = (14) + (24) + (34). \)
Finally, apply Murphy’s formula to get

\[
\prod_{i=1}^{3} \left(1 - \frac{1}{(u - c_i)^2}\right) \frac{u}{u - c_4} \cdot \phi \left. \frac{u - c_4}{u - x_4} \right|_{u = c_4} = \phi u.
\]
Finally, apply Murphy’s formula to get

\[ \prod_{i=1}^{3} \left( 1 - \frac{1}{(u - c_i)^2} \right) \frac{u}{u - c_4} \cdot \phi \left. \frac{u - c_4}{u - x_4} \right|_{u = c_4} = \phi_U. \]

Thus,

\[ \phi_U = \phi(0, 1, -1, 0) \]

\[ = \frac{1}{2} \left( 1 + (1 2) \right) \left( 1 - (1 3) \right) \left( 2 - (2 3) \right) \]

\[ \times \left( 2 - (1 4) - (2 4) - (3 4) \right) \left( 2 + (1 4) + (2 4) + (3 4) \right). \]
The symmetric group $\mathfrak{S}_k$ acts naturally on the tensor product space

$$\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N,$$  \quad  \text{$k$ factors,}

by permuting the factors. On the other hand, $\mathbb{C}^N$ carries the vector representation of the Lie algebra $\mathfrak{gl}_N$ so that the tensor product space is a representation of $\mathfrak{gl}_N$. 
The symmetric group $\mathcal{S}_k$ acts naturally on the tensor product space

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by permuting the factors. On the other hand, $\mathbb{C}^N$ carries the vector representation of the Lie algebra $\mathfrak{gl}_N$ so that the tensor product space is a representation of $\mathfrak{gl}_N$.

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a partition of $k$ with $\ell(\lambda) \leq N$. Consider an arbitrary standard $\lambda$-tableau $\mathcal{U}$ and let $\Phi_\mathcal{U} \in \text{End} (\mathbb{C}^N)^{\otimes k}$ denote the image of the matrix element $\phi_\mathcal{U}$ under the action of $\mathcal{S}_k$ on the tensor product space.
Then the subspace

$$L_U = \Phi_U(C^N \otimes k)$$

is a $\mathfrak{gl}_N$-submodule of the tensor product module. This submodule is irreducible and isomorphic to $L(\lambda)$. 
Then the subspace

$$L_U = \Phi_U (\mathbb{C}^N)^\otimes k$$

is a $\mathfrak{gl}_N$-submodule of the tensor product module. This submodule is irreducible and isomorphic to $L(\lambda)$.

If $U = U^r$ is the row tableau of shape $\lambda$, then the subspace $L_{U^r}$ coincides with the image of the Young symmetrizer,

$$L_{U^r} = H_{U^r} A_{U^r} (\mathbb{C}^N)^\otimes k,$$

where $H_{U^r}$ and $A_{U^r}$ are the row symmetrizer and column anti-symmetrizer of $U^r$. 
In the vector representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$ we have $E_{ij} \mapsto e_{ij}$ and so the image of the matrix $E = \sum_{i,j=1}^N e_{ij} \otimes E_{ij}$ under the action of $\mathfrak{gl}_N$ can be written as

$$
\sum_{a=1}^k \sum_{i,j=1}^N e_{ij} \otimes 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (k-a)} \in \text{End} \mathbb{C}^N \otimes \text{End} (\mathbb{C}^N)^{\otimes k}.
$$
In the vector representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$ we have $E_{ij} \mapsto e_{ij}$ and so the image of the matrix $E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij}$ under the action of $\mathfrak{gl}_N$ can be written as

$$\sum_{a=1}^{k} \sum_{i,j=1}^{N} e_{ij} \otimes 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (k-a)} \in \text{End } \mathbb{C}^N \otimes \text{End } (\mathbb{C}^N)^{\otimes k}.$$ 

Hence, under the evaluation homomorphism

$$T(u) \mapsto 1 + E u^{-1},$$

the image of $T^t(u)$ in the representation $L_{\mathcal{U}}$ is

$$T^t(u) \mapsto 1 + (P_{01} + P_{02} + \cdots + P_{0k}) u^{-1}.$$
In particular, if $k = 1$ then this takes the form

$$T^t(u) \mapsto R_{01}(-u),$$

where we have used the Yang $R$-matrix.
In particular, if $k = 1$ then this takes the form

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where we have used the Yang $R$-matrix.

For any complex number $z$ we can make the vector space $\mathbb{C}^N$ into a representation of $\mathcal{Y}(\mathfrak{gl}_N)$ by the assignment

$$T^t(u) \mapsto R_{01}(-u - z).$$
In particular, if \(k = 1\) then this takes the form

\[
T^t(u) \mapsto R_{01}(-u),
\]

where we have used the Yang \(R\)-matrix.

For any complex number \(z\) we can make the vector space \(\mathbb{C}^N\) into a representation of \(\mathcal{Y}(\mathfrak{gl}_N)\) by the assignment

\[
T^t(u) \mapsto R_{01}(-u - z).
\]

More generally, \(\mathcal{Y}(\mathfrak{gl}_N)\) acts on \((\mathbb{C}^N)^\otimes k\) by

\[
T^t(u) \mapsto R_{01}(-u - z_1) R_{02}(-u - z_2) \ldots R_{0k}(-u - z_k),
\]

where \(z_1, \ldots, z_k\) are fixed complex numbers.
Consider a standard $\lambda$-tableau $\mathcal{U}$ and for any index $r = 1, \ldots, k$ denote by $c_r = c_r(\mathcal{U})$ the content of the cell of $\mathcal{U}$ occupied by $r$. 
Consider a standard $\lambda$-tableau $\mathcal{U}$ and for any index $r = 1, \ldots, k$ denote by $c_r = c_r(\mathcal{U})$ the content of the cell of $\mathcal{U}$ occupied by $r$.

Proposition

The subspace $L_U$ of $(\mathbb{C}^N)^\otimes k$ is stable under the action of $\mathcal{Y}(\mathfrak{gl}_N)$ defined by

$$T^t(u) \mapsto R_{01}(-u - c_1) R_{02}(-u - c_2) \ldots R_{0k}(-u - c_k).$$

Moreover, the representation of $\mathcal{Y}(\mathfrak{gl}_N)$ on $L_U$ obtained by restriction is isomorphic to the evaluation module $L(\lambda)$. 
Proof.

Observe that $R_{ij}(u - v)$ coincides with the image of the element $\rho_{ij}(u, v)$ under the action of the symmetric group $\mathfrak{S}_{k+1}$ on the tensor product of the vector spaces $\mathbb{C}^N$. Hence, applying the fusion procedure, we get

$$R_{01}(-u - c_1)R_{02}(-u - c_2)\cdots R_{0k}(-u - c_k)\Phi_U$$

$$= \Phi_U \left( 1 + \frac{P_{01} + P_{02} + \cdots + P_{0k}}{u} \right).$$

This implies the first part of the proposition. The second part follows by taking into account that $P_{01} + P_{02} + \cdots + P_{0k}$ commutes with $\Phi_U$. \qed
Gelfand–Tsetlin bases

Given any finite-dimensional irreducible representation of the Yangian $Y(\mathfrak{gl}_N)$, there exists an automorphism of $Y(\mathfrak{gl}_N)$ of the form $T(u) \mapsto f(u) T(u)$ such that its composition with the representation is isomorphic to a subquotient of a tensor product module

$$L(\lambda^{(1)}) \otimes \ldots \otimes L(\lambda^{(p)}),$$

where $L(\lambda^{(i)})$ is the irreducible representation of $\mathfrak{gl}_N$ with the highest weight $\lambda^{(i)}$. 
Gelfand–Tsetlin bases

Given any finite-dimensional irreducible representation of the Yangian $Y(\mathfrak{gl}_N)$, there exists an automorphism of $Y(\mathfrak{gl}_N)$ of the form $T(u) \mapsto f(u) \cdot T(u)$ such that its composition with the representation is isomorphic to a subquotient of a tensor product module

$$L(\lambda^{(1)}) \otimes \ldots \otimes L(\lambda^{(p)}),$$

where $L(\lambda^{(i)})$ is the irreducible representation of $\mathfrak{gl}_N$ with the highest weight $\lambda^{(i)}$.

All generators $t_{ij}^{(r)}$ with $r \geq p + 1$ act as zero operators.
Definition

For any positive integer $p$, the Yangian of level $p$ is the quotient $Y_p(\mathfrak{gl}_N)$ of the algebra $Y(\mathfrak{gl}_N)$ by the ideal generated by all elements $t^{(r)}_{ij}$ with $r \geq p + 1$ and $1 \leq i, j \leq N$. 
Definition

For any positive integer $p$, the Yangian of level $p$ is the quotient $Y_p(\mathfrak{gl}_N)$ of the algebra $Y(\mathfrak{gl}_N)$ by the ideal generated by all elements $t^{(r)}_{ij}$ with $r \geq p + 1$ and $1 \leq i, j \leq N$.

The composition of any finite-dimensional irreducible representation of $Y(\mathfrak{gl}_N)$ with an appropriate automorphism $T(u) \mapsto f(u) T(u)$ can be regarded as a representation of $Y_p(\mathfrak{gl}_N)$ for some $p \geq 1$. If $p = 1$ then the algebra $Y_1(\mathfrak{gl}_N)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{gl}_N)$.
\( \mathcal{Y}_p(\mathfrak{gl}_N) \) can be regarded as an algebra with generators \( t_{ij}^{(r)} \) for \( 1 \leq r \leq p \) and \( 1 \leq i, j \leq N \), subject to the defining relations

\[
(u - \nu) \left[ T_{ij}(u), T_{kl}(\nu) \right] = T_{kj}(u) T_{il}(\nu) - T_{kj}(\nu) T_{il}(u),
\]

where

\[
T_{ij}(u) = \delta_{ij} u^p + t_{ij}^{(1)} u^{p-1} + \cdots + t_{ij}^{(p)}.
\]
The irreducible representation $L(\lambda(u))$ is generated by a nonzero vector $\zeta$ such that

$$T_{ij}(u)\zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq N,$$

$$T_{ii}(u)\zeta = \lambda_i(u)\zeta \quad \text{for} \quad 1 \leq i \leq N,$$

where $\lambda_i(u)$ is a monic polynomial in $u$ of degree $p$. Write

$$\lambda_i(u) = (u + \lambda_i^{(1)})(u + \lambda_i^{(2)}) \ldots (u + \lambda_i^{(p)}), \quad i = 1, \ldots, N.$$
The irreducible representation $L(\lambda(u))$ is generated by a nonzero vector $\zeta$ such that

$$T_{ij}(u) \zeta = 0 \quad \text{for} \quad 1 \leq i < j \leq N,$$

and

$$T_{ii}(u) \zeta = \lambda_i(u) \zeta \quad \text{for} \quad 1 \leq i \leq N,$$

where $\lambda_i(u)$ is a monic polynomial in $u$ of degree $p$. Write

$$\lambda_i(u) = (u + \lambda_i^{(1)})(u + \lambda_i^{(2)}) \ldots (u + \lambda_i^{(p)}), \quad i = 1, \ldots, N.$$

Impose the generality condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \quad \text{for all} \quad i, j \quad \text{and all} \quad k \neq m.$$
The Gelfand–Tsetlin pattern $\Lambda(u)$ (associated with the highest weight $\lambda(u)$) is an array of monic polynomials in $u$ of degree $p$ of the form

\[
\begin{array}{cccc}
\lambda_{N1}(u) & \lambda_{N2}(u) & \ldots & \lambda_{NN}(u) \\
\lambda_{N-1,1}(u) & \ldots & \lambda_{N-1,N-1}(u) \\
\ldots & \ldots & \ldots \\
\lambda_{21}(u) & \lambda_{22}(u) \\
\lambda_{11}(u)
\end{array}
\]
Here the top row coincides with $\lambda(u)$, and we have the betweenness conditions

$$\lambda_{r+1,i}(u) \rightarrow \lambda_{ri}(u) \rightarrow \lambda_{r+1,i+1}(u)$$

for $r = 1, \ldots, N - 1$ and $i = 1, \ldots, r$. 

$\lambda_i(u) \rightarrow \mu_i(u)$ means that there exists a uniquely determined decomposition

$$\mu_i(u) = (u + \mu_1(u)) (u + \mu_2(u)) \cdots (u + \mu_p(u)),$$

$i = 1, \ldots, N - 1$, such that \[\lambda(k)i - \mu(k)i \in \mathbb{Z}^+\] for all $i$ and $k$. 


Here the top row coincides with $\lambda(u)$, and we have the betweenness conditions

$$
\lambda_{r+1,i}(u) \longrightarrow \lambda_{ri}(u) \longrightarrow \lambda_{r+1,i+1}(u)
$$

for $r = 1, \ldots, N - 1$ and $i = 1, \ldots, r$.

Notation

$$
\lambda_i(u) \longrightarrow \mu_i(u)
$$

means that there exists a uniquely determined decomposition

$$
\mu_i(u) = (u + \mu_i^{(1)})(u + \mu_i^{(2)}) \cdots (u + \mu_i^{(p)}), \quad i = 1, \ldots, N - 1,
$$

such that $\lambda_i^{(k)} - \mu_i^{(k)} \in \mathbb{Z}_+$ for all $i$ and $k$. 
Theorem

The representation $L(\lambda(u))$ of $Y_p(\mathfrak{gl}_N)$ admits a basis $\{\zeta_\Lambda\}$ parameterized by all patterns $\Lambda(u)$ associated with the highest weight $\lambda(u)$. 

Corollary (Branching rule).

$L(\lambda(u))|_{Y_p(\mathfrak{gl}_{N-1})} \sim L(\mu(u))L'(\mu(u))$, where $\mu(u)$ runs over all tuples of monic polynomials $\mu(u) = (\mu_1(u), ..., \mu_{N-1}(u))$ of degree $p$ satisfying the betweenness conditions.
Theorem

The representation $L(\lambda(u))$ of $\mathcal{Y}_p(\mathfrak{gl}_N)$ admits a basis $\{\zeta_\Lambda\}$ parameterized by all patterns $\Lambda(u)$ associated with the highest weight $\lambda(u)$.

Corollary (Branching rule).

$$L(\lambda(u))|_{\mathcal{Y}_p(\mathfrak{gl}_{N-1})} \cong \bigoplus_{\mu(u)} L'(\mu(u)),$$

where $\mu(u)$ runs over all tuples of monic polynomials $\mu(u) = (\mu_1(u), \ldots, \mu_{N-1}(u))$ of degree $p$ satisfying the betweenness conditions.
Introduce the polynomials with coefficients in $\mathcal{Y}_p(\mathfrak{gl}_N)$ by

\[
A_r(u) = T_{\frac{1}{1} \cdots \frac{r}{r}}(u), \quad B_r(u) = T_{\frac{1}{1} \cdots \frac{r}{r-1}, \frac{r+1}{r+1}}(u),
\]

\[
C_r(u) = T_{\frac{1}{1} \cdots \frac{r-1}{r-1}, \frac{r+1}{r+1}}(u).
\]
Introduce the polynomials with coefficients in $Y_p(gl_N)$ by

$$A_r(u) = T_{1\ldots r}^1(u), \quad B_r(u) = T_{1\ldots r}^{1\ldots r-1, r+1}(u), \quad C_r(u) = T_{1\ldots r}^{1\ldots r-1,r+1}(u).$$

The coefficients of the polynomials $A_r(u)$ for $r = 1, \ldots, N$ and the polynomials $B_r(u)$ and $C_r(u)$ for $r = 1, \ldots, N - 1$ generate the algebra $Y_p(gl_N)$. 
For a pattern $\Lambda(u)$ due to the generality condition there exist uniquely determined decompositions

$$
\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \ldots (u + \lambda_{ri}^{(p)}), \quad 1 \leq i \leq r \leq N,
$$

such that $\lambda_{Ni}^{(k)} = \lambda_{i}^{(k)}$,

$$
\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \quad \text{and} \quad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+
$$

for $k = 1, \ldots, p$ and $1 \leq i \leq r \leq N - 1$. 
For a pattern $\Lambda(u)$ due to the generality condition there exist uniquely determined decompositions

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \ldots (u + \lambda_{ri}^{(p)}), \quad 1 \leq i \leq r \leq N,$$

such that $\lambda_{Ni}^{(k)} = \lambda_{i}^{(k)}$,

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}^+ \quad \text{and} \quad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}^+$$

for $k = 1, \ldots, p$ and $1 \leq i \leq r \leq N - 1$.

Set

$$l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1, \quad k = 1, \ldots, p \quad \text{and} \quad i = 1, \ldots, r.$$
Theorem

We have

\[ A_r(u) \zeta_\Lambda = \lambda_{r1}(u) \ldots \lambda_{rr}(u - r + 1) \zeta_\Lambda, \]

for \( r = 1, \ldots, N \), and
Theorem

We have

\[ A_r(u) \zeta_\Lambda = \lambda_{r1}(u) \ldots \lambda_{rr}(u - r + 1) \zeta_\Lambda, \]

for \( r = 1, \ldots, N \), and

\[ B_r(-l_{ri}^{(k)}) \zeta_\Lambda = -\lambda_{r+1,1}(-l_{ri}^{(k)}) \ldots \lambda_{r+1,r+1}(-l_{ri}^{(k)} - r) \zeta_{\Lambda + \delta_{ri}^{(k)}}, \]

\[ C_r(-l_{ri}^{(k)}) \zeta_\Lambda = \lambda_{r-1,1}(-l_{ri}^{(k)}) \ldots \lambda_{r-1,r-1}(-l_{ri}^{(k)} - r + 2) \zeta_{\Lambda - \delta_{ri}^{(k)}}, \]

for \( r = 1, \ldots, N - 1 \).
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