

Bases for certain cohomologies with local coefficients of a hyperplane complement

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We consider **regular** functions and differential forms on M :

$$\mathcal{O}(M) = \mathbb{C}[z_1, \dots, z_n][\ell_H^{-1} \mid H \in \mathcal{A}],$$

$$\Omega^p(M) = \bigoplus_{i_1 < \dots < i_p} \mathcal{O}(M) dz_{i_1} \wedge \dots \wedge dz_{i_p}.$$

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$$\text{Then } H^p(M, \mathbb{C}) = \frac{\ker(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M))}{\text{im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M))}.$$

Let $A^\bullet(M)$ be the \mathbb{C} -subalgebra of $\Omega^\bullet(M)$ generated by

$$\omega_H = \frac{d\ell_H}{\ell_H} \in \Omega^1(M), \text{ for } H \in \mathcal{A}.$$

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Theorem. (Brieskorn) $(A^\bullet(M), 0) \xrightarrow{\text{q-iso}} (\Omega^\bullet(M), d)$, i.e.

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Orlik and Solomon found defining relations and a basis for $A^\bullet(M)$, which is now called the **Orlik-Solomon algebra**.

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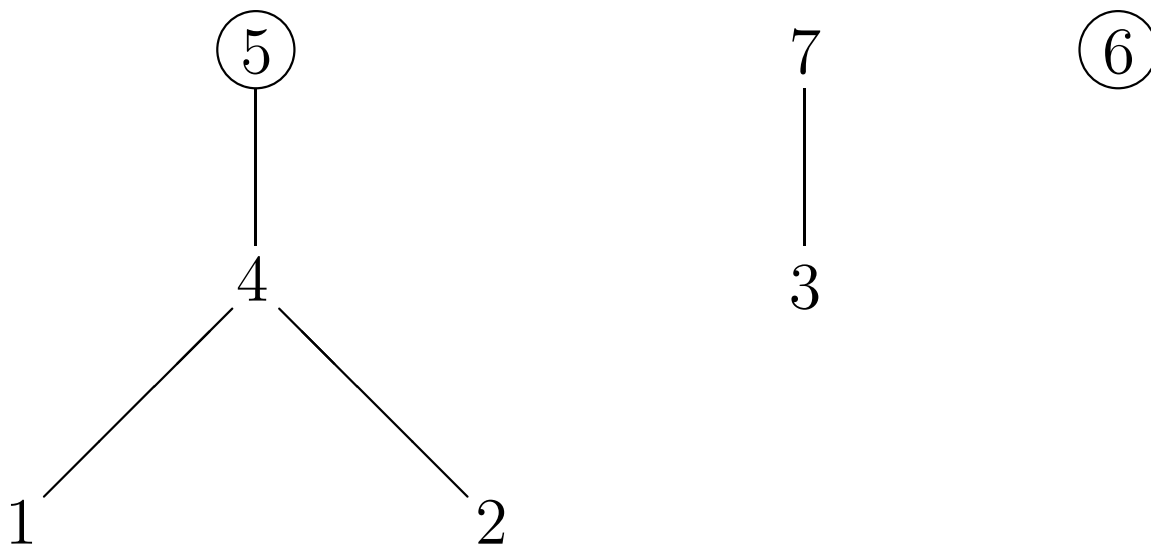
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The Orlik-Solomon presentation of $A^\bullet(M)$ is:

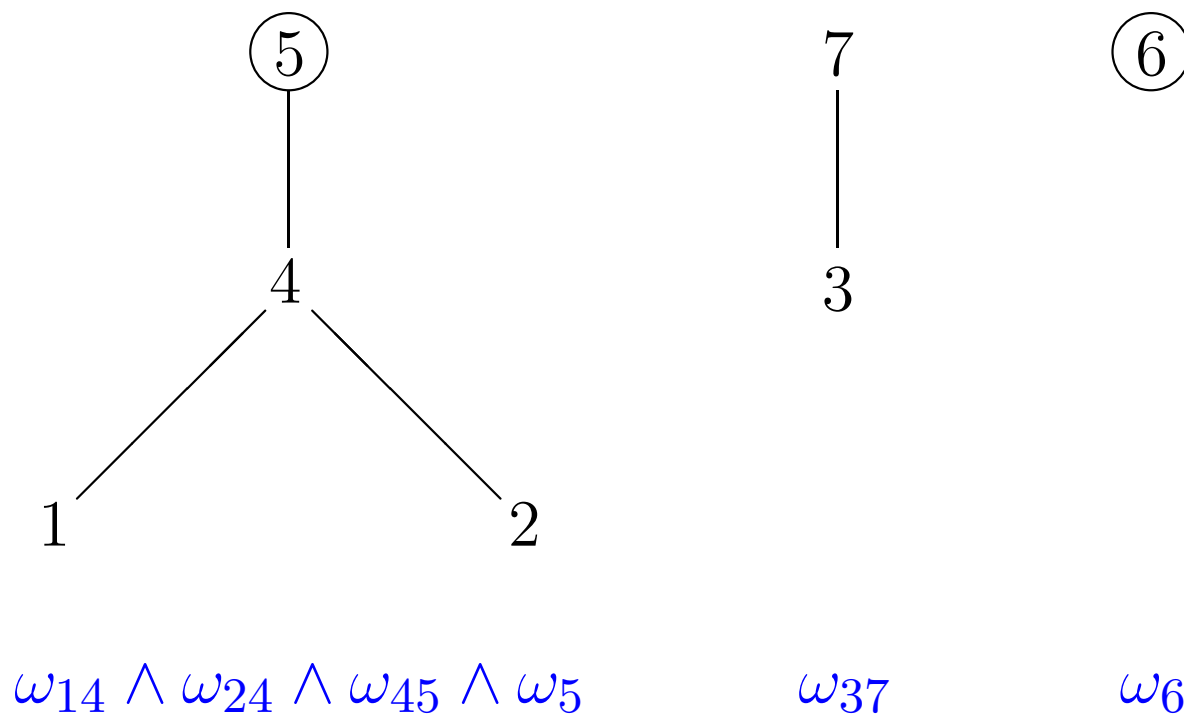
$$\begin{aligned} \langle \omega_i = \frac{dz_i}{z_i}, \omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j} \mid \\ \omega_i \wedge \omega_j = \omega_i \wedge \omega_{ij} - \omega_j \wedge \omega_{ij}, \\ \omega_{ij} \wedge \omega_{jk} + \omega_{ik} \wedge \omega_{ij} + \omega_{jk} \wedge \omega_{ik} = 0 \rangle. \end{aligned}$$

(This case was actually done by Arnol'd.)

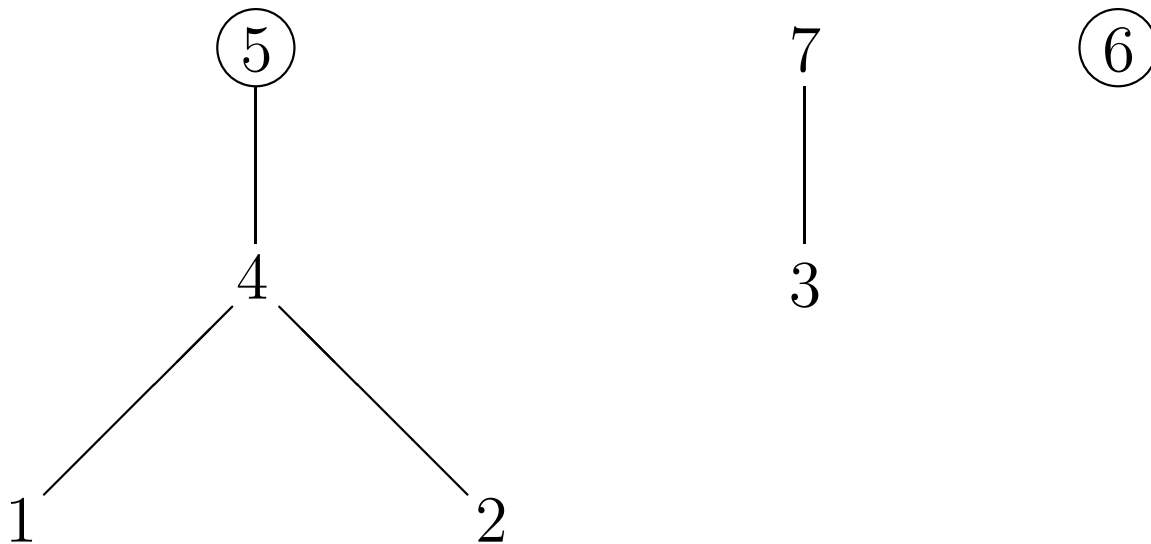
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$$\omega_{14} \wedge \omega_{24} \wedge \omega_{45} \wedge \omega_5 \quad \wedge \quad \omega_{37} \quad \wedge \quad \omega_6$$

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Up to isomorphism, \mathcal{L}_ω depends only on $\omega + \sum \mathbb{Z} \omega_H$, and

$$H^p(M, \mathcal{L}_\omega) = \frac{\ker(d + \omega \wedge - : \Omega^p(M) \rightarrow \Omega^{p+1}(M))}{\operatorname{im}(d + \omega \wedge - : \Omega^{p-1}(M) \rightarrow \Omega^p(M))}.$$

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Theorem. (Esnault, Schechtman, Viehweg)

$(A^\bullet(M), \omega \wedge -) \xrightarrow{\text{q-iso}} (\Omega^\bullet(M), d + \omega \wedge -)$, i.e.

$$\frac{\ker(\omega \wedge - : A^p(M) \rightarrow A^{p+1}(M))}{\operatorname{im}(\omega \wedge - : A^{p-1}(M) \rightarrow A^p(M))} \xrightarrow{\sim} H^p(M, \mathcal{L}_\omega),$$

unless ω satisfies a certain **resonance** condition.

Example continued. Take $\omega = \sum a_i \omega_i$. This is resonant if:

$$\sum_{i \in J} a_i \in \mathbb{Z}^+ \text{ for some } J \subseteq \{1, \dots, n\} \text{ or } \sum_i a_i \in \mathbb{Z}^-.$$

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Theorem. Suppose all $a_i \in \mathbb{Q}$. Then $H^\bullet(M, \mathcal{L}_\omega)$ has a basis given by marked rectified forests on $\{1, \dots, n\}$ which satisfy

$$\sum_{i \in T} a_i \in \mathbb{Z}, \text{ for all trees } T. \quad (*)$$

The differential form associated to a forest is usually **not** in $A^\bullet(M)$, because there are two changes to the definition.

1. If an edge $i—j$ is **unbreakable** (deleting it violates $(*)$), it contributes $\tilde{\omega}_{ij} = \frac{z_j dz_i - z_i dz_j}{z_i z_j (z_i - z_j)}$ rather than ω_{ij} .

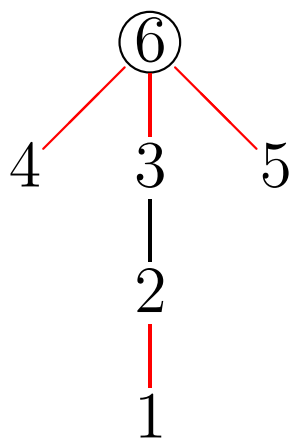
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2. Each tree T carries the coefficient $\prod_{j \in T} z_j^{b_{j,T}}$, where

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E.g. if $a_i = \frac{1}{2}$,



$\longrightarrow z_6$

$$\times (\tilde{\omega}_{12} \wedge \omega_{23} \wedge \tilde{\omega}_{36} \wedge \tilde{\omega}_{46} \wedge \tilde{\omega}_{56} \wedge \omega_6)$$

Proof. Let $r \in \mathbb{Z}^+$ be such that $ra_i \in \mathbb{Z}$ for all i . Define

$$\widetilde{M} = \{(z_i) \in \mathbb{C}^n \mid z_i \text{ nonzero and } z_i^r \text{ distinct}\}.$$

The group $(\mathbb{Z}/r\mathbb{Z})^n$ acts by $(c_i).(z_i) = (\exp(2\pi i \frac{c_i}{r})z_i)$. The quotient map is $\widetilde{M} \rightarrow M : (z_i) \mapsto (z_i^r)$.

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Since the pull-back of \mathcal{L}_ω to \widetilde{M} is trivial,

$$H^p(M, \mathcal{L}_\omega) \xrightarrow{\sim} \{x \in H^p(\widetilde{M}, \mathbb{C}) \mid (c_i).x = \exp(-2\pi\mathbf{i} \sum c_i a_i)x\}.$$

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Moreover, \widetilde{M} is the hyperplane complement for $\mathcal{A} = \{\{z_i = 0\}, \{z_i = \zeta z_j\}\}$. The result follows easily using the Orlik-Solomon basis of $A^\bullet(\widetilde{M})$.