K^F-invariants in Irreducible Representations of G^F, when $G = GL_n$

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Finite Reductive Groups

Let $\mathbb{F}_q$ be a finite field, q odd. Let G be a connected reductive group, defined $/\mathbb{F}_q$, with Frobenius $F : G \to G$. So G^F is a *finite reductive group*.

E.g. $GL_n(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$, $U_n(\mathbb{F}_{q^2})$ (GL_n, $F(a_{ij}) = (a_{ji}^q)^{-1}$)

Irreducible representations of G^F (over $\overline{\mathbb{Q}}$) classified by:
Green (1955) for $GL_n(\mathbb{F}_q)$,
Kawanaka/Lusztig-Srinivasan (1977) for $U_n(\mathbb{F}_{q^2})$,
Lusztig (1983) in general.

Key tool: Deligne-Lusztig virtual reps R_T^λ for F-stable maximal torus T, $\lambda \in \widehat{T^F}$.

E.g. $R_{T_{\mathsf{max}}}^\lambda = \mathrm{Ind}_{B^F}^{G^F}(\lambda)$, where $B \supset T_{\mathsf{max}}$, $FB = B$.
$R_{T_{\mathsf{max}}}^1 = \mathcal{F}(G^F/B^F)$.
$R_T^1 = \sum (-1)^i H_c^i(X_B, \overline{\mathbb{Q}_l})$, where $B \supset T$,

$$X_B = \{gBg^{-1} \in \mathcal{B} \mid g^{-1}F(g) \in F(B)\}.$$

Finite Reductive Symmetric Spaces

Let $\theta : G \to G$ be an involn with $\theta F = F\theta$.
Let K be an F-stable subgroup, $(G^\theta)^\circ \leq K \leq G^\theta$.
So G^F/K^F is a *finite reductive symmetric space*.
Have representation of G^F on $\mathcal{F}(G^F/K^F)$.

Problem For any irrep V of G^F, calculate

$$\boxed{V} := [V : \mathcal{F}(G^F/K^F)] = \dim V^{K^F}.$$

Now solved for all cases where $G = GL_n$.

E.g. When $G^F = GL_n(\mathbb{F}_q)$, possible K^F:

- $Sp_n(\mathbb{F}_q)$ (n even), [Bannai-Kawanaka-Song(1990)]
- $GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)$ ($n^+ + n^- = n$),
- $GL_{n/2}(\mathbb{F}_{q^2})$ (n even),
- $O_n(\mathbb{F}_q)$ or $SO_n(\mathbb{F}_q)$ (n odd),
- $O_n^\pm(\mathbb{F}_q)$ or $SO_n^\pm(\mathbb{F}_q)$ (n even).

Example: Principal-series unipotent reps

Def $V \in \widehat{G^F}$ is *principal-series unipotent* if $V^{B^F} \neq 0$.

If so, $\mathrm{Hom}(V^*, \mathcal{F}(G^F/B^F)) = V^{B^F}$ is a simple module for

$$\mathrm{End}_{G^F}(\mathcal{F}(G^F/B^F)) \cong e_{B^F} G^F e_{B^F} =: \mathcal{H}(G^F, B^F),$$

the *Hecke algebra* of the pair (G^F, B^F).
Moreover $[V : W]_{G^F} = [V^{B^F} : W^{B^F}]_{\mathcal{H}(G^F,B^F)}$ for all W. So

$$\boxed{V} = [V^{B^F} : \mathcal{F}(B^F \backslash G^F/K^F)]_{\mathcal{H}(G^F,B^F)}.$$

E.g. If $G^F = GL_n(\mathbb{F}_q)$, $\mathcal{H}(G^F, B^F) = \mathcal{H}(S_n)_q$, the usual q-deformation of $\overline{\mathbb{Q}}S_n$. Hence unipotent reps $\{V_\rho \,|\, \rho \vdash n\}$. If $K^F = GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)$, $\mathcal{F}(B^F \backslash G^F/K^F) = M_q^K$, q-deformation of *coherent continuation* rep M^K of S_n attached to $U(n^+, n^-)$. So by Barbasch-Vogan (1982),

$$\boxed{V_\rho} = |\{(n^+, n^-)\text{-signed tableaux of shape } \rho'\}|.$$

Example: Discrete-series representations

Def $V \in \widehat{G^F}$ is *discrete-series* if $[V : \mathrm{Ind}_{P^F}^{G^F}(W)] = 0$ for all proper F-stable parabolics P, reps W of P^F/U_P^F.

E.g. For $GL_n(\mathbb{F}_q)$, discrete-series reps are $(-1)^{n-1} R_{T_{\min}}^{\lambda}$ where $T_{\min}^F \cong \mathbb{F}_{q^n}^{\times}$, λ, λ^q, . . . , $\lambda^{q^{n-1}}$ distinct.

Lusztig, "Symmetric Spaces over a Finite Field" (1990): Let $\mathcal{I}$ be the set of θ-stable maximal tori T_0 such that for some $B_0 \supset T_0$, $\theta(B_0)$ is opposed to B_0. Let

$$\Theta_{T_{\min},K,\lambda}^F = \{f \in G^F \mid f^{-1} T_{\min} f \in \mathcal{I}, \lambda|_{(T_{\min} \cap fKf^{-1})^F} = 1\}.$$

Then

$$\boxed{(-1)^{n-1} R_{T_{\min}}^{\lambda}} = |T_{\min}^F \backslash \Theta_{T_{\min},K,\lambda}^F / K^F|.$$

E.g. (1) If $K^F = Sp_n(\mathbb{F}_q)$, $\mathcal{I} = \emptyset$, so this is 0.
(2) If $K^F = GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)$, this is

$$\begin{cases} 1, & \text{if } n^+ = n^- = n/2,\ \lambda^{-1} = \lambda^{q^{n/2}}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $K^F = SO_n^{\pm}(\mathbb{F}_q)$ (n even), this is

$$\begin{cases} 2, & \text{if } \lambda(-1) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

General Formula for $\boxed{R_T^\lambda}$

Generally, Lusztig defines a union of T^F–K^F double cosets $\Theta^F_{T,K,\lambda} \subset G^F$, and proves that

$$\boxed{R_T^\lambda} = \sum_{T^F \backslash \Theta^F_{T,K,\lambda}/K^F} \pm 1.$$

E.g. $G^F = GL_n(\mathbb{F}_q)$, $K^F = GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)$.
If $T \leftrightarrow w \in S_n$, $\lambda = 1$,

$$T^F \backslash \Theta^F_{T,K,\lambda}/K^F \leftrightarrow \begin{array}{l}\text{involutions } \sigma \text{ in } Z_{S_n}(w),\\ (n^+ - s, n^- - s)\text{-signed fixed pts,}\\ \text{signs stable under } w\end{array}$$

$$\pm 1 = (-1)^{|\{\text{cycles of } w \text{ fixed by } \sigma, \text{ not pointwise}\}|}$$

Since $R_T^1 = \sum_{\rho \vdash n} \chi^\rho(w) V_\rho$, earlier formula for $\boxed{V_\rho}$ $\Leftrightarrow$

$$\sum_{\substack{\text{involutions } \sigma \text{ in } Z_{S_n}(w),\\ (n^+ - s, n^- - s)\text{-signed fixed pts,}\\ \text{signs stable under } w}} (-1)^{|\{\text{cycles of } w \text{ fixed by } \sigma, \text{ not pointwise}\}|}$$

$$= \sum_{\rho \vdash n} |\{(n^+, n^-)\text{-signed tableaux of shape } \rho'\}| \chi^\rho(w).$$

Proving analogous facts gives all other cases.