Geometric modular representation theory

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The idea

- A representation of a group G over a field k is a vector space V over k on which G acts by linear transformations.
- The modular case, often more difficult to study, is when k has characteristic ℓ (a prime).
- Some of the hard questions in the modular case can be re-formulated in geometric terms.
- This actually helps!

Representations of $GL_n(k)$

One very important group is the general linear group $GL_n(k)$ of $n \times n$ invertible matrices over k. Its representations over k include:

- k^n (column vectors). Let $\{e_1, \cdots, e_n\}$ be the standard basis.
- $S^i(k^n)$, the ith symmetric power of k^n, for $i \geq 0$. This has dimension $\binom{n+i-1}{i}$, with the standard basis

$$\{e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} \mid i_1, \cdots, i_n \in \mathbb{N}, i_1 + \cdots + i_n = i\}.$$

- $\Lambda^i(k^n)$, the ith exterior power of k^n, for $0 \leq i \leq n$. This has dimension $\binom{n}{i}$, with the standard basis

$$\{e_{a_1} \wedge e_{a_2} \wedge \cdots \wedge e_{a_i} \mid 1 \leq a_1 < a_2 < \cdots < a_i \leq n\}.$$

- $S^3(\Lambda^6(k^n)) \otimes (S^2(k^n))^*$, and so forth.

Irreducible representations

The most basic representations are those that are irreducible, meaning that they have no nontrivial invariant subspaces.

- k^n is an irreducible representation of $GL_n(k)$.
- More generally, $\Lambda^i(k^n)$ is irreducible for all i.
- If $\mathrm{char}(k) = 0$, then $S^i(k^n)$ is irreducible for all i.
- If $\mathrm{char}(k) = \ell$, then $S^i(k^n)$ is irreducible only when $i < \ell$.

Example ($i = \ell = 2$)

$S^2(k^n)$ has an invariant subspace $\mathrm{span}\{e_1^2, \cdots, e_n^2\}$, because

$$(a_1 e_1 + \cdots + a_n e_n)^2 = a_1^2 e_1^2 + \cdots + a_n^2 e_n^2 \quad \text{when } \mathrm{char}(k) = 2.$$

Highest weight theory for $GL_n(k)$

Assume the field k is algebraically closed. We usually only consider representations V of $GL_n(k)$ that are rational in the sense that the homomorphism $GL_n(k) \to GL(V)$ is a morphism of varieties over k. Any such V has a weight space decomposition

$$V = \bigoplus_{\lambda=(\lambda_1,\cdots,\lambda_n)\in\mathbb{Z}^n} V_\lambda, \text{ where}$$

$$V_\lambda = \{v \in V \mid \mathrm{diag}(a_1, \cdots, a_n) \cdot v = a_1^{\lambda_1} \cdots a_n^{\lambda_n} v \text{ for all } a_1, \cdots, a_n \in k^\times\}.$$

The weights of V are those λ such that $V_\lambda \neq 0$.

Theorem (Cartan 1913, Chevalley 1958)

For every $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$, there is an irreducible representation $L(\lambda, k)$ of $GL_n(k)$ with highest weight λ. Every irreducible representation $\cong L(\lambda, k)$ for a unique such λ.

Describing the irreducible representations

The description of $L(\lambda, k)$ usually depends on $\mathrm{char}(k)$:

- $L((1, 0, \cdots, 0), k) = k^n$.
- $L((\underbrace{1, \cdots, 1}_{i}, 0, \cdots, 0), k) = \Lambda^i(k^n)$.
- $L((i, 0, \cdots, 0), k) = \mathrm{span}\{(a_1 e_1 + \cdots + a_n e_n)^i\} \subset S^i(k^n)$. This has a basis $\{e_1^{i_1} \cdots e_n^{i_n} \mid \binom{i}{i_1, \cdots, i_n} \neq 0 \text{ in } k\}$.
- When $\mathrm{char}(k) = 0$, we understand $L(\lambda, k)$ quite well, e.g. the Weyl dimension formula says that

$$\dim L(\lambda, k) = \prod_{1 \leq a < b \leq n} \frac{\lambda_a - \lambda_b + b - a}{b - a},$$

  the Weyl character formula describes the weight multiplicities, and there are explicit constructions with explicit bases.
- When $\mathrm{char}(k) = \ell$, even $\dim L(\lambda, k)$ is unknown in general.

Topology enters the story

There is another well-known definition of a vector space over k whose explicit description depends on $\mathrm{char}(k)$: the homology $H_*(X, k)$ of a topological space X with coefficients in k.

Example

The real projective plane $\mathbb{RP}^2$ can be constructed as a cell complex with one 2-cell, one 1-cell, and one 0-cell, and the boundary maps

$$k^1 \xrightarrow{\times 2} k^1 \xrightarrow{\times 0} k^1.$$

Hence $\dim H_*(\mathbb{RP}^2, k) = 1$ if $\mathrm{char}(k) \neq 2$, and 3 if $\mathrm{char}(k) = 2$.

A dream: what if there were topological spaces $X(\lambda)$, one for each λ as above, such that $\dim H_*(X(\lambda), k) = \dim L(\lambda, k)$ for any k?

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Complex geometry takes over

Theorem (part of 'geometric Satake', Mirković–Vilonen 2007)

For any λ as above, there is a complex projective variety $X(\lambda)$, usually singular but with a stratification into smooth pieces, such that $\dim IH_*(X(\lambda), k) = \dim L(\lambda, k)$ for any k, where IH_* is intersection homology (a version of homology for stratified spaces).

- $X(1, 0, \cdots, 0) = \mathbb{CP}^{n-1}$ is smooth, $\dim H_*(\mathbb{CP}^{n-1}, k) = n$.
- $X(\underbrace{1, \cdots, 1}_{i}, 0, \cdots, 0) = \mathrm{Gr}_i(\mathbb{C}^n) = \{V \subset \mathbb{C}^n \mid \dim V = i\}$.

  This is also smooth, and $\dim H_*(\mathrm{Gr}_i(\mathbb{C}^n), k) = \binom{n}{i}$.
- $X(i, 0, \cdots, 0)$ is the singular variety

$$\{V \subset \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{i} \mid \dim V = i,\ \varphi(V) \subset V\}$$

  where $\varphi(v_1, \cdots, v_i) = (v_2, \cdots, v_i, 0)$.

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Example ($n = 2,\ \lambda = (2,0)$)

As a special case of the above definition,

$$X(2,0) = \{V \subset \mathbb{C}^2 \oplus \mathbb{C}^2 \mid \dim V = 2,\ \varphi(V) \subset V\}$$

where $\varphi(v_1, v_2) = (v_2, 0)$. It has an isolated singularity at the point

$$V_0 = \ker(\varphi) = \mathbb{C}^2 \oplus \{0\}.$$

This point has a conical open neighbourhood

$$\begin{aligned} U(2,0) &= \{V \in X(2,0) \mid V \cap (\{0\} \oplus \mathbb{C}^2) = \{0\}\} \\ &= \{\{(v, xv) \mid v \in \mathbb{C}^2\} \mid x \in \mathrm{Mat}_2(\mathbb{C}),\ x^2 = 0\}. \end{aligned}$$

The definition of $IH_*(X(2,0), k)$ incorporates the homology of

$$U(2,0) \setminus \{V_0\} \cong \{x \in \mathrm{Mat}_2(\mathbb{C}),\ x^2 = 0,\ x \neq 0\} \sim \mathbb{RP}^3,$$

and $H_*(\mathbb{RP}^3, k)$ is different when $\mathrm{char}(k) = 2$.

An early success story

Similar geometric ideas appeared first in characteristic-0 theory, in studying irreducible highest-weight representations $L(\sigma, \mathbb{C})$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ in the same block as the trivial representation. These are indexed by permutations σ of $[n] = \{1, 2, \cdots, n\}$.

Theorem (Beilinson–Bernstein, Brylinski–Kashiwara 1981)

The weight multiplicities in $L(\sigma, \mathbb{C})$ can be determined from $IH_*(Y(\sigma), \mathbb{C})$ where $Y(\sigma)$ is the Schubert variety

$$\{V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim(V_i \cap \mathrm{span}\{e_1, \cdots, e_j\}) \geq |\sigma[i] \cap [j]|\}.$$

This solved the problem: Kazhdan–Lusztig proved a combinatorial formula for $IH_*(Y(\sigma), \mathbb{C})$, using Deligne's theory of weights which applies only to characteristic-0 coefficients. This was the prototype for many subsequent results, e.g. the varieties $X(\lambda)$ appearing in geometric Satake are Schubert varieties for the group $GL_n(\mathbb{C}((t)))$.

A recent success(?) story

Recall the representations $L(\lambda, k)$ of $GL_n(k)$. Take $\mathrm{char}(k) = \ell$.

1979: Lusztig wrote down a complicated combinatorial formula depending on λ and ℓ and conjectured that it equalled $\dim L(\lambda, k)$, as long as $\ell \geq 2n - 3$ (this was the lower bound required for the formula to make sense for all λ).

1994: Andersen–Jantzen–Soergel proved that the Lusztig conjecture is true when ℓ is sufficiently large relative to n, but with no specific lower bound on ℓ.

2011: Using the varieties $X(\lambda)$ and their moment graphs, Fiebig found a sufficient lower bound, much larger than $2n - 3$.

2000: Soergel showed that Lusztig's conjecture implies a statement about $IH_*(Y(\sigma), k)$ (note: now we have modular coefficients).

2013: Williamson found a family of counterexamples to the latter statement where ℓ grows exponentially in n. So the conjectured bound of $2n - 3$ is nowhere near sufficient.