ICE-EM/AMSI Summer School 2007 Differential Geometry Course
Assignment 0: A warm-up

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Here are some review questions on concepts that will be used during the course. By the start of the Wednesday lecture in the first week, please hand in your answer to **one** of the following questions, ideally the one that you found most challenging. Also, please indicate very briefly whether there were any questions that you don't know how to answer, and how hard/easy you found the others. This assignment is *not for credit*, but is designed to help you revise and help me get to know the class.

1. Using spherical coordinates, calculate the surface area of the unit sphere.

2. Suppose that $k : \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable and $Dk(3,0) = \begin{pmatrix} 1 & 5 \\ -1 & 0 \end{pmatrix}$ and $k(3,0) = (1,2)$. If $h(x,y) = \|k(x,y)\|$, compute $\nabla h(3,0)$.

3. Consider $f : \mathbf{R}^3 \to \mathbf{R}^2$ given by $f(x,y,z) = \left(x^2 + y^2, y^2 + z^2\right)$.
   (a) Sketch some of the level sets of f, i.e. $f^{-1}(a,b)$ for various (a,b). Calculate the rank of $Df(x,y,z)$ at all points (x,y,z). What do you notice?
   (b) Use the Implicit Function Theorem to show that, in some neighbourhood of $(1,0,1)$ it is possible to solve the equation $f(x,y,z) = (1,1)$ for x and z as functions of y. Now find those functions explicitly (the Implicit Function Theorem is not needed here).

4. Let V be a finite-dimensional vector space, with dual space V^*. Given any basis on V, define the corresponding dual basis on V^*, and prove that it's a basis. What if V is infinite-dimensional?

5. This question concerns matrices with real entries. (*What would be different for complex matrices?*)
   (a) Prove that $GL(n)$ is a group with respect to matrix multiplication. Prove that $O(n)$ and $SO(n)$ are subgroups of $GL(n)$.
   (b) Let $M(n)$ be the set of $n \times n$ matrices, with its standard topology induced via its isomorphism with $\mathbf{R}^{n^2}$. Prove that $GL(n)$ is an open, non-compact subset of $M(n)$.
   (c) Prove that $O(n)$ is not connected.

6. (a) Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Sketch the vector field $\dot{\mathbf{x}} = A\mathbf{x}$ and find its general solution.
   (b) Do the same for $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

7. Given a differentiable function $H : \mathbf{R}^{2n} \to \mathbf{R}$, with the standard coordinates labelled $q_1, \ldots, q_n, p_1, \ldots, p_n$, *Hamilton's equations* are
$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \text{for } i = 1, \ldots n.$$
   Let $(\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$.
   (a) Prove that H is conserved by every solution to Hamilton's equations, i.e. $\dot{H} = 0$, i.e., if $(\mathbf{q}(t), \mathbf{p}(t))$ is a solution to the equations, then $\frac{d}{dt} H\left(\mathbf{q}(t), \mathbf{p}(t)\right) = 0$ for all t.
   (b) Calculate Hamilton's equations for $H(\mathbf{q}, \mathbf{p}) := \dfrac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{q})$, where m is a constant (the "mass"), $\|\cdot\|$ is the standard Euclidean norm, and $V : \mathbf{R}^n \to \mathbf{R}$ is differentiable function (the "potential"). Write down the equivalent second order system in the $\mathbf{q}$ variables only.
   (c) Solve the equations in part (b) for $n = 1$ and $V(q) = mq$, and sketch the solutions in the (q,p) plane (the "phase plane"). (*These equations model a falling object.*)