ICE-EM/AMSI 2007: MEASURE THEORY
Handout 0
Background On Sets and Real Analysis

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We give here a very brief survey of set theory and real analysis. We've made no attempt at logical rigor or logical minimalism. For more details see, for example, *Real Analysis* by Royden (Prentice Hall, 3rd ed., 1988) or *Foundations of Real and Abstract Analysis* by Bridges (Springer-Verlag, 2005). For a very thorough treatment of set theory see, for example, *Naive Set Theory* by Halmos (Springer-Verlag, 1998) or *Basic Set Theory* by Levy (Dover, 2002).

1 SET THEORY

We take as given the notion of a **set** as a collection of objects. These objects, which can be numbers, functions, sets, animals, politicians, whatever, are the **elements** of the set. (We'll sometimes talk of "collections" of sets or "classes" of sets, not with any logical import, but simply to avoid the clumsy expression "set of sets"). If x is an element of the set A, we write

$$x \in A\,.$$

We also have the notion of one set being a **subset** (or **superset**) of another:

$$\begin{cases} A \subset B \ \text{ if } \ x \in A \implies x \in B\,, \\ A \supset B \ \text{ if } \ x \in B \implies x \in A\,. \end{cases}$$

A set is determined entirely by its elements,

$$(A \subset B \text{ and } A \supset B) \iff A = B\,.$$

The **empty set** $\emptyset$ is the (unique) set with no elements. We have the following standard sets of numbers (discussed briefly in §2 of this handout).

$$\begin{cases} \text{Natural Numbers} & \mathbb{N} = \{1, 2, 3, \ldots\} \\ \text{Integers} & \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\ \text{Rational Numbers} & \mathbb{Q} = \left\{\dfrac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\} \\ \text{Real Numbers} & \mathbb{R} = \{???\} = \overline{\mathbb{Q}} \\ \text{Complex Numbers} & \mathbb{C} = \{x + yi : x, y \in \mathbb{R}\} \end{cases}$$

We construct new sets from old in the standard algebraic manner:

$$\begin{cases} \text{Union} & A \cup B = \{x : x \in A \text{ or } x \in B, \text{ or both}\} \\ \text{Intersection} & A \cap B = \{x : x \in A \text{ and } x \in B\} \\ \text{Relative Complement} & A{\sim}B = \{x : x \in A \text{ and } x \notin B\} \end{cases}$$

If the context is clear, we'll use the abbeviation ${\sim}B$ for $A{\sim}B$. For example, we'll write ${\sim}\mathbb{Q}$ for the set $\mathbb{R}{\sim}\mathbb{Q}$ of irrational numbers.

We have the standard algebraic rules:

$$\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \\ (A \cup B) \cup C = A \cup (B \cup C) & \text{(so we can unambiguously write } A \cup B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) & \text{(so we can unambiguously write } A \cap B \cap C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ {\sim}{\sim}A = A \\ \left.\begin{array}{l} {\sim}(A \cap B) = {\sim}A \cup {\sim}B \\ {\sim}(A \cup B) = {\sim}A \cap {\sim}B \end{array}\right\} & (\textbf{De Morgan's Laws}) \end{cases}$$

The **direct product** of two sets A and B is defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}\,.$$

(Here (a, b) is an **ordered pair**, which we can take as a given notion, but can be formally defined as $\{a, \{a, b\}\}$). As a special case,

$$A^2 = A \times A = \{(a, b) : a \in A, b \in A\}\,.$$

Identifying $(a,(b,c)) \approx ((a,b),c) \approx (a,b,c)$, we have

$$A \times B \times C = (A \times B) \times C = A \times (B \times C) = \{(a,b,c) : a \in A, b \in B, c \in C\}\,,$$

and similarly for further products.

The **power set** of a set X is the set of all subsets of X:

$$\wp(X) = \{A : A \subset X\}\,.$$

Given a **function** $f : X \to Y$ we define

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in X\}\,.$$

(More precisely, we define a function to be such a collection of ordered pairs). Also

$$\begin{cases} \mathrm{domain}(f) = X\,, \\ \mathrm{range}(f) = f(X) = \{f(x) : x \in X\}\,. \end{cases}$$

f is **surjective** if $f(X) = Y$, and f is **injective** if

$$f(x) = f(z) \implies x = z\,.$$

If f is surjective and injective then f is **bijective**. In this case, f has an **inverse function** $f^{-1} : Y \to X$, for which

$$x = f^{-1}(y) \iff y = f(x)\,.$$

If $A \subset X$ then

$$f(A) = \{f(x) : x \in A\} = \text{the } \textbf{image} \text{ of } A \text{ under } f.$$

If $B \subset Y$ then

$$f^{-1}(B) = \{x : f(x) \in B\} = \text{the } \textbf{preimage} \text{ or } \textbf{inverse image} \text{ of } B \text{ under } f.$$

Note that f^{-1} here does not refer to the the inverse function of f. If f is a bijection then $f^{-1}(B)$ is ambiguous, but then the image of B under f^{-1} is identical to the preimage of B under f.

Note that

$$\begin{cases} f(A \cup B) = f(A) \cup f(B), \\ f(A \cap B) \subset f(A) \cap f(B)\,. \end{cases}$$

The preimage is better behaved:

$$\begin{cases} f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \\ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)\,, \\ f^{-1}(\sim A) = \sim f^{-1}(A)\,. \end{cases}$$

We also have

$$\begin{cases} f\left(f^{-1}(A)\right) \subset A \text{ with equality if } f \text{ is surjective} \\ f^{-1}\left(f(B)\right) \supset B \text{ with equality if } f \text{ is injective.} \end{cases}$$

A **finite sequence** $a_1, a_2, \ldots, a_n = \{a_j\}_{j=1}^n$ is a function with domain $\{1, 2, \ldots, n\}$. An **infinite sequence** $a_1, a_2, \cdots = \{a_j\}_{j=1}^\infty$ is a function with domain $\mathbb{N}$. Thus an infinite sequence $A_1, A_2, \ldots$ of subsets of X is a function $f : \mathbb{N} \to \wp(X)$. Given such a sequence, we can consider infinite (and finite) unions and intersections:

$$\begin{cases} \displaystyle\bigcup_{j=1}^\infty A_j = \bigcup_{j \in \mathbb{N}} A_j = \{x : x \in A_j \text{ for some } j \in \mathbb{N}\}\,, \\ \displaystyle\bigcap_{j=1}^\infty A_j = \bigcap_{j \in \mathbb{N}} A_j = \{x : x \in A_j \text{ for all } j \in \mathbb{N}\}\,. \end{cases}$$

Here we think of $\mathbb{N}$ as an **indexing set**: we have one set A_j for each $j \in \mathbb{N}$. Other indexing sets are possible - $\mathbb{R}$ being an important example - and we can consider the analogous unions and intersections. Also, if $\mathcal{I}$ is a collection of sets then $\mathcal{I}$ is itself an indexing set for that collection; we write

$$\begin{cases} \displaystyle\bigcup \mathcal{I} = \bigcup_{A \in \mathcal{I}} A = \{x : x \in A \text{ for some } A \in \mathcal{I}\}\,, \\ \displaystyle\bigcap \mathcal{I} = \bigcap_{A \in \mathcal{I}} A = \{x : x \in A \text{ for all } A \in \mathcal{I}\}\,. \end{cases}$$

The rules for images and preimages of functions extend to arbitrary unions and intersections.

For any indexed collection of sets, we have the generalization of the distribution and De Morgan laws above:

$$\begin{cases} \displaystyle A \cap \left(\bigcup_{\alpha \in \mathcal{I}} B_\alpha\right) = \bigcup_{\alpha \in \mathcal{I}} (A \cap B_\alpha)\,, \\ \displaystyle A \cup \left(\bigcap_{\alpha \in \mathcal{I}} B_\alpha\right) = \bigcap_{\alpha \in \mathcal{I}} (A \cup B_\alpha)\,, \\ \displaystyle \sim\left(\bigcap_{\alpha \in \mathcal{I}} B_\alpha\right) = \bigcup_{\alpha \in \mathcal{I}} (\sim B_\alpha)\,, \\ \displaystyle \sim\left(\bigcup_{\alpha \in \mathcal{I}} B_\alpha\right) = \bigcap_{\alpha \in \mathcal{I}} (\sim B_\alpha)\,. \end{cases}$$

A set is **countable** if either $A = \emptyset$ or $A = \text{range}(f)$ for some $f : \mathbb{N} \to X$. This includes the possibility that A is finite (since f need not be injective). A is **countably infinite** if A is countable and not finite. If A is countably infinite then we can write $A = \text{range}(f)$ for some injective $f : \mathbb{N} \to A$. A countable union of countable sets is again countable. A set is **uncountable** if it is not countable.

Next, we have the notion of a **relation** on a set X. For example "x is less than y", or "$x - y$ is a rational number". For a general relation R we write xRy if x holds in that relation to y.[1] R is an **equivalence relation** on X if:

$$\begin{cases} xRx & \text{(reflexivity);} \\ xRy \implies yRx & \text{(symmetry);} \\ xRy \text{ and } yRz \implies xRz & \text{(transitivity).} \end{cases}$$

For example, "$x - y$ is a rational number" is an equivalence relation on $\mathbb{R}$ (and on $\mathbb{Q}$).

If R is an equivalence relation on X and $x \in X$ then we define the **equivalence class** of x by

$$E_x = \{y \in X : xRy\}\,.$$

Then, for any x and y in X, either $E_x = E_y$ or $E_x \cap E_y = \emptyset$. Thus, every element of X is in exactly one equivalence class of R. We write

$$X/R = \{E_x : x \in X\}$$

for the set of equivalence classes of R.

A relation R on X is a **partial ordering** if

$$\begin{cases} xRy \text{ and } yRx \implies x = y & \text{(antisymmetry);} \\ xRy \text{ and } yRz \implies xRz & \text{(transitivity).} \end{cases}$$

For example, $<$ on $\mathbb{R}$ and $\subset$ on $\wp(X)$ are partial orderings (the latter being a reflexive partial ordering).[2] A partial ordering is **linear** if, as well,

$$\text{for all } x, y \in X, \text{ either } x = y \text{ or } xRy \text{ or } yRx.$$

So, $<$ on $\mathbb{R}$ is linear, but $\subset$ on $\wp(X)$ is not linear (if X has more than one element).

An important and deep principle is:

[1] Formally, a relation R is a collection of ordered pairs in $X \times X$, and we write xRy if $(x, y) \in R$.

[2] Some texts require that a partial ordering be reflexive; thus, they would consider $\leq$, but not $<$, to be a partial ordering on $\mathbb{R}$.

Hausdorff Maximal Principle. *Suppose R is a partial ordering on a set X. Then there is a maximal linear ordered $Y \subset X$. That is:*

(i) *Y is linearly ordered by R;*

(ii) *If $Y \subset Z \subset X$ and if Z is linearly ordered by R, then $Z = Y$.*

The Hausdorff Maximal Principle cannot really be proved: the Principle, or something equivalent to it, must be accepted as an axiom (or simply rejected from use). The most famous equivalence is:

The Axiom of Choice. *Suppose $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of non-empty and* **pairwise disjoint** *sets (that is, $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$). Then there is a set A which contains exactly one element from each A_α.*

The finite axiom of choice (i.e. the case where $\mathcal{I}$ is finite) is trivial. The countable axiom of choice is generally considered intuitive, but nonetheless is unprovable from the other standard axioms of set theory. Stared at for long enough, the uncountable axiom of choice can make one feel very uneasy. See *Halmos* and *Levy* for details.

2 REAL ANALYSIS

2.1 The real number system

We take as given the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ of, respectively, **natural numbers, integers** and **rational numbers**; we merely note that $\mathbb{N}$ is effectively characterized by **mathematical induction**: if $A \subset \mathbb{N}$ and

$$\begin{cases} 1 \in A \\ j \in A \implies j+1 \in A \end{cases}$$

then $A = \mathbb{N}$. The constructions of $\mathbb{Z}$ and $\mathbb{Q}$ from $\mathbb{N}$ are then quite natural and easy.[3]

The big step of analysis is to fill in the gaps, to extend $\mathbb{Q}$ to the set $\mathbb{R}$ of **real numbers**. This can be done either *constructively* (where we somehow *make* the real numbers out of the rational numbers), or *axiomatically* (where we simply list certain fundamental properties which we are willing to accept as true). Here we'll do neither: we'll be content to recall the signature properties of $\mathbb{R}$, without regard to whether they are assumed or proved.[4]

The quickest method of characterizing $\mathbb{R}$ is with the concept of **least upper bound**. Given a set $A \subset \mathbb{R}$, we say A is **bounded above** if there is an **upper bound** M for A: that is, for every a,

$$a \in A \implies a \leq M\,.$$

Then α is the **least upper bound** for A if:

$$\begin{cases} \alpha \text{ is an upper bound for } A; \\ \text{If } \beta \text{ is another upper bound for } A \text{ then } \alpha \leq \beta\,. \end{cases}$$

Least Upper Bound Property. *If $A \subset \mathbb{R}$ is non-empty, and if A is bounded above, then A has a least upper bound α.* In this case, we write

$$\alpha = \sup A\,.$$

Similarly, we can consider sets which are **bounded below**, and a non-empty set A which is bounded below will have a **greatest lower bound** $\beta = \inf A$.

[3] See, e.g., Part 1 of *Calculus* by Spivak (Publish or Perish, 3rd ed., 1994).

[4] An axiomatic system of the real numbers consists of **algebraic** laws, such as the **Commutative Law** $(a+b=b+a)$, and laws of **ordering**, such as $a<b \Rightarrow a+c<b+c$. However the system $\mathbb{Q}$ will also satisfy all such laws. What is needed is one last axiom, to capture the "no gaps" property of the real numbers. Any of the limit-type properties we now describe could be taken as this final axiom.

The Least Upper Bound property leads easily to

Archimedean Property. *$\mathbb{N}$ is not bounded above.*

An alternative characterization of $\mathbb{R}$ can be made via infinite sequences. A real-valued sequence $\{a_j\}_{j=1}^{\infty}$ (or $\{a_j\}$ for short) is **increasing** if $a_{j+1} \geq a_j$ for all $j \in \mathbb{N}$. (Note that an "increasing" sequence need not be **strictly** increasing). Analogously, we can define $\{a_j\}$ to be **decreasing**, and $\{a_j\}$ is **monotonic** if $\{a_j\}$ is either increasing or decreasing. $\{a_j\}$ **converges** to $a \in \mathbb{R}$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j \geq N \implies |a_j - a| < \epsilon$. In this case, we write $a = \lim\limits_{j\to\infty} a_j$ or $a_j \to a$. if a_n increases (decreases) to a, we also write $a_n \nearrow a$ $(a_n \searrow a)$.

We now have

Monotonic Sequence Property. *If $\{a_j\}$ is a monotonic and bounded sequence of real numbers then $\{a_j\}$ converges to some $a \in \mathbb{R}$.*

We can also combine the previous concepts: if $\{a_j\}$ is bounded (above and below), but not necessarily monotonic, we can still define

$$\begin{cases} \limsup\limits_{j\to\infty} a_j = \lim\limits_{j\to\infty} \sup\limits_{k\geq j} a_k\,, \\ \liminf\limits_{j\to\infty} a_j = \lim\limits_{j\to\infty} \inf\limits_{k\geq j} a_k\,. \end{cases}$$

Note that

$$\lim_{j\to\infty} a_j = a \quad \iff \quad \begin{cases} \limsup\limits_{j\to\infty} a_j = a \\ \liminf\limits_{j\to\infty} a_j = a\,. \end{cases}$$

We can also characterize $\mathbb{R}$ in terms of **continuous functions**. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, $|x-a| < \delta \implies |f(x)-f(a)| < \epsilon$. Then the no-gaps property of $\mathbb{R}$ is captured by:

Intermediate Value Theorem. *If $f : [a,b] \to \mathbb{R}$ is continuous, and if $f(a) < 0$ and $f(b) > 0$, then there is a number $c \in (a,b)$ such that $f(c) = 0$.*

Further key properties of $\mathbb{R}$ are implied in the results below. As a final property here, we note that, whereas $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable: there is no bijection between $\mathbb{N}$ and $\mathbb{R}$.

2.2 Euclidean space

Given $\mathbb{R}$ and $m \in \mathbb{N}$, we define m**-dimensional Euclidean Space**:

$$\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_m) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, m\} .$$

Of course, choosing $m = 1$ includes $\mathbb{R}$ as a special case. On $\mathbb{R}^m$ we can define the **inner product**,

$$\langle x, y \rangle = \sum_{j=1}^{m} x_j y_j, \qquad x, y \in \mathbb{R}^m ,$$

and the associated **norm**,

$$\|x\| = \sqrt{\langle x, x \rangle} .$$

We can then define

$$d(x, y) = \|x - y\| ,$$

the **distance** between $x, y \in \mathbb{R}^m$. Then, as for $\mathbb{R}$, we can define the concepts of convergent sequences, and **Cauchy** sequences: a sequence $\{x_j\}$ in $\mathbb{R}^m$ is Cauchy if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j, k \geq N \implies \|x_j - x_k\| < \epsilon$. The simplest no-gaps property of Euclidean space is then:

Completeness of Euclidean Space. *Every Cauchy sequence in* $\mathbb{R}^m$ *converges.*

An equivalent characterization can be made in terms of the convergence of **infinite series**. An infinite series $\sum_{j=1}^{\infty} x_j$ in $\mathbb{R}^m$ is defined in terms of the **sequence** $\{s_j\}$ **of partial sums**, where $s_j = \sum_{i=1}^{j} x_i$. Then the series $\sum_{j=1}^{\infty} x_j$ (or $\sum x_j$ for short) is defined to converge if the sequence $\{s_j\}$ converges. A series $\sum x_j$ in $\mathbb{R}^m$ is said to **converge absolutely** if the series $\sum_{j=1}^{\infty} \|x_j\|$ (of real numbers) converges. We then have

Absolute Convergence Test. *Every absolutely convergent series in* $\mathbb{R}^m$ *converges.*

2.3 Normed spaces and inner product spaces

A real **vector space** X is a **normed space** if there is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

$$\begin{cases} \|x\| \geq 0, \text{ and } \|x\| = 0 \text{ iff } x = 0 \\ \|\alpha x\| = |\alpha| \cdot \|x\|, \text{ for all } x \in X \text{ and } \alpha \in \mathbb{R} \\ \|x + y\| \leq \|x\| + \|y\| \qquad \textbf{(Triangle Inequality)} \end{cases}$$

A real vector space X is a (real) **inner product space** if there is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that

$$\begin{cases} \langle x, x \rangle \geq 0, \text{ and } \langle x, x \rangle = 0 \text{ iff } x = 0 \\ \langle x, y \rangle = \langle y, x \rangle \\ \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \text{ for all } x, y, z \in X \text{ and } \alpha, \beta \in \mathbb{R} \end{cases}$$

$\mathbb{R}^m$ is an inner product space, as indicated above. As for $\mathbb{R}^m$, any inner product is a normed space by defining

$$\|x\| = \sqrt{\langle x, x \rangle} .$$

We then have the **Cauchy-Schwartz Inequality**:

$$\langle x, y \rangle \leq \|x\| \cdot \|y\| .$$

2.4 Metric spaces

A **metric space** (X, d) is a set X together with a **distance function** $d : X \times X \to \mathbb{R}$ satisfying:

$$\begin{cases} d(x, y) \geq 0, \text{ and } d(x, y) = 0 \text{ iff } x = y \\ d(x, y) = d(y, x) \\ d(x, z) \leq d(x, y) + d(y, z). \qquad \textbf{(Triangle Inequality)} \end{cases}$$

We write X for (X, d) if the associated metric d is clear. Any normed space, and $\mathbb{R}^m$ in particular, is naturally a metric space by defining

$$d(x, y) = \|x - y\| .$$

A subset $A \subset X$ of a metric space X is a metric **subspace** with the **induced metric** d_A in the obvious manner. On any set X, we can define the **discrete metric** d_s:

$$d_s(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

We can define convergent and Cauchy sequences in a metric space X, as in Euclidean space. X is said to be **complete** if every Cauchy sequence is convergent. Thus, $\mathbb{R}^m$ is complete; by comparison, the interval $(0, 1)$ with the induced metric is not complete. Any discrete metric space is complete.

A complete normed space is called a **Banach Space**, and a complete inner product space is called a **Hilbert Space**. In a normed space, we also have the notion of infinite series being convergent; then a normed space X is a Banach Space iff every absolutely convergent series is convergent.

Given a metric space X, $x \in X$ and $r \in \mathbb{R}$, we define, respectively, the **open ball** and **closed ball** of radius r about x:

$$\begin{cases} B_r(x) = \{y \in X : d(x, y) < r\}\,, \\ C_r(x) = \{y \in X : d(x, y) \leq r\}\,. \end{cases}$$

A set $U \subset X$ is **open** if for every $x \in U$ there is an $r > 0$ such that $B_r(x) \subset U$. A set $C \subset X$ is **closed** if its **complement** $X \sim C$ is open. Equivalently, C is closed if, for any sequence $\{x_j\}$ in C, whenever x_j converges to $x \in X$ then $x \in C$. Note that open balls are open and closed balls are closed, and that X and $\emptyset$ are both open and closed. A closed subset of a complete metric space is complete. With respect to the discrete metric, all sets are both open and closed.

If $A \subset X$ is non-empty and $x \in X$, the **distance** from x to A

$$\mathrm{dist}(A, x) = \inf\{d(a, x) : a \in A\}\,.$$

Given $r \in \mathbb{R}$, $\{x \in X : \mathrm{dist}(A, x) < r\}$ is open and $\{x \in X : \mathrm{dist}(A, x) \leq r\}$ is closed. The **diameter** of a non-empty set A is defined by

$$\mathrm{diam}(A) = \sup\{d(a, b) : a, b \in A\}$$

if this sup exists, and $\mathrm{diam}(A) = \infty$ otherwise; A is **bounded** if $\mathrm{diam}(A) < \infty$.

The **interior** A° of $A \subset X$ is the union of all open sets contained in A, and the **closure** $\bar{A}$ of A is the intersection of all closed sets containing A. Note that $\bar{A}$ is the union of A together with any limit $x \in X$ of a sequence $\{x_j\}$ in A which converges. A is **dense** in X if $\bar{A} = X$; note that $\mathbb{Q}^m$ is dense in $\mathbb{R}^m$, and in particular $\mathbb{Q}$ is dense in $\mathbb{R}$. The **boundary** of $A \subset X$ is defined by

$$\partial A = \overline{A} \cap \overline{\sim A}\,.$$

A metric space X is **compact** if any of the following equivalent conditions hold:

- Any sequence in X has a convergent **subsequence**[5] (**Bolzano-Weierstrass**).
- X is complete and **totally bounded**;[6]
- From any collection $\{O_\alpha\}$ of open sets which covers X, there is a finite subcollection which also covers X (**Heine-Borel**).
- If $\{C_\alpha\}$ is a collection of closed sets in X such that any finite subcollection from $\{C_\alpha\}$ hs non-empty intersection, then all of $\{C_\alpha\}$ has non-empty intersection.

Note, in particular, that a compact metric space is bounded. An important result is that a subset of $\mathbb{R}^m$ is compact iff it is closed and bounded; in particular, any closed and bounded interval $[a, b]$ on $\mathbb{R}$ is compact. If a set X is given the discrete metric, then X is compact iff X is finite.

A real-valued function $f : X \to \mathbb{R}$ on a metric space X is **continuous** if any of the following equivalent conditions hold:

- For any $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$;
- For any $\{x_j\}$ in X, if $x_j \to x$ then $f(x_j) \to f(x)$;
- For any open $U \subset \mathbb{R}$, the inverse image $f^{-1}(U)$ is open in X.
- For any closed $C \subset \mathbb{R}$, the inverse image $f^{-1}(C)$ is closed in X.

If $A \subset X$ then the function $x \mapsto \mathrm{dist}(A, x)$ is continuous. If $f : X \to \mathbb{R}$ is continuous and X is a compact metric space, then f is **uniformly continuous**: for any $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$.

[5] A subsequence $\{x_{j_k}\}_{k=1}^\infty$ of sequence $\{x_j\}_{j=1}^\infty$ is determined by a strictly increasing function (sequence) $k \mapsto j_k$ of natural numbers.

[6] X is totally bounded if, for any $\epsilon > 0$, X is contained in finitely many open balls of radius $\epsilon > 0$.

2.5 Topological spaces

A **topological space** $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T} = \{U_\alpha\}$ of subsets of X, designated as the **open sets**. This collection must satisfy:

$$\begin{cases} \emptyset \text{ and } X \text{ are open} \\ \text{If } A \text{ and } B \text{ are open then } A \cap B \text{ is open} \\ \text{The union of any collection of open sets is open} \end{cases}$$

We write X for the topological space if the topology $\mathcal{T}$ is clear. A **base** for the topology $\mathcal{T}$ is a subcollection $\mathcal{B}$ of open sets such that: for each $U \in \mathcal{T}$ and each $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B \subset U$. Any base completely determines the topology $\mathcal{T}$. A collection $\mathcal{B}$ of subsets of X will be a base for some topology on X if $\cup\mathcal{B} = X$ and if, for any $U, V \in \mathcal{B}$, there is a $W \in \mathcal{B}$ with $W \subset U \cap V$.

For $x \in X$, a **base at** x is a collection $\mathcal{B}_x$ of open sets such that: for each $U \in \mathcal{T}$ containing x there is a $B \in \mathcal{B}_x$ such that $x \in B \subset U$. A metric space X, with open sets as defined above, is a topological space; the open balls are a base for X, and the open balls centred at a given x are a base at x.

The **discrete topology** $\mathcal{T}_\mathbf{s} = \wp(X)$ on X consists of all subsets of X; this is the topology derived from the discrete metric. At the other extreme, the **indiscrete topology** $\mathcal{T}_\mathbf{i} = \{\emptyset, X\}$ consists solely of $\emptyset$ and X.

Closedness, compactness and continuity in a topological space can be defined as for metric spaces above, where we restrict to those characterizations made in terms of open sets or closed sets. In particular, if X and Y are topological spaces then $f : X \to Y$ is defined to be continuous if $f^{-1}(U)$ is open in X for every open $U \subset Y$; if X and Y are metric spaces, this is equivalent to $f(x_j) \to f(x)$ in Y whenever $x_j \to x$ in X, and is equivalent to the $\epsilon - \delta$ definition of continuity. If X is discrete then any $f : X \to Y$ is continuous; at the other extreme, if X is indiscrete, then f is continuous iff f is constant.

If $A \subset X$ is a subset of a topological space X, then A can be given the **subspace topology** by declaring

$$U \text{ is open in } A \quad \Longleftrightarrow \quad U = V \cap A \text{ for some open } V \subset X.$$

The corresponding property then holds for closed sets. If $A \subset X$ is open (closed) in X then $U \subset A$ is open (closed) in A iff U is open (closed) in X. If X is a metric space, then the induced metric d_A give the subspace topology on $A \subset X$.

We note the following elementary properties:

- A closed subset of a compact topological space is compact;
- If $f : X \to Y$ is continuous then
$$\begin{cases} f : X \to B \text{ is continuous for any } B \supset f(X) \\ \text{the } \textbf{restriction } f_{|A} : A \to Y \text{ of } f \text{ is continuous for any } A \subset X \end{cases}$$
- If $A \subset X$ is closed, $f : A \to \mathbb{R}$ is continuous and $f = 0$ on ∂A, then $g : X \to \mathbb{R}$ is continuous where
$$g(x) = \begin{cases} f(x) & x \in A \\ 0 & x \in \sim A \end{cases}$$
- If $f : X \to Y$ is continuous and X is compact then $f(X)$ is compact.
- If $f : X \to \mathbb{R}$ is continuous and X is compact then f is bounded above (and below), and f achieves its maximum (and minimum) on X.
- If $\{f_j : X \to \mathbb{R}\}$ is a sequence of continuous functions **converging uniformly** to f,[7] then f is continuous.

For a topological space X, define

$$C(X) = \{f : f \text{ is a continuous real-valued function on } X\}.$$

We then define $C_0(X)$ as the set of functions in $C(X)$ with **compact support**;[8] of course, if X is compact then $C_0(X) = C(X)$. A consequence of the above results is that $C_0(X)$ is a normed space with the norm

$$\|f\| = \max\{f(x) : x \in X\}.$$

$C_0(X)$ is complete with respect to the metric associated to this norm.

A topological space X is **locally compact** if for every $x \in X$ there in an open set U containing x with $\overline{U}$ compact. Clearly, $\mathbb{R}^m$ is locally compact. If X is an infinite metric space then $C(X)$ is not locally compact; X itself is not locally compact if given the discrete metric.

If X and Y are topological spaces then the product $X \times Y$ can be given the **product topology**. A base for the product topology is the collection of all sets of the form $U_X \times U_Y$,

[7] A sequence $\{f_j : X \to \mathbb{R}\}$ converges uniformly to f if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j \geq N \Longrightarrow \sup_{x \in X} |f_j(x) - f(x)| < \epsilon$.

[8] The support of a function $f : X \to \mathbb{R}$ is $\mathrm{spt}(f) = \overline{f(X)}$.

where U_X is open in X and U_Y is open in Y. The product topology is the weakest topology (i.e. the topology including the fewest "open" sets) such that the **projections** $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous. If X and Y are compact (locally compact) then $X \times Y$ is compact (locally compact).

A topological space X is **metrizable** if there is a metric d on X which gives rise to exactly the open sets of X; of course this will be the case when the topology arises from a metric in the first place. Not all topological spaces are metrizable; as a trivial example, if X contains at least two points, then the indiscrete topology on X is not metrizable (see below). For many more interesting examples, see *Topology* by Munkres (Prentice Hall, 2nd ed.,1999) or *Counterexamples in Topology* by Steen and Seebach (Dover, 1995).

Even if a topological space is metrizable, there will be many metrics which give rise to the same topology. If X and Y are metrizable, then so is $X \times Y$: if d_X metrizes X and d_Y metrizes Y, then

$$d((a,b),(c,d)) = d_X(a,c) + d_Y(b,d)$$

defines a metric which gives rise to the product topology on $X \times Y$. Note that this procedure does *not* give the standard metric on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, introduced previously: "balls" with respect to this product metric are diamond-shaped. However, the product metric and the standard metric do give rise to the same topology on $\mathbb{R}^2$ (and the metrics are also equivalent in stronger, geometric, senses).

Countability properties and **separation properties** indicate the extent to which a topological space behaves like a metric space. A topological space X is **separable** if it has a countable dense subset. X is **first countable** if there is a countable base at every $x \in X$, and X is **second countable** if X has a countable base. We note:

- $\mathbb{R}^m$ is separable;
- Any metric space is first countable;
- A metric space is second countable iff it is separable;

A topological space X is **Hausdorff** if, for any distinct points $c, d \in X$, there are disjoint open sets A and B with $c \in A$ and $d \in B$. A Hausdorff topological space is **normal** if, for any disjoint closed sets C and D, there are disjoint open sets A and B with $C \subset A$ and $D \subset B$. We note that:

- Any compact subset of a Hausdorff space is closed;
- Any singleton subset $\{a\}$ of a Hausdorff space is closed;
- Any metrizable topological space is normal;

Note that if X is indiscrete and contains at least two points, then X is not Hausdorff: it is thus immediate that (X, d_i) is not metrizable. A deeper result is:

Urysohn's Lemma. *Suppose C and D are disjoint closed subsets of a normal space X. Then there is a continuous function $f : X \to \mathbb{R}, 0 \le f \le 1$, such that $f = 0$ on C and $f = 1$ on D.*

Note that Urysohn's Lemma is trivial for a metric space: we can simply define

$$f(x) = \frac{\mathrm{dist}(x,C)}{\mathrm{dist}(x,C) + \mathrm{dist}(x,D)}.$$

If X is locally compact and Hausdorff then Urysohn's Lemma still holds, as long as we assume one of the disjoint closed sets, C or D, is compact.

One use of Urysohn's Lemma is to prove that many normal spaces are in fact metrizable:

Urysohn Metrization Theorem. *If X is normal and second countable then X is metrizable.*

Urysohn's Metrization Theorem continues to hold if X is locally compact, Hausdorff and second countable.