Enhancing the Jordan canonical form

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Outline

The Jordan canonical form theorem

Let $\mathrm{Mat}_n(\mathbb{C})$ denote the ring of $n \times n$ complex matrices, and $GL_n(\mathbb{C})$ the group of invertible matrices in $\mathrm{Mat}_n(\mathbb{C})$. Recall that $A, B \in \mathrm{Mat}_n(\mathbb{C})$ are similar if $B = gAg^{-1}$ for some $g \in GL_n(\mathbb{C})$.

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Theorem (Jordan canonical form)

Every $A \in \mathrm{Mat}_n(\mathbb{C})$ is similar to a block-diagonal matrix with diagonal blocks $J_{\ell_1}(a_1), J_{\ell_2}(a_2), \cdots, J_{\ell_k}(a_k)$, where

$$J_\ell(a) = \begin{pmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 1 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \in \mathrm{Mat}_\ell(\mathbb{C}).$$

Moreover, the multiset of pairs (a_i, ℓ_i) is uniquely determined by A.

Note that A is diagonalizable if and only if all ℓ_i equal 1.

An alternative way to state the theorem is as follows:

Theorem (Jordan canonical form)

For any linear transformation A of an n-dimensional $\mathbb{C}$-vector space V, there is a basis of V of the form $\coprod_i \{v_i^{(1)}, \cdots, v_i^{(\ell_i)}\}$ where

$$(A - a_i)v_i^{(1)} = 0,\ (A - a_i)v_i^{(2)} = v_i^{(1)},\ \cdots,\ (A - a_i)v_i^{(\ell_i)} = v_i^{(\ell_i - 1)}.$$

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The first step in the proof is to show that V is the direct sum of the generalized eigenspaces

$$V_a^{\text{gen}} = \{v \in V \mid (A - a)^k v = 0 \text{ for some } k \geq 1\},$$

where a runs over all the eigenvalues of A. This reduces the problem to the case $V = V_a^{\text{gen}}$, and then replacing A with $A - a$ reduces to the case where A is nilpotent (only eigenvalue is 0).

So the main content of the Jordan canonical form theorem is the classification of nilpotent matrices. Define the nilpotent cone

$$\mathcal{N}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) \mid x \text{ nilpotent}\}.$$

The group $GL_n(\mathbb{C})$ acts on the nilpotent cone $\mathcal{N}_n$ by $g.x = gxg^{-1}$.

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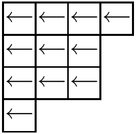
Theorem (Jordan canonical form – nilpotent case)

The $GL_n(\mathbb{C})$-orbits in $\mathcal{N}_n$ are in bijection with the partitions of n. If λ is a partition with parts $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{Z}^+$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k, \ \ \lambda_1 + \lambda_2 + \cdots + \lambda_k = n,$$

the orbit $\mathcal{O}_\lambda$ is the one containing $\mathrm{diag}(J_{\lambda_1}(0), \cdots, J_{\lambda_k}(0))$.

Pictorially:



$\square$ = basis element of $\mathbb{C}^{11}$

$\longleftarrow$ = action of $x \in \mathcal{O}_{(4,3,3,1)}$

Generalities part 1

When an algebraic group G acts on a variety X, there are some standard problems, whose answers often have broader significance.

Problems

1. Parametrize the G-orbits in X (e.g. combinatorially).
2. Determine the closure order on the orbits: that is, give a condition for when $Gy \subseteq \overline{Gx}$ (meaning the Zariski closure).
3. The orbits Gx are smooth, but their closures need not be. For each singular orbit closure $\overline{Gx}$, describe a resolution.
4. Describe the singularities in other ways, e.g. by computing the intersection cohomology of $\overline{Gx}$ at points y in its boundary.

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Example

When G is simple and X is the product of two copies of its flag variety, the answer to 4 is described by the Kazhdan–Lusztig polynomials, which turn up throughout representation theory.

Orbit closures in the nilpotent cone

Theorem (Gerstenhaber)

Let π, λ be partitions of n. Then

$$\mathcal{O}_\pi \subseteq \overline{\mathcal{O}_\lambda} \qquad \Longleftrightarrow \qquad \begin{array}{rcl} \pi_1 & \leq & \lambda_1, \\ \pi_1 + \pi_2 & \leq & \lambda_1 + \lambda_2, \\ \pi_1 + \pi_2 + \pi_3 & \leq & \lambda_1 + \lambda_2 + \lambda_3, \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \end{array}$$

We write $\pi \leq \lambda$ for this dominance partial order of partitions.

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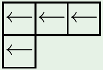
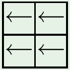
Example

Note that $(2,2) \leq (3,1)$. The corresponding orbits are:

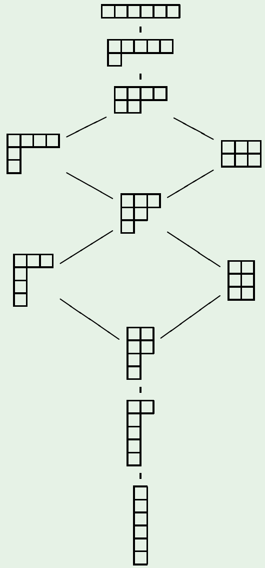


$$\mathcal{O}_{(3,1)} = \{x \in \mathcal{N}_4 \,|\, \mathrm{rk}(x) = 2,\ x^2 \neq 0\},$$



$$\mathcal{O}_{(2,2)} = \{x \in \mathcal{N}_4 \,|\, \mathrm{rk}(x) = 2,\ x^2 = 0\}.$$

Example (Hasse diagram of $\mathcal{P}_6$)



It is easy to see that for $x \in \mathcal{O}_\lambda$,

$$\dim \ker(x^i) = \lambda_1^* + \cdots + \lambda_i^*, \text{ for all } i \geq 1,$$

where $\lambda_i^* = |\{j \mid \lambda_j \geq i\}|$ (length of the ith column). We have

$$\overline{\mathcal{O}_\lambda} = \{x \in \mathcal{N}_n \mid \dim \ker(x^i) \geq \lambda_1^* + \cdots + \lambda_i^*, \forall i \geq 1\}.$$

This has a resolution of singularities

$$\{(x, 0 = V_0 \subset V_1 \subset \cdots \subset \mathbb{C}^n) \mid \dim V_i/V_{i-1} = \lambda_i^*, x(V_i) \subseteq V_{i-1}\},$$

which is actually the cotangent bundle of $\{(V_i)\} = GL_n(\mathbb{C})/P$.

Example

$n = 10$, $\lambda = (3,3,3,1)$, $\lambda^* = (4,3,3)$.

←	←	←
←	←	←
←	←	←
←		

$$\begin{array}{c} T^*\{0 \subset V_1 \subset V_2 \subset \mathbb{C}^{10} \mid \dim V_1 = 4, \dim V_2 = 7\} \\ \downarrow \\ \overline{\mathcal{O}_{(3,3,3,1)}} \end{array}$$

The resolution of the whole nilpotent cone $\mathcal{N}_n = \overline{\mathcal{O}_{(n)}}$ is the Springer resolution $\pi : \widetilde{\mathcal{N}_n} \to \mathcal{N}_n$ where

$$\pi^{-1}(x) = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \,|\, x(V_i) \subseteq V_{i-1}\}.$$

The cohomology of the fibres $\pi^{-1}(x)$ is encapsulated in $R\pi_*\mathbb{C}$.

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Theorem (Springer correspondence for the symmetric group)

The endomorphism ring of $R\pi_*\mathbb{C}$ is $\mathbb{C}S_n$. Hence the irreducible representations of S_n are in bijection with the simple summands of $R\pi_*\mathbb{C}$, which are the intersection cohomology complexes $IC(\overline{\mathcal{O}_\lambda})$.

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More generally, there is a procedure for calculating the intersection cohomology of the orbit closures inductively, when one knows the cohomology of all the fibres of their resolutions.

Theorem (Lusztig 1981)

The intersection cohomology of $\overline{\mathcal{O}_\lambda}$ at points of $\mathcal{O}_\pi$ is described by the combinatorial Kostka polynomial $K_{\lambda,\pi}(t)$.

Generalities part 2

For a general reductive G acting linearly on a vector space V, the role of the nilpotent cone is played by the Hilbert nullcone

$$\mathcal{N}_G(V) = \{v \in V \mid 0 \in \overline{Gv}\},$$

which is the subvariety defined by the G-invariant polynomial functions on V with zero constant term. In general, G may have infinitely many orbits in $\mathcal{N}_G(V)$, and little is known about them.

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Example

The case most closely analogous to $\mathcal{N}_n$ is when $V = \mathrm{Lie}(G)$, and

$$\mathcal{N}_G(\mathrm{Lie}(G)) = \{x \in \mathrm{Lie}(G) \mid x \text{ nilpotent}\}.$$

Here there is still a Springer resolution π, but the stabilizers G_x are not all connected, so non-trivial local systems arise in $R\pi_*\mathbb{C}$. The upshot is that the G-orbits in $\mathcal{N}_G(\mathrm{Lie}(G))$ are in bijection with a subset of the irreducible representations of the Weyl group of G.

The enhanced nilpotent cone

Consider $GL_n(\mathbb{C})$ acting on $\mathbb{C}^n \oplus \mathrm{Mat}_n(\mathbb{C})$. Here parametrizing the orbits amounts to classifying pairs (v, A), $v \in \mathbb{C}^n$, $A \in \mathrm{Mat}_n(\mathbb{C})$, up to the equivalence relation

$$(v, A) \sim (w, B) \Leftrightarrow w = gv \text{ and } B = gAg^{-1} \text{ for some } g \in GL_n(\mathbb{C}).$$

From the alternative viewpoint, this means finding, for a pair (v, A) of a vector and a linear transformation, a basis which simultaneously puts both v and A in as nice a form as possible.

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From the alternative viewpoint, this means finding, for a pair (v, A) of a vector and a linear transformation, a basis which simultaneously puts both v and A in as nice a form as possible. Arguing as before, we reduce to the case where A is nilpotent, i.e. we consider only $GL_n(\mathbb{C})$-orbits in the enhanced nilpotent cone

$$\mathbb{C}^n \times \mathcal{N}_n = \mathcal{N}_{GL_n(\mathbb{C})}(\mathbb{C}^n \oplus \mathrm{Mat}_n(\mathbb{C})).$$

So we are dealing with pairs (v, x) of a vector and a nilpotent matrix. Note that whatever relationships hold between v and x must be preserved, e.g. that $x^3 v = 0$, or that v lies in $\mathrm{im}(x^2)$.

We can certainly find a basis which puts x in Jordan form. As mentioned before, this basis is not unique: we can still act by

$$Z_{GL_n(\mathbb{C})}(x) = \{g \in GL_n(\mathbb{C}) \mid gx = xg\}.$$

So the problem is to classify $Z_{GL_n(\mathbb{C})}(x)$-orbits in $\mathbb{C}^n$, for $x \in \mathcal{O}_\lambda$.

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So the problem is to classify $Z_{GL_n(\mathbb{C})}(x)$-orbits in $\mathbb{C}^n$, for $x \in \mathcal{O}_\lambda$. We can display vectors by filling in the boxes with the coefficient of the corresponding basis vector; we aim to simplify by applying invertible 'row operations' which commute with x.

Example

$n = 11,\ \lambda = (4, 3, 3, 1)$.

$$\begin{matrix} 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & \\ 0 & 3 & 0 & \\ 0 & & & \end{matrix} \xrightarrow{\text{scale } R_3} \begin{matrix} 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & & & \end{matrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{matrix} 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & \\ 0 & 1 & 0 & \\ 0 & & & \end{matrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1^{\text{shifted}}} \begin{matrix} 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & \\ 0 & 1 & 0 & \\ 0 & & & \end{matrix} \xrightarrow{R_1 \to R_1 - 5R_1^{\text{shifted}}} \begin{matrix} 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & & & & \end{matrix}$$

Proposition (Achar–H., Adv. Math. 2008)

For any (v, x), there is a basis $\coprod_i \{v_i^{(1)}, \cdots, v_i^{(\mu_i+\nu_i)}\}$ of $\mathbb{C}^n$ where

$$xv_i^{(1)} = 0,\ xv_i^{(2)} = v_i^{(1)},\ \cdots,\ xv_i^{(\mu_i+\nu_i)} = v_i^{(\mu_i+\nu_i-1)},\ v = \sum_{\mu_i \neq 0} v_i^{(\mu_i)},$$

and μ and ν are partitions whose sizes add up to n. Moreover, the bipartition $(\mu; \nu)$ is uniquely determined by the pair (v, x). So the $GL_n(\mathbb{C})$-orbits in $\mathbb{C}^n \times \mathcal{N}_n$ are in bijection with the bipartitions of n.

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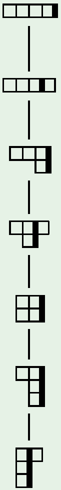
and μ and ν are partitions whose sizes add up to n. Moreover, the bipartition $(\mu;\nu)$ is uniquely determined by the pair (v, x). So the $GL_n(\mathbb{C})$-orbits in $\mathbb{C}^n \times \mathcal{N}_n$ are in bijection with the bipartitions of n.

Independently, Travkin proved the same result using a more intrinsic description of the orbits:

$$\mathcal{O}_{\mu;\nu} = \{(v, x) \in \mathbb{C}^n \times \mathcal{N}_n \,|\, x|_{E^x v} \in \mathcal{O}_\mu \text{ and } x|_{\mathbb{C}^n/E^x v} \in \mathcal{O}_\nu\},$$

where $E^x = \{y \in \mathrm{Mat}_n(\mathbb{C}) \,|\, yx = xy\}$. Curiously, the set of bipartitions of n parametrize the irreducible representations of the hyperoctahedral group (Coxeter type B_n), which has nothing much to do with $\mathbb{C}^n \times \mathcal{N}_n$ – see our paper for the explanation.

Example (descriptions of some orbits in $\mathbb{C}^4 \times \mathcal{N}_4$)



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The solutions to the other problems in this case are as follows.

Theorem (Achar–H., Adv. Math. 2008)

1. *For two bipartitions* $(\rho;\sigma)$ *and* $(\mu;\nu)$ *of n, we have*

$$\mathcal{O}_{\rho;\sigma} \subseteq \overline{\mathcal{O}_{\mu;\nu}} \iff \begin{array}{rcl} \rho_1 & \leq & \mu_1, \\ \rho_1+\sigma_1 & \leq & \mu_1+\nu_1, \\ \rho_1+\sigma_1+\rho_2 & \leq & \mu_1+\nu_1+\mu_2, \\ \rho_1+\sigma_1+\rho_2+\sigma_2 & \leq & \mu_1+\nu_1+\mu_2+\nu_2, \\ \vdots \quad \vdots & \vdots & \vdots \quad \vdots \end{array}$$

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2. $\overline{\mathcal{O}_{\mu;\nu}}$ *consists of those* (v,x) *such that there is an* x*-invariant subspace* $W \subset \mathbb{C}^n$ *with* $v \in W$, $x|_W \in \overline{\mathcal{O}_\mu}$, $x|_{\mathbb{C}^n/W} \in \overline{\mathcal{O}_\nu}$. *Hence a resolution can be easily obtained from those for the ordinary nilpotent orbit closures.*

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3. *The intersection cohomology of* $\overline{\mathcal{O}_{\mu;\nu}}$ *at points of* $\mathcal{O}_{\rho;\sigma}$ *is described by Shoji's Kostka polynomials* $K_{(\mu;\nu),(\rho;\sigma)}(t)$.