Poset Homology and the Cohomology of Real DeConcini-Procesi Models

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Outline

Let P be a finite poset and $x \leq y$ in P. For $i \geq 0$, let $C_i(x, y)$ be the $\mathbb{Q}$-v. space with basis the chains $x = x_0 < x_1 < \cdots < x_i = y$.

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$$\cdots C_{i+1}(x,y) \xrightarrow{\partial} C_i(x,y) \xrightarrow{\partial} C_{i-1}(x,y) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(x,y) \to 0$$

where $\partial(x_0 < \cdots < x_i) = \sum_{j=1}^{i-1}(-1)^j(x_0 < \cdots < \widehat{x_j} < \cdots < x_i)$.

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Lemma

We have $\sum(-1)^i h_i(x,y) = \mu(x,y)$ (the Möbius function of P).

Recall that $\sum_{z\in[x,y]} \mu(x,z) = \delta_{x,y}$.

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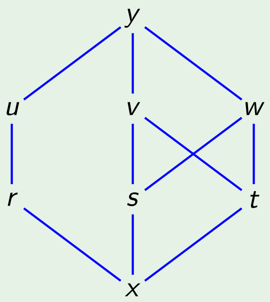
- The following poset has $h_2(x, y) = h_3(x, y) = 1$:



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- For x in a pure bounded poset, the rank $\mathrm{rk}(x)$ is the length of a maximal chain from $\hat{0}$ to x.
- A pure bounded poset P is Cohen-Macaulay if for all $x \leq y$,

$$h_i(x, y) = 0 \text{ for } i < \mathrm{rk}(y) - \mathrm{rk}(x).$$

It follows that $h_{\mathrm{rk}(y)-\mathrm{rk}(x)}(x, y) = (-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)}\mu(x, y)$.

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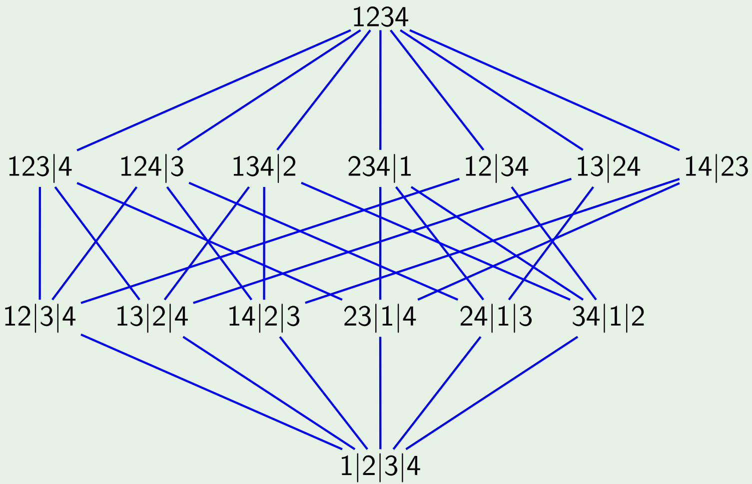
Example

The partition lattice Π_n, whose elements are the partitions of the set $\{1, \cdots, n\}$ ordered by refinement, is Cohen-Macaulay.

The diagram of Π_4:



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Its Poincaré polynomial $P(M_{\mathcal{A}}, t) := \sum_i \dim H^i(M_{\mathcal{A}}; \mathbb{Q})\,(-t)^i$.

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Definition

The intersection lattice $L_{\mathcal{A}}$ is the poset of all intersections of elements of $\mathcal{A}$, ordered by reverse inclusion.

$L_{\mathcal{A}}$ is bounded ($\hat{0} = V$, $\hat{1} = \bigcap_{W \in \mathcal{A}} W$), but not always pure.

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Aim

To express $P(M_{\mathcal{A}}, t)$ in terms of the combinatorics of $L_{\mathcal{A}}$.

Example

For $n \geq 2$, the arrangement $A_{n-1}^{\mathbb{R}}$ consists of the hyperplanes

$$\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, x_i = x_j\} \text{ for } 1 \leq i < j \leq n.$$

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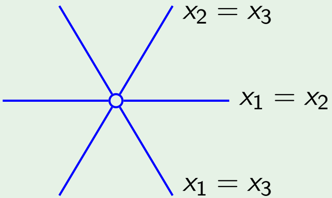
e.g. $A_2^{\mathbb{R}}$:



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For $n \geq 2$, the arrangement $A^{\mathbb{R}}_{n-1}$ consists of the hyperplanes

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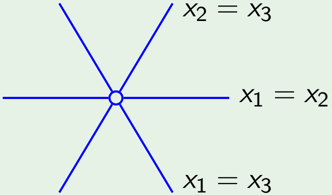

The complement $M_{A^{\mathbb{R}}_{n-1}} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, x_i\text{'s distinct}\}$ consists of $n!$ chambers (open cones), one for each possible order of the coordinates. Thus it is homotopy equivalent to $n!$ points.

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The arrangement $A_{n-1}^{\mathbb{C}}$ is defined similarly in $\mathbb{C}^n$ (viewed as a real arrangement, it consists of subspaces of codimension 2).

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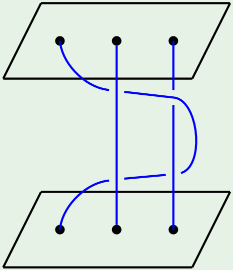
The complement $M_{A_{n-1}^{\mathbb{C}}}$ is connected; it is an Eilenberg-MacLane space for the pure braid group.



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The arrangement $A^{\mathbb{C}}_{n-1}$ is defined similarly in $\mathbb{C}^n$ (viewed as a real arrangement, it consists of subspaces of codimension 2).
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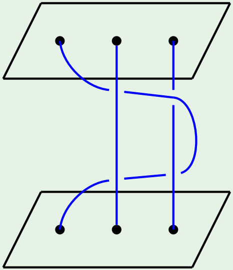

Despite the difference in topology, $L_{A^{\mathbb{R}}_{n-1}}$ and $L_{A^{\mathbb{C}}_{n-1}}$ are both isomorphic to Π_n. Clearly we need more than just the poset.

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Theorem (Goresky-MacPherson, 1988)

For any $\mathcal{A}$, the Poincaré polynomial $P(M_{\mathcal{A}}, t)$ equals

$$\sum_{W \in L_{\mathcal{A}}} \sum_{j} h_j(\hat{0}, W)\,(-t)^{\operatorname{codim} W - j}.$$

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If $\mathcal{A}$ is a real hyperplane arrangement, then:

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So the Theorem becomes the result of Zaslavsky:

$$\#\text{ of regions of } M_{\mathcal{A}} = \sum_{W \in L_{\mathcal{A}}} (-1)^{\operatorname{rk}(W)} \mu(\hat{0}, W).$$

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For instance, this is one way to prove the famous fact:

$$P(M_{A_{n-1}^{\mathbb{C}}}, t) = (1-t)(1-2t)\cdots(1-(n-1)t).$$

Let $\mathcal{A}$ be a real subspace arrangement in V.

Definition

We say $W \in L_{\mathcal{A}} \setminus \{V\}$ is reducible if there exist $W_1, \cdots, W_s \in L_{\mathcal{A}} \setminus \{V\}$, $s \geq 2$, such that:

- $W = W_1 \cap \cdots \cap W_s$, a direct intersection;
- every $W' \in L_{\mathcal{A}}$ such that $W' \supseteq W$ is of the form $W'_1 \cap \cdots \cap W'_s$ for some $W'_i \in L_{\mathcal{A}}$ with $W'_i \supseteq W_i$.

Let $\mathcal{F}_{\mathcal{A}}$ be the set of irreducible elements in $L_{\mathcal{A}} \setminus \{V\}$.

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Definition

The DeConcini-Procesi model $\overline{M}_{\mathcal{A}}$ is the closure of the image of the obvious map:

$$M_{\mathcal{A}} \to \prod_{W \in \mathcal{F}_{\mathcal{A}}} \mathbb{P}(V/W).$$

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$\overline{M}_{\mathcal{A}}$ is a smooth compact real variety; the image of $M_{\mathcal{A}}$ is an open subvariety whose complement is a normal-crossings divisor.

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When $n = 3$: $\overline{M}_{A_2^{\mathbb{R}}} = \mathbb{P}(\mathbb{R}^3/\mathbb{R}^{\text{diag}})$ is a circle.

When $n = 4$: the map involved is

$$M_{A_3^{\mathbb{R}}} \to \mathbb{P}(\mathbb{R}^4/\mathbb{R}^{\mathrm{diag}}) \times \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}),$$

where the $\mathbb{P}^1$'s record the 3-element subsets of coordinates.

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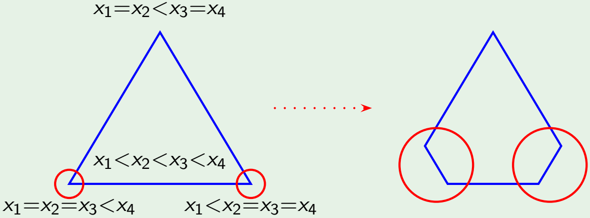
The closure of each chamber in $\mathbb{P}(\mathbb{R}^4/\mathbb{R}^{\text{diag}})$ is a triangle, but the extra information blows up two of the vertices:



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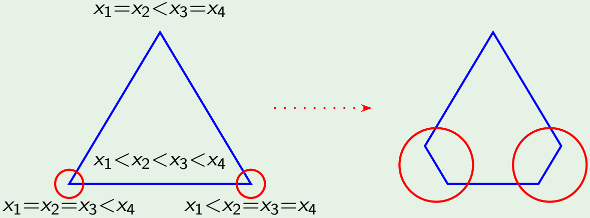

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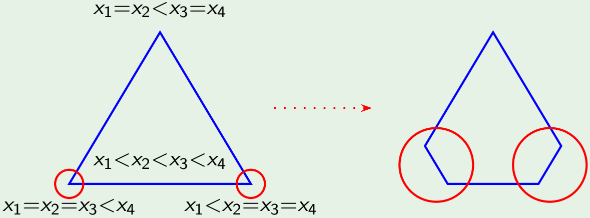

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In general $\overline{M}_{A_{n-1}^{\mathbb{R}}}$ is the moduli space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$.

Note that when $\mathcal{A} = \{W\}$,

$$P(\overline{M}_{\mathcal{A}}, t) = \begin{cases} 1 + (-t)^{\operatorname{codim} W - 1}, & \text{if } \operatorname{codim} W \text{ is even,} \\ 1, & \text{if } \operatorname{codim} W \text{ is odd.} \end{cases}$$

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Theorem (Rains, 2006)

For any $\mathcal{A}$, the Poincaré polynomial $P(\overline{M}_{\mathcal{A}}, t)$ equals

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Even for hyperplane arrangements, $L_{\mathcal{A}}^{\mathrm{ev}}$ need not be Cohen-Macaulay. However ...

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Take $\mathcal{A} = A^{\mathbb{R}}_{n-1}$. Then $L^{\mathrm{ev}}_{\mathcal{A}} \cong \Pi^{\mathrm{odd}}_n$, the poset of partitions of $\{1, \cdots, n\}$ into blocks of odd size.

$$P(\overline{M}_{A^{\mathbb{R}}_{n-1}}, t) = \sum_{\pi \in \Pi^{\mathrm{odd}}_n} \mu_{\Pi^{\mathrm{odd}}_n}(\hat{0}, \pi)\, t^{\mathrm{rk}_{\Pi^{\mathrm{odd}}_n}(\pi)}$$

$$= \begin{cases} (1-t)(1-9t)\cdots(1-(n-3)^2 t), & \text{if } n \text{ is even,} \\ (1-4t)(1-16t)\cdots(1-(n-3)^2 t), & \text{if } n \text{ is odd.} \end{cases}$$

The second equality is due to Calderbank-Hanlon-Robinson, 1986. This example was in Etingof-Henriques-Kamnitzer-Rains, 2005; Rains also found the character of S_n on $H^i(\overline{M}_{A^{\mathbb{R}}_{n-1}}; \mathbb{Q})$.

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Take $\mathcal{A} = B_n^{\mathbb{R}}$, the reflecting hyperplanes of the Weyl gp $W(B_n)$. Then $L_{\mathcal{A}}^{\text{ev}}$ is the poset of parabolic subgroups of $W(B_n)$ all of whose components have even rank.

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In this case we have the generating function:

$$1 + \frac{x}{2} + \sum_{n\geq 2} P(\overline{M}_{B_n^{\mathbb{R}}}, t)\frac{x^n}{2^n n!}$$
$$= \mathrm{sech}(\frac{1}{2}\mathrm{arcsinh}(t^{1/2}x))\exp(\frac{t^{-1/2}}{2}\mathrm{arcsinh}(t^{1/2}x)),$$

and an analogue giving the characters of $W(B_n)$ on $H^i(\overline{M}_{B_n^{\mathbb{R}}};\mathbb{Q})$.

Example

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There is a similar formula for type D_n.