Orbit closures in nilpotent cones

Anthony Henderson

University of Sydney

June 2010

Outline

The nullcone of a representation

Let G be a connected reductive algebraic group over $\mathbb{C}$, and V a finite-dimensional representation of G. In invariant theory, one studies the ring of invariant polynomial functions $\mathbb{C}[V]^G$ and the resulting algebro-geometric quotient

$$\psi : V \twoheadrightarrow G \backslash\backslash V = \operatorname{Spec} \mathbb{C}[V]^G.$$

Usually this is not a quotient map in the strongest sense, because one may have v, w in different G-orbits such that $\psi(v) = \psi(w)$ (i.e. v and w cannot be distinguished by G-invariant polynomials).

The nullcone of a representation

Let G be a connected reductive algebraic group over $\mathbb{C}$, and V a finite-dimensional representation of G. In invariant theory, one studies the ring of invariant polynomial functions $\mathbb{C}[V]^G$ and the resulting algebro-geometric quotient

$$\psi : V \twoheadrightarrow G \backslash\!\backslash V = \operatorname{Spec} \mathbb{C}[V]^G.$$

Usually this is not a quotient map in the strongest sense, because one may have v, w in different G-orbits such that $\psi(v) = \psi(w)$ (i.e. v and w cannot be distinguished by G-invariant polynomials). The Hilbert nullcone of V is

$$\mathcal{N}_G(V) = \psi^{-1}(\psi(0)) = \{v \in V \mid 0 \in \overline{Gv}\}.$$

Thus $\mathcal{N}_G(V)$ is the subvariety (possibly reducible, usually singular) defined by the G-invariant polynomial functions which vanish at 0. It is a ‘cone’ in that it is stable under scalar multiplication. Other fibres of ψ can often be seen as deformations of $\mathcal{N}_G(V)$.

Example

Take $G = GL_2$, $V = \mathbb{C}^2$. The only GL_2-invariant polynomial functions are constants, so $GL_2 \backslash\backslash \mathbb{C}^2$ is a point, $\mathcal{N}_{GL_2}(\mathbb{C}^2) = \mathbb{C}^2$. There are two orbits, $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$.

Example

Take $G = GL_2$, $V = \mathbb{C}^2$. The only GL_2-invariant polynomial functions are constants, so $GL_2 \backslash\backslash \mathbb{C}^2$ is a point, $\mathcal{N}_{GL_2}(\mathbb{C}^2) = \mathbb{C}^2$. There are two orbits, $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$.

Example

Take $G = GL_2$, $V = \mathfrak{gl}_2 = \mathrm{Mat}_2$, on which GL_2 acts by conjugation. The GL_2-equivariant polynomial functions on $\mathfrak{gl}_2$ are generated by trace and determinant, so $GL_2 \backslash\backslash \mathfrak{gl}_2 \cong \mathbb{C}^2$ and

$$\mathcal{N}(\mathfrak{gl}_2) = \{x \in \mathfrak{gl}_2 \text{ nilpotent}\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a^2 + bc = 0 \right\}.$$

Again there are two orbits, $\{0\}$ and $\mathcal{N}(\mathfrak{gl}_2) \setminus \{0\}$. The surface $\mathcal{N}(\mathfrak{gl}_2)$ has a quotient singularity at 0: there is an isomorphism

$$\{\pm 1\} \setminus \mathbb{C}^2 \xrightarrow{\sim} \mathcal{N}(\mathfrak{gl}_2) : \pm(u, v) \mapsto \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}.$$

General concepts and problems

Problems

1. Parametrize the G-orbits in $\mathcal{N}_G(V)$. Are there finitely many?
2. Determine the closure order on the orbits: that is, give a condition for when $Gw \subseteq \overline{Gv}$.
3. If $\mathcal{N}_G(V)$ is singular, find a G-equivariant resolution of singularities, preferably one with nice fibres.
4. Do the same for the singular orbit closures $\overline{Gv}$.
5. Use these resolutions to describe the singularities further, e.g. by computing the local intersection cohomology.
6. Describe the categories of G-equivariant perverse or coherent sheaves on $\mathcal{N}_G(V)$.

The motivation comes from the case of the nilpotent cone $\mathcal{N}(\mathfrak{g})$, where the answers have great significance for representation theory.

Example

Let $V = \mathbb{C}^n$ ($n \geq 3$), and let $G = SO_n$ be the connected stabilizer of the usual nondegenerate symmetric bilinear form. Then

$$\mathcal{N}_{SO_n}(\mathbb{C}^n) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \,|\, x_1^2 + x_2^2 + \cdots + x_n^2 = 0\}.$$

There are two SO_n-orbits: $\{0\}$ and $\mathcal{N}_{SO_n}(\mathbb{C}^n) \setminus \{0\}$.

Example

Let $V = \mathbb{C}^n$ $(n \geq 3)$, and let $G = SO_n$ be the connected stabilizer of the usual nondegenerate symmetric bilinear form. Then

$$\mathcal{N}_{SO_n}(\mathbb{C}^n) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 0\}.$$

There are two SO_n-orbits: $\{0\}$ and $\mathcal{N}_{SO_n}(\mathbb{C}^n) \setminus \{0\}$.
The singularity at 0 is 'only conical': $\mathcal{N}_{SO_n}(\mathbb{C}^n)$ is the affine cone over a nonsingular quadric hypersurface Q in $\mathbb{P}^{n-1}$, which is a homogeneous space for SO_n. Hence we have a resolution $\pi : L_Q \to \mathcal{N}_{SO_n}(\mathbb{C}^n)$ where L_Q is the tautological line bundle on Q.

Example

Let $V = \mathbb{C}^n$ ($n \geq 3$), and let $G = SO_n$ be the connected stabilizer of the usual nondegenerate symmetric bilinear form. Then

$$\mathcal{N}_{SO_n}(\mathbb{C}^n) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 0\}.$$

There are two SO_n-orbits: $\{0\}$ and $\mathcal{N}_{SO_n}(\mathbb{C}^n) \setminus \{0\}$.
The singularity at 0 is ‘only conical’: $\mathcal{N}_{SO_n}(\mathbb{C}^n)$ is the affine cone over a nonsingular quadric hypersurface Q in $\mathbb{P}^{n-1}$, which is a homogeneous space for SO_n. Hence we have a resolution $\pi : L_Q \to \mathcal{N}_{SO_n}(\mathbb{C}^n)$ where L_Q is the tautological line bundle on Q.
In this situation, the intersection cohomology at 0 is given by:

$$\begin{aligned} IH_0^i(\mathcal{N}_{SO_n}(\mathbb{C}^n)) &\cong \operatorname{coker}(H^{i-2}(Q) \xrightarrow{\text{Lef}} H^i(Q)) \\ &\cong \begin{cases} \mathbb{C}, & \text{if } i = 0, \\ \mathbb{C}, & \text{if } i = n-2 \text{ and } n \text{ is even}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example

The Sp_4-invariant polynomial functions on $(\mathbb{C}^4)^k$, for $k \geq 3$, are generated by the symplectic form $\langle \cdot, \cdot \rangle$, so

$$\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k) = \{(v^1, \cdots, v^k) \in (\mathbb{C}^4)^k \mid \langle v^i, v^j \rangle = 0, \forall i, j\}.$$

Let Z be the subvariety where all v^i lie in a single line; then Sp_4 clearly has infinitely many orbits in Z, and hence in $\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$.

Example

The Sp_4-invariant polynomial functions on $(\mathbb{C}^4)^k$, for $k \geq 3$, are generated by the symplectic form $\langle \cdot, \cdot \rangle$, so

$$\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k) = \{(v^1, \cdots, v^k) \in (\mathbb{C}^4)^k \,|\, \langle v^i, v^j \rangle = 0, \forall i, j\}.$$

Let Z be the subvariety where all v^i lie in a single line; then Sp_4 clearly has infinitely many orbits in Z, and hence in $\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$. In fact, Z is the singular locus of $\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$; we have a resolution

$$\pi : B_\Lambda^k \to \mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$$

where Λ is the 3-dimensional variety of Lagrangian subspaces of $\mathbb{C}^4$, and B_Λ^k is the sum of k copies of its tautological bundle. The nontrivial fibres are $\pi^{-1}(0) \cong \Lambda$ and $\pi^{-1}(x) \cong \mathbb{P}^1$ for $0 \neq x \in Z$.

Example

The Sp_4-invariant polynomial functions on $(\mathbb{C}^4)^k$, for $k \geq 3$, are generated by the symplectic form $\langle \cdot, \cdot \rangle$, so

$$\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k) = \{(v^1, \cdots, v^k) \in (\mathbb{C}^4)^k \,|\, \langle v^i, v^j \rangle = 0, \forall i, j\}.$$

Let Z be the subvariety where all v^i lie in a single line; then Sp_4 clearly has infinitely many orbits in Z, and hence in $\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$. In fact, Z is the singular locus of $\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$; we have a resolution

$$\pi : B_\Lambda^k \to \mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)$$

where Λ is the 3-dimensional variety of Lagrangian subspaces of $\mathbb{C}^4$, and B_Λ^k is the sum of k copies of its tautological bundle. The nontrivial fibres are $\pi^{-1}(0) \cong \Lambda$ and $\pi^{-1}(x) \cong \mathbb{P}^1$ for $0 \neq x \in Z$. In this situation, the local intersection cohomology is given by:

$$IH_x^i(\mathcal{N}_{Sp_4}((\mathbb{C}^4)^k)) \cong H^i(\pi^{-1}(x)).$$

Definition (Goresky–MacPherson)

Let X be a variety with a good finite stratification (e.g. into group orbits). A stratified resolution $\pi : \widetilde{X} \to X$ is said to be semismall if for every non-open stratum S, $\dim \pi^{-1}(x) \leq \frac{1}{2}\mathrm{codim}\,_X S, \ \forall x \in S$, and is said to be small if the inequality is always strict.

Definition (Goresky–MacPherson)

Let X be a variety with a good finite stratification (e.g. into group orbits). A stratified resolution $\pi : \widetilde{X} \to X$ is said to be semismall if for every non-open stratum S, $\dim \pi^{-1}(x) \leq \frac{1}{2}\mathrm{codim}_X S, \ \forall x \in S$, and is said to be small if the inequality is always strict.

If $\pi : \widetilde{X} \to X$ is small, then we simply have $IH^i_x(X) \cong H^i(\pi^{-1}(x))$. Otherwise, $IH^i_x(X)$ is obtained from $H^i(\pi^{-1}(x))$ by 'deleting the contributions from' $IH^*_x(\overline{S}, \mathcal{L})$ for local systems $\mathcal{L}$ on non-open S. If we have a resolution of each $\overline{S}$, and no non-trivial $\mathcal{L}$'s occur, then the intersection cohomology can be calculated recursively. This calculation is simpler if the resolutions are semismall.

Definition (Goresky–MacPherson)

Let X be a variety with a good finite stratification (e.g. into group orbits). A stratified resolution $\pi : \widetilde{X} \to X$ is said to be semismall if for every non-open stratum S, $\dim \pi^{-1}(x) \leq \frac{1}{2}\mathrm{codim}_X S$, $\forall x \in S$, and is said to be small if the inequality is always strict.

If $\pi : \widetilde{X} \to X$ is small, then we simply have $IH^i_x(X) \cong H^i(\pi^{-1}(x))$. Otherwise, $IH^i_x(X)$ is obtained from $H^i(\pi^{-1}(x))$ by 'deleting the contributions from' $IH^*_x(\overline{S}, \mathcal{L})$ for local systems $\mathcal{L}$ on non-open S. If we have a resolution of each $\overline{S}$, and no non-trivial $\mathcal{L}$'s occur, then the intersection cohomology can be calculated recursively. This calculation is simpler if the resolutions are semismall.

Example

Resolutions of Schubert varieties can be used to prove that their intersection cohomology is given by Kazhdan–Lusztig polynomials:

$$\sum_i \dim IH^{2i}_{X_y}(\overline{X_w})\, q^i = P_{y,w}(q).$$

The nilpotent cone of type A

The GL_n-orbits in $\mathcal{N}(\mathfrak{gl}_n) = \{x \in \mathfrak{gl}_n \text{ nilpotent}\}$ are determined by:

$$\begin{aligned}\mathcal{P}_n = \{\text{partitions of } n\} &\longleftrightarrow \{GL_n\text{-orbits in } \mathcal{N}(\mathfrak{gl}_n)\} \\ \lambda = (\lambda_1, \lambda_2, \cdots) &\mapsto \{x \text{ with Jordan blocks of sizes } \lambda_i\} = \mathcal{O}_\lambda.\end{aligned}$$

The nilpotent cone of type A

The GL_n-orbits in $\mathcal{N}(\mathfrak{gl}_n) = \{x \in \mathfrak{gl}_n \text{ nilpotent}\}$ are determined by:

$$\begin{aligned} \mathcal{P}_n = \{\text{partitions of } n\} &\longleftrightarrow \{GL_n\text{-orbits in } \mathcal{N}(\mathfrak{gl}_n)\} \\ \lambda = (\lambda_1, \lambda_2, \cdots) &\mapsto \{x \text{ with Jordan blocks of sizes } \lambda_i\} = \mathcal{O}_\lambda. \end{aligned}$$

Theorem (Gerstenhaber 1961)

Let π, λ be partitions of n. Then

$$\mathcal{O}_\pi \subseteq \overline{\mathcal{O}_\lambda} \qquad \Longleftrightarrow \qquad \begin{array}{rcl} \pi_1 & \leq & \lambda_1, \\ \pi_1 + \pi_2 & \leq & \lambda_1 + \lambda_2, \\ \pi_1 + \pi_2 + \pi_3 & \leq & \lambda_1 + \lambda_2 + \lambda_3, \\ \vdots \quad \vdots \quad & \vdots & \quad \vdots \quad \vdots \end{array}$$

Partitions of n, and this ‘dominance’ partial order, also play a major role in the representation theory of S_n, the Weyl group of GL_n.

The orbit closure $\overline{\mathcal{O}_\lambda}$ can be described as

$$\{x \in \mathcal{N}(\mathfrak{gl}_n) \mid \dim\ker(x^i) \geq \lambda_1^* + \cdots + \lambda_i^*, \forall i \geq 1\},$$

where $\lambda_i^* = |\{j \mid \lambda_j \geq i\}|$. Let $\mathcal{F}_\lambda$ be the partial flag variety

$$\{(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n) \mid \dim V_i = \lambda_1^* + \cdots + \lambda_i^*, \forall i \geq 1\},$$

a homogeneous space for GL_n. We have a resolution

$$\pi : \{(x, (V_i)) \in \mathfrak{gl}_n \times \mathcal{F}_\lambda \mid x(V_i) \subseteq V_{i-1}\} \to \overline{\mathcal{O}_\lambda} : (x, (V_i)) \mapsto x,$$

which turns out to be semismall. The fibre $\pi^{-1}(x)$ is the variety of partial flags $(V_i) \in \mathcal{F}_\lambda$ such that $x(V_i) \subseteq V_{i-1}$.

The orbit closure $\overline{\mathcal{O}_\lambda}$ can be described as

$$\{x \in \mathcal{N}(\mathfrak{gl}_n) \mid \dim \ker(x^i) \geq \lambda_1^* + \cdots + \lambda_i^*, \forall i \geq 1\},$$

where $\lambda_i^* = |\{j \mid \lambda_j \geq i\}|$. Let $\mathcal{F}_\lambda$ be the partial flag variety

$$\{(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n) \mid \dim V_i = \lambda_1^* + \cdots + \lambda_i^*, \forall i \geq 1\},$$

a homogeneous space for GL_n. We have a resolution

$$\pi : \{(x, (V_i)) \in \mathfrak{gl}_n \times \mathcal{F}_\lambda \mid x(V_i) \subseteq V_{i-1}\} \to \overline{\mathcal{O}_\lambda} : (x, (V_i)) \mapsto x,$$

which turns out to be semismall. The fibre $\pi^{-1}(x)$ is the variety of partial flags $(V_i) \in \mathcal{F}_\lambda$ such that $x(V_i) \subseteq V_{i-1}$.

Example

In the case $\lambda = (n)$, we get the Springer resolution of the whole nilpotent cone $\mathcal{N}(\mathfrak{gl}_n) = \overline{\mathcal{O}_{(n)}}$, in which the fibre over x is

$$\mathcal{B}_x = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim V_i = i, x(V_i) \subseteq V_{i-1}\}.$$

The Weyl group S_n has a Springer representation on $H^i(\mathcal{B}_x)$, which links nilpotent orbits to representations of S_n.

These resolutions of orbit closures can be used to prove:

Theorem (Lusztig 1981)

The intersection cohomology of $\overline{\mathcal{O}_\lambda}$ is given by Kostka polynomials:

$$\sum_i \dim IH^{2i}_{\mathcal{O}_\pi}(\overline{\mathcal{O}_\lambda})\, q^i = q^{n(\pi)-n(\lambda)} K_{\lambda\pi}(q^{-1}).$$

These resolutions of orbit closures can be used to prove:

Theorem (Lusztig 1981)

The intersection cohomology of $\overline{\mathcal{O}_\lambda}$ is given by Kostka polynomials:

$$\sum_i \dim IH^{2i}_{\mathcal{O}_\pi}(\overline{\mathcal{O}_\lambda})\, q^i = q^{n(\pi)-n(\lambda)} K_{\lambda\pi}(q^{-1}).$$

More information about the singularities of $\overline{\mathcal{O}_\lambda}$ comes from viewing them as examples of Nakajima's quiver varieties of type A:

Theorem (Kraft–Procesi 1979)

There is a variety Z_λ (defined using a type-A quiver) which is a normal complete intersection and has an action of $GL_n \times H_\lambda$, where H_λ is a product of general linear groups, such that

$$H_\lambda \backslash\backslash Z_\lambda \cong \overline{\mathcal{O}_\lambda}.$$

It follows that $\overline{\mathcal{O}_\lambda}$ is a normal variety with rational singularities.

Hesselink resolutions

For general G and V, the Hilbert–Mumford criterion says that

$$v \in \mathcal{N}_G(V) \iff \lim_{t\to 0} \varphi(t)v = 0 \text{ for some } \varphi : \mathbb{C}^\times \to G.$$

Any such 1-parameter subgroup φ gives rise to a $\mathbb{Z}$-grading

$$V = \bigoplus_{a\in\mathbb{Z}} V_a^\varphi \text{ where } V_a^\varphi = \{v \in V \mid \varphi(t)v = t^a v, \forall t \in \mathbb{C}^\times\},$$

and $\lim_{t\to 0} \varphi(t)v = 0$ if and only if $v \in V_{\geq 1}^\varphi$.

Hesselink resolutions

For general G and V, the Hilbert–Mumford criterion says that

$$v \in \mathcal{N}_G(V) \iff \lim_{t\to 0} \varphi(t)v = 0 \text{ for some } \varphi : \mathbb{C}^\times \to G.$$

Any such 1-parameter subgroup φ gives rise to a $\mathbb{Z}$-grading

$$V = \bigoplus_{a\in\mathbb{Z}} V_a^\varphi \text{ where } V_a^\varphi = \{v \in V \mid \varphi(t)v = t^a v, \forall t \in \mathbb{C}^\times\},$$

and $\lim_{t\to 0}\varphi(t)v = 0$ if and only if $v \in V_{\geq 1}^\varphi$.
Let $P(\varphi)$ be the parabolic subgroup of G whose Lie algebra is $\mathfrak{g}_{\geq 0}^\varphi$.

Theorem (Hesselink 1979)

There is a finite stratification of $\mathcal{N}_G(V)$ *into smooth* G*-stable strata* S *such that each closure* $\overline{S}$ *has a resolution*

$$\pi : G \times_{P(\varphi)} V_{\geq a}^\varphi \to G.V_{\geq a}^\varphi = \overline{S}$$

for a suitable 1*-parameter subgroup* φ *and positive integer* a*.*

In particular, $\mathcal{N}_G(V)$ itself has such a resolution, if it is irreducible.

Example

When $V = \mathfrak{g}$, the Jacobson–Morozov theorem guarantees that any nilpotent $x \in \mathcal{N}(\mathfrak{g})$ forms part of an $\mathfrak{sl}_2$-triple $\{x, h, y\}$. If we define $\varphi_x : \mathbb{C}^\times \to G$ by $\varphi_x(t) = \exp(\log(t)h)$, then $x \in V_2^{\varphi_x}$. The strata given by Hesselink's construction are the nilpotent orbits, and the resolution of $\overline{G.x}$ is

$$G \times_{P(\varphi_x)} V_{\geq 2}^{\varphi_x} \to G.V_{\geq 2}^{\varphi_x} = \overline{G.x}.$$

In particular, when $G.x$ is the open orbit in $\mathcal{N}(\mathfrak{g})$, we get the Springer resolution of $\mathcal{N}(\mathfrak{g})$, in which the fibre over x is

$$\mathcal{B}_x = \{\text{Borel subalgebras } \mathfrak{b} \subset \mathfrak{g} \,|\, x \in \mathfrak{b}\}.$$

For other orbit closures, the resolutions are usually not semismall. (For $\mathfrak{gl}_n$, they differ from the semismall resolutions used previously.)

Example

When $V = \mathfrak{g}$, the Jacobson–Morozov theorem guarantees that any nilpotent $x \in \mathcal{N}(\mathfrak{g})$ forms part of an $\mathfrak{sl}_2$-triple $\{x, h, y\}$. If we define $\varphi_x : \mathbb{C}^\times \to G$ by $\varphi_x(t) = \exp(\log(t)h)$, then $x \in V_2^{\varphi_x}$. The strata given by Hesselink's construction are the nilpotent orbits, and the resolution of $\overline{G.x}$ is

$$G \times_{P(\varphi_x)} V_{\geq 2}^{\varphi_x} \to G.V_{\geq 2}^{\varphi_x} = \overline{G.x}.$$

In particular, when $G.x$ is the open orbit in $\mathcal{N}(\mathfrak{g})$, we get the Springer resolution of $\mathcal{N}(\mathfrak{g})$, in which the fibre over x is

$$\mathcal{B}_x = \{\text{Borel subalgebras } \mathfrak{b} \subset \mathfrak{g} \,|\, x \in \mathfrak{b}\}.$$

For other orbit closures, the resolutions are usually not semismall. (For $\mathfrak{gl}_n$, they differ from the semismall resolutions used previously.)

In general, the Hesselink strata need not be single orbits, even when there are finitely many orbits.

The enhanced nilpotent cone of type A

When $G = GL_n$, $V = \mathbb{C}^n \oplus \mathfrak{gl}_n$, the nullcone is just $\mathbb{C}^n \times \mathcal{N}(\mathfrak{gl}_n)$. Classifying GL_n-orbits of such pairs (v, x) requires considering not just the Jordan type of x, but how the vector v interacts with x.

The enhanced nilpotent cone of type A

When $G = GL_n$, $V = \mathbb{C}^n \oplus \mathfrak{gl}_n$, the nullcone is just $\mathbb{C}^n \times \mathcal{N}(\mathfrak{gl}_n)$. Classifying GL_n-orbits of such pairs (v, x) requires considering not just the Jordan type of x, but how the vector v interacts with x.

Theorem (Achar–H., Advances in Math. 2008)

1. *The GL_n-orbits in $\mathbb{C}^n \times \mathcal{N}(\mathfrak{gl}_n)$ are in bijection with $\mathcal{Q}_n$, the set of ordered pairs of partitions $(\mu; \nu)$ whose total size is n.*
2. *The orbit closure $\overline{\mathcal{O}_{\mu;\nu}}$ consists of those (v, x) for which there is an x-invariant $|\mu|$-dimensional subspace $W \subset \mathbb{C}^n$, containing v, such that $x|_W \in \overline{\mathcal{O}_\mu}$ and $x|_{\mathbb{C}^n/W} \in \overline{\mathcal{O}_\nu}$.*
3. *For $(\rho; \sigma), (\mu; \nu) \in \mathcal{Q}_n$,*

$$\mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}} \iff \begin{array}{rcl} \rho_1 & \leq & \mu_1, \\ \rho_1 + \sigma_1 & \leq & \mu_1 + \nu_1, \\ \rho_1 + \sigma_1 + \rho_2 & \leq & \mu_1 + \nu_1 + \mu_2, \\ \vdots \quad \vdots & \vdots & \vdots \quad \vdots \end{array}$$

From (2) we deduce that each orbit closure $\overline{\mathcal{O}_{\mu;\nu}}$ in $\mathbb{C}^n \times \mathcal{N}(\mathfrak{gl}_n)$ has a semismall resolution. (These are again not the Hesselink resolutions; in fact, here the Hesselink strata are not single orbits.)

Theorem (Achar–H. as above)

The IC of $\overline{\mathcal{O}_{\mu;\nu}}$ is given by Kostka–Shoji polynomials:

$$\sum_i \dim IH^i_{\mathcal{O}_{\rho;\sigma}}(\overline{\mathcal{O}_{\mu;\nu}})\, q^i = q^{b(\rho;\sigma)-b(\mu;\nu)} K_{(\mu;\nu),(\rho;\sigma)}(q^{-1}).$$

From (2) we deduce that each orbit closure $\overline{\mathcal{O}_{\mu;\nu}}$ in $\mathbb{C}^n \times \mathcal{N}(\mathfrak{gl}_n)$ has a semismall resolution. (These are again not the Hesselink resolutions; in fact, here the Hesselink strata are not single orbits.)

Theorem (Achar–H. as above)

The IC of $\overline{\mathcal{O}_{\mu;\nu}}$ is given by Kostka–Shoji polynomials:

$$\sum_i \dim IH^i_{\mathcal{O}_{\rho;\sigma}}(\overline{\mathcal{O}_{\mu;\nu}})\, q^i = q^{b(\rho;\sigma)-b(\mu;\nu)} K_{(\mu;\nu),(\rho;\sigma)}(q^{-1}).$$

Theorem (Achar–H.–Jones, arXiv:1004.3822)

There is a variety $Z_{\mu;\nu}$ (similar to Z_λ) with an action of $GL_n \times H_{\mu;\nu}$, where $H_{\mu;\nu}$ is a product of general linear groups, such that

$$H_{\mu;\nu} \backslash\backslash Z_{\mu;\nu} \cong \overline{\mathcal{O}_{\mu;\nu}}.$$

We conjecture, and prove in special cases, that $Z_{\mu;\nu}$ is a normal complete intersection, implying that $\overline{\mathcal{O}_{\mu;\nu}}$ is normal.