SPHERICAL FUNCTIONS OF THE SYMMETRIC SPACE $GL_n(\mathbb{F}_{q^2})/GL_n(\mathbb{F}_q)$

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$\mathbb{F}_q$ finite field, k algebraic closure of $\mathbb{F}_q$.
Let $G = GL_n(k)$, with Frobenius map $F : G \to G$,

$$F(g_{ij}) = (g_{ij}^q).$$

So $G^F = GL_n(\mathbb{F}_q)$, $G^{F^2} = GL_n(\mathbb{F}_{q^2})$.

Problem 1 Calculate the irred chars $\chi : G^F \to \overline{\mathbb{Q}}$.

Solved by Green in 1955.

Problem 2 For any irred const χ of $\mathrm{Ind}_{G^F}^{G^{F^2}}(1)$, calculate the *spherical function* $\mathrm{Ave}(\chi) : G^{F^2}/G^F \to \overline{\mathbb{Q}}$,

$$\mathrm{Ave}(\chi)(gG^F) = \frac{1}{|G^F|} \sum_{g' \in G^F} \chi(gg').$$

Problem 3 For any irred const $\chi \otimes \overline{\chi}$ of $\mathrm{Ind}_{G^F}^{G^F \times G^F}(1)$, calculate the spherical function

$$\mathrm{Ave}(\chi \otimes \overline{\chi}) : (G^F \times G^F)/G^F \to \overline{\mathbb{Q}}$$
$$= \frac{\chi \star \chi}{|G^F|} = \frac{\chi}{\chi(1)} : G^F \to \overline{\mathbb{Q}}.$$

Green's Solution of Problem 1

Define $\mathcal{P}_n = \{\underline{\lambda} = (\lambda_\alpha)_{\alpha\in k^\times} \mid \sum_{\alpha\in k^\times} |\lambda_\alpha| = n\}$.
Let $\mathcal{P}_n^F = \{\underline{\lambda} \in \mathcal{P}_n \mid \lambda_{F(\alpha)} = \lambda_\alpha,\ \forall\alpha\}$, where $F(\alpha) = \alpha^q$.

Lemma 4 *$\{$conjugacy classes in $G^F\} \longleftrightarrow \mathcal{P}_n^F$.*

E.g. $\{$unipotent classes in $G^F\}$

$$\longleftrightarrow \{\underline{\lambda} \in \mathcal{P}_n^F \text{ unipotent (i.e. } |\lambda_1| = n)\}$$
$$\longleftrightarrow \{\text{partitions of } n\}$$

To any pair (T,θ) of an F-stable maximal torus T and a character $\theta : T^F \to \overline{\mathbb{Q}}^\times$, Green associates a *basic character* $B_{(T,\theta)} : G^F \to \overline{\mathbb{Q}}$, a virtual character whose values are explicitly computable.

E.g. (1) If $T =$ split $T_1 \subset B$, then $B_{(T_1,\theta)} = \mathrm{Ind}_{B^F}^{G^F}(\theta)$.
(2) If $T = T_w$ where w has cycle type μ, and u is in the conjugacy class corresponding to unipotent $\underline{\lambda}$, then

$$B_{(T_w,\theta)}(u) = Q_\mu^{\lambda_1}(q). \text{ (Green polynomial)}$$

$B_{(T,\theta)}$ only depends on the G^F-conjugacy class of (T,θ).

Lemma 5 *$\{G^F$-conjugacy classes of $(T,\theta)\} \longleftrightarrow \mathcal{P}_n^F$.*

For $\underline{\nu} \in \mathcal{P}_n^F$, let $B_{\underline{\nu}} = B_{(T,\theta)}$ for $(T,\theta) \longleftrightarrow \underline{\nu}$.

E.g. Unipotent $\underline{\nu} \leftrightarrow (T_w, 1)$ where w has cycle type ν_1, *unipotent basic character* $B_{\underline{\nu}} = B_{(T_w,1)}$.

Prop 6 **(1)** *$\{B_{\underline{\nu}} \,|\, \underline{\nu} \in \mathcal{P}_n^F\}$ is an orthogonal basis of the space of class functions $\mathcal{F}^{G^F}(G^F)$.*
(2) $\langle B_{(T,\theta)}, B_{(T,\theta)} \rangle = |W(T)_\theta^F|$.

Thm 7 (Green) *For any $\underline{\rho} \in \mathcal{P}_n^F$,*

$$\chi^{\underline{\rho}} := (-1)^{n+\sum_{\alpha\in\langle F\rangle\backslash k^\times}|\rho_\alpha|} \sum_{\substack{\underline{\nu}\in\mathcal{P}_n^F \\ |\nu_\alpha|=|\rho_\alpha|,\forall\alpha}} \left(\prod_{\alpha\in\langle F\rangle\backslash k^\times} z_{\nu_\alpha}^{-1}\chi_{\nu_\alpha}^{\rho_\alpha}\right) B_{\underline{\nu}}$$

is an irreducible character of G^F, and thus $\widehat{G^F} \longleftrightarrow \mathcal{P}_n^F$.

E.g. Unipotent $\underline{\rho} \rightsquigarrow$ unipotent character $\chi^{\underline{\rho}}$:

$$\chi^{\underline{\rho}} = \sum_{\underline{\nu}\text{ unip.}} z_{\nu_1}^{-1}\chi_{\nu_1}^{\rho_1} B_{\underline{\nu}} = \frac{1}{n!}\sum_{w\in S_n}\chi^{\rho_1}(w)B_{(T_w,1)}$$

$$B_{(T_w,1)} = \sum_{\underline{\rho}\text{ unip.}} \chi^{\rho_1}(w)\chi^{\underline{\rho}}$$

$$(w=1:) \quad \mathrm{Ind}_{B^F}^{G^F}(1) = \sum_{\underline{\rho}\text{ unip.}} (\deg\ \chi^{\rho_1})\chi^{\underline{\rho}}$$

If $\rho_1 = (n)$, $\chi^{\underline{\rho}} =$ trivial; if $\rho_1 = (1^n)$, $\chi^{\underline{\rho}} =$ Steinberg.

Analogous Treatment of Problem 2

Identify G^{F^2}/G^F with $G^{\tilde{F}}$, where $\tilde{F} : g \mapsto F(g)^{-1}$,

$$gG^F \longleftrightarrow gF(g)^{-1}.$$

Let $\mathcal{P}_n^{\tilde{F}} = \{\underline{\lambda} \in \mathcal{P}_n \,|\, \lambda_{\tilde{F}(\alpha)} = \lambda_\alpha,\ \forall\alpha\}$, where $\tilde{F}(\alpha) = \alpha^{-q}$.

Lemma 8 *$\{G^F$-conjugacy classes in $G^{\tilde{F}}\} \longleftrightarrow \mathcal{P}_n^{\tilde{F}}$.*

Thm 9 (Gow 1984) *For $\underline{\rho} \in \mathcal{P}_n^{F^2}$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{G^F}^{G^{F^2}}(1) \rangle = \begin{cases} 1, & \text{if } \underline{\rho} \in \mathcal{P}_n^{\tilde{F}} \\ 0, & \text{otherwise.} \end{cases}$$

Cor 10 (1) *If $\underline{\rho} \in \mathcal{P}_n^{\tilde{F}}$, $\mathrm{Ave}(\chi^{\underline{\rho}})(1) = 1$.*
(2) *$\{\mathrm{Ave}(\chi^{\underline{\rho}}) \mid \underline{\rho} \in \mathcal{P}_n^{\tilde{F}}\}$ is an orthogonal basis of the space of G^F-invariant functions $\mathcal{F}^{G^F}(G^{\tilde{F}})$.*

To any pair $(T, \tilde{\theta})$, T as above, character $\tilde{\theta} : T^{\tilde{F}} \to \overline{\mathbb{Q}}^{\times}$, associate analogous *basic function* $\widetilde{B}_{(T,\tilde{\theta})} \in \mathcal{F}^{G^F}(G^{\tilde{F}})$, only depending on the G^F-conjugacy class of $(T, \tilde{\theta})$.

Lemma 11 *$\{G^F$-conjugacy classes of $(T, \tilde{\theta})\} \longleftrightarrow \mathcal{P}_n^{\tilde{F}}$.*

For $\underline{\nu} \in \mathcal{P}_n^{\tilde{F}}$, let $\widetilde{B}_{\underline{\nu}} = \widetilde{B}_{(T,\tilde{\theta})}$ for $(T, \tilde{\theta}) \longleftrightarrow \underline{\nu}$.

Prop 12 **(1)** *$\{\widetilde{B}_{\underline{\nu}} \,|\, \underline{\nu} \in \mathcal{P}_n^{\tilde{F}}\}$ is an orthog b. of $\mathcal{F}^{G^F}(G^{\tilde{F}})$.*
(2) $\langle \widetilde{B}_{(T,\tilde{\theta})}, \widetilde{B}_{(T,\tilde{\theta})} \rangle = \frac{|G^F|}{|G^{\tilde{F}}|}\frac{|T^{\tilde{F}}|}{|T^F|}|W(T)_{\tilde{\theta}}^F|$.

For $\underline{\tau} \in \mathcal{P}_n^{\tilde{F}}$, let

$$C_{\underline{\tau}} = \sum_{\substack{\underline{\nu} \in \mathcal{P}_n^{\tilde{F}} \\ |\nu_\alpha| = |\tau_\alpha|, \forall \alpha}} \left(\prod_{\alpha \in \langle \tilde{F} \rangle \backslash k^\times} z_{\nu_\alpha}^{-1} \chi_{\nu_\alpha}^{\tau_\alpha} \right) \widetilde{B}_{\underline{\nu}}.$$

Then $\{C_{\underline{\tau}} \mid \underline{\tau} \in \mathcal{P}_n^{\tilde{F}}\}$ is a basis of $\mathcal{F}^{G^F}(G^{\tilde{F}})$, <u>not</u> orthog.

Thm 13 (H.) *There are* $b_{\underline{\rho},\underline{\tau}} \in \mathbb{Q}$ *for* $\underline{\rho}, \underline{\tau} \in \mathcal{P}_n^{\tilde{F}}$ *such that*

$$\text{Ave}(\chi^{\underline{\rho}}) = \sum_{\substack{\underline{\tau} \in \mathcal{P}_n^{\tilde{F}} \\ \tau_\alpha \geq \rho_\alpha, \forall \alpha}} b_{\underline{\rho},\underline{\tau}} \; C_{\underline{\tau}}.$$

Since we know the inner products of the $C_{\underline{\tau}}$'s, and $\{\text{Ave}(\chi^{\underline{\rho}})\}$ is an orthogonal basis with $\text{Ave}(\chi^{\underline{\rho}})(1) = 1$, Thm 13 determines $\text{Ave}(\chi^{\underline{\rho}})$ uniquely.

E.g. For $\underline{\rho}$ unipotent, Thm 13 implies that

$$\text{Ave}(\chi^{\underline{\rho}}) = \frac{(1-q)^{-1}(1-q^2)^{-1}\cdots(1-q^n)^{-1}}{\sum_{|\nu_1|=n} z_{\nu_1}^{-1}\Omega_{\nu_1}^{\rho_1}(-q,q)\prod_{j=1}^{\ell(\nu_1)}(1-q^{(\nu_1)_j})^{-1}} \times \sum_{\underline{\nu}\text{ unip.}} z_{\nu_1}^{-1}\Omega_{\nu_1}^{\rho_1}(-q,q)\widetilde{B}_{\underline{\nu}},$$

where $\Omega_\mu^\lambda(r,t) \in \mathbb{Q}(r,t)$ is related to Macdonald's symmetric functions of two parameters by:

$$P_{\lambda'}(x;r,t) = \sum_\mu \epsilon_\mu z_\mu^{-1}\Omega_\mu^\lambda(r,t)p_\mu(x).$$

Lusztig's Interpretation of Green's Work

(T, θ) as above. Let $\mathcal{L}$ be the F-stable rk-1 $\overline{\mathbb{Q}_l}$-local system on T such that $\theta(t) = \mathrm{tr}(\phi_t, \mathcal{L}_t)$, for $\phi : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$.

Consider the diagram

$$T \xleftarrow{\alpha} \{(g, xT) \in G_{\mathsf{rss}} \times G/T \,|\, x^{-1}gx \in T\} \xrightarrow{\pi} G_{\mathsf{rss}}$$

where $\alpha(g, xT) = x^{-1}gx$, $\pi(g, xT) = g$.

Define the *basic complex* $K_{(T,\mathcal{L})} = IC(G, \pi_*\alpha^*\mathcal{L})[\dim G]$, a G-equivariant semisimple perverse sheaf on G with induced $\phi_{(T,\mathcal{L})} : F^*K_{(T,\mathcal{L})} \xrightarrow{\sim} K_{(T,\mathcal{L})}$. Define the *characteristic function* $\chi(K_{(T,\mathcal{L})}, \phi_{(T,\mathcal{L})}) : G^F \to \overline{\mathbb{Q}_l}$ by

$$\chi(K_{(T,\mathcal{L})}, \phi_{(T,\mathcal{L})})(g) = \sum_i (-1)^i \ \mathrm{tr}(\mathcal{H}^i_g \phi_{(T,\mathcal{L})}, \mathcal{H}^i_g K_{(T,\mathcal{L})}).$$

Thm 14 (Lusztig) $B_{(T,\theta)} = (-1)^n \chi(K_{(T,\mathcal{L})}, \phi_{(T,\mathcal{L})})$.

Prop 15 *$K_{(T,\mathcal{L})} \cong \bigoplus_{E \in \widehat{W(T)_{\mathcal{L}}}} E \otimes K_{(T,\mathcal{L},E)}$, where $K_{(T,\mathcal{L},E)}$ is a G-equivariant simple perverse sheaf on G.*

These $K_{(T,\mathcal{L},E)}$ are called the *character sheaves* of G. $K_{(T,\mathcal{L},E)}$ is F-stable $\Longleftrightarrow F^*E \cong E$.

Thm 16 *For any $\underline{\rho} \in \mathcal{P}_n^F$, there is a triple $(T, \mathcal{L}, E)$ with $F^*E \cong E$, and an isom $\phi_{(T,\mathcal{L},E)} : F^*K_{(T,\mathcal{L},E)} \xrightarrow{\sim} K_{(T,\mathcal{L},E)}$, such that*

$$\chi^{\underline{\rho}} = (-1)^{\sum_{\alpha \in \langle F \rangle \backslash k^\times} |\rho_\alpha|} \chi(K_{(T,\mathcal{L},E)}, \phi_{(T,\mathcal{L},E)}).$$

So the irreducible characters are (up to sign) the characteristic functions of the F-stable character sheaves.

E.g. $\underline{\rho}$ unipotent $\rightsquigarrow T = T_1$, $\mathcal{L} = \overline{\mathbb{Q}_l}$ (trivial sheaf), $W(T)_{\mathcal{L}} = S_n$ (F acting trivially), $E = E^{\rho_1}$.

Analogous Interpretation of Theorem 13

$(T, \tilde{\theta})$ as above. Let $\widetilde{\mathcal{L}}$ be the $\tilde{F}$-stable rk-1 $\overline{\mathbb{Q}_l}$-local system on T such that $\tilde{\theta} = \chi(\widetilde{\mathcal{L}}, \widetilde{\varphi})$ for $\widetilde{\varphi} : \tilde{F}^* \widetilde{\mathcal{L}} \xrightarrow{\backsim} \widetilde{\mathcal{L}}$. Get induced $K_{(T,\widetilde{\mathcal{L}})}$, isom $\widetilde{\varphi}_{(T,\widetilde{\mathcal{L}})} : \tilde{F}^* K_{(T,\widetilde{\mathcal{L}})} \xrightarrow{\backsim} K_{(T,\widetilde{\mathcal{L}})}$.

Prop 17 $\widetilde{B}_{(T,\tilde{\theta})} = (-1)^n \chi(K_{(T,\widetilde{\mathcal{L}})}, \widetilde{\varphi}_{(T,\widetilde{\mathcal{L}})})$.

For $E \in \widehat{W(T)}_{\widetilde{\mathcal{L}}}$, $K_{(T,\widetilde{\mathcal{L}},E)}$ is $\tilde{F}$-stable $\Longleftrightarrow F^* E \cong E$.

Prop 18 *For any $\underline{\tau} \in \mathcal{P}_n^{\tilde{F}}$, there is a triple $(T, \widetilde{\mathcal{L}}, E)$ with $F^* E \cong E$, and an isom $\widetilde{\varphi}_{(T,\widetilde{\mathcal{L}},E)} : \tilde{F}^* K_{(T,\widetilde{\mathcal{L}},E)} \xrightarrow{\backsim} K_{(T,\widetilde{\mathcal{L}},E)}$, such that*

$$C_{\underline{\tau}} = (-1)^n \chi(K_{(T,\widetilde{\mathcal{L}},E)}, \widetilde{\varphi}_{(T,\widetilde{\mathcal{L}},E)}).$$

If $\chi^{\underline{\rho}} = \pm\chi(K_{(T',\mathcal{L}',E')}, \phi_{(T',\mathcal{L}',E')}) : G^{F^2} \to \overline{\mathbb{Q}}$, then

$$\text{Ave}(\chi^{\underline{\rho}}) \;\propto\; \chi(m_!(K_{(T',\mathcal{L}',E')} \boxtimes \tilde{F}^* K_{(T',\mathcal{L}',E')})) : G^{\tilde{F}} \to \overline{\mathbb{Q}},$$

where $m : G \times G \to G$ is mult. Define *convolution*

$$K_1 \star K_2 := m_!(K_1 \boxtimes K_2).$$

So Thm 13 is a statement about char. functions, relative to $\tilde{F}$, of character sheaves and their convolutions.

E.g. Unipotent case of Thm 13: for unipotent $\underline{\rho}$,

$$\chi(K_{(T,\overline{\mathbb{Q}}_l,E^{\rho_1})} \star K_{(T,\overline{\mathbb{Q}}_l,E^{\rho_1})}) = \sum_{\substack{\underline{\tau} \text{ unip.} \\ \tau_1 \geq \rho_1}} c_{\underline{\rho},\underline{\tau}} \; \chi(K_{(T,\overline{\mathbb{Q}}_l,E^{\tau_1})}).$$

So it suffices to prove that all simple perverse constituents of $K_{(T,\overline{\mathbb{Q}}_l,E^{\rho_1})} \star K_{(T,\overline{\mathbb{Q}}_l,E^{\rho_1})}$ are isomorphic to some $K_{(T,\overline{\mathbb{Q}}_l,E^{\tau_1})}$ for $\tau_1 \geq \rho_1$.

Convolution of Character Sheaves

Let G be a connected reductive group over k. Let $\widehat{G}$ denote the set of (isomorphism classes of) character sheaves on G, as defined by Lusztig.

First step in classification:

$$\widehat{G} \quad = \coprod_{\{(T,\mathcal{L})\}/G-\text{conj.}} \widehat{G}_{(T,\mathcal{L})}.$$

Second step:

$$\widehat{G}_{(T,\mathcal{L})} \;= \coprod_{\substack{\mathcal{C}\text{ two-sided}\\ \text{cell in } W(T)_{\mathcal{L}}}} \widehat{G}^{\,\mathcal{C}}_{(T,\mathcal{L})}.$$

E.g. $G = GL_n$, $\mathcal{L} = \overline{\mathbb{Q}_l}$, $W(T)_{\mathcal{L}} = S_n$,

$$\{\text{two-sided cells}\} \leftrightarrow \{\text{partitions of } n\}.$$

$K_{(T,\overline{\mathbb{Q}_l},E^\rho)}$ is the unique character sheaf in $\widehat{G}^{\,\mathcal{C}(\rho)}_{(T,\mathcal{L})}$.

Recall the partial order $\leqslant$ on two-sided cells.

Thm 19 (Grojnowski, H.) $A_1 \in \widehat{G}^{\leqslant \mathcal{C}_1}_{(T_1,\mathcal{L}_1)}$, $A_2 \in \widehat{G}^{\leqslant \mathcal{C}_2}_{(T_2,\mathcal{L}_2)}$.
(1) *If* $(T_1, \mathcal{L}_1) \nsim_{G\text{-conj.}} (T_2, \mathcal{L}_2)$, $A_1 \star A_2 \cong 0$.
(2) *If* $(T_1, \mathcal{L}_1) = (T_2, \mathcal{L}_2) = (T, \mathcal{L})$, *then all simple perverse constituents of* $A_1 \star A_2$ *are in* $\widehat{G}^{\leqslant \mathcal{C}_1, \mathcal{C}_2}_{(T,\mathcal{L})}$.

E.g. If $G = GL_n$, Thm 19 $\Rightarrow$ Thm 13.

Proof (sketch).

Fix T, Borel $B = TU$. Can define convolution on

$$\mathbf{D}^G_{\mathcal{L} \boxtimes \mathcal{L}^{-1}}(G/U \times G/U).$$

This makes the "mixed Grothendieck group" of this category into an algebra, which by Mars-Springer is $\mathcal{H}(T)_{\mathcal{L}}$, the Hecke algebra of $W(T)_{\mathcal{L}}$.

Say $K \in \mathbf{D}^{G,\text{mixed}}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)^{\leqslant\mathcal{C}}$ if $[K]^{\text{mixed}} \in \mathcal{H}(T)^{\leqslant\mathcal{C}}_{\mathcal{L}}$. Since each $\mathcal{H}(T)^{\leqslant\mathcal{C}}_{\mathcal{L}}$ is a two-sided ideal,

$$K_i \in \mathbf{D}^{G,\text{mixed}}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)^{\leqslant\mathcal{C}_i},\ i = 1, 2 \Rightarrow$$
$$K_1 \star K_2 \in \mathbf{D}^{G,\text{mixed}}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)^{\leqslant\mathcal{C}_1,\mathcal{C}_2}.$$

Connection with character sheaves (Ginzburg):

$$G/U \times G/U \xleftarrow{r} G \times G/U \xrightarrow{q} G \times G/B \xrightarrow{p} G,$$

where $r(g, g'U) = (gg'U, g'U)$, $q(g, g'U) = (g, g'B)$, and $p(g, g'B) = g$. If $K \in \mathbf{D}^{G}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)$, define

$$\text{Ch}(K) = p_! K',\ \text{where } q^* K' \cong r^* K.$$

Prop 20 *$A \in \widehat{G}^{\leqslant\mathcal{C}}_{(T,\mathcal{L})} \iff A$ is a simple perverse constituent of* Ch(K) *for some $K \in \mathbf{D}^{G,\text{mixed}}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)^{\leqslant\mathcal{C}}$.*

Then need to find relationship between convolution on $\mathbf{D}^{G}_{\mathcal{L}\boxtimes\mathcal{L}^{-1}}(G/U \times G/U)$ and convolution of character sheaves.