Nilpotent orbits and Kazhdan-Lusztig polynomials of type A

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Nilpotent orbits of the linear quiver

Let $V = V_1 \oplus \cdots \oplus V_n$ be a graded vector space,

$$d = \dim V,\ \ d_i = \dim V_i.$$

Consider

$$\mathcal{N}_V = \{\varphi \in \mathrm{End}(V) \mid \varphi(V_i) \subseteq V_{i+1}\},$$

and the orbits of $G_V = \{g \in GL(V) \mid g(V_i) = V_i\}$, acting on $\mathcal{N}_V$ by conjugation.

These are in bijection with **multisegments**

$$\mathbf{m} = \sum_{1 \le i \le j \le n} m_{i,j}\,[i,j],$$

where $\boxed{\sum_{i \le k, j \ge k} m_{i,j} = d_k \text{ for all } k.}$ Let

- $\mathcal{O}_{\mathbf{m}}$ be the orbit corresponding to $\mathbf{m}$,
- $IC_{\mathbf{m},\mathbf{m}'}$ be $\sum_i \dim \mathcal{H}^{2i}_{\mathbf{m}} IC(\overline{\mathcal{O}_{\mathbf{m}'}})\, q^i$,
- $(IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'})$ be the inverse matrix to $(IC_{\mathbf{m},\mathbf{m}'})$.

Representation-theoretic significance

Let $r \geq n$, and let $U_v(\widehat{\mathfrak{sl}_r})$ be **quantum affine** $\mathfrak{sl}_r$, specialized at v which is not a root of 1.

To each multisegment $\mathbf{m}$ we attach a $U_v(\widehat{\mathfrak{sl}_r})$-module

$$M_{\mathbf{m}} := \bigotimes_{i \leq j} (\wedge^{j-i+1} \mathbb{C}^r)_i^{\otimes m_{i,j}}.$$

These have simple constituents $\{L_{\mathbf{m}}\}$, with multiplicities

$$[M_{\mathbf{m}} : L_{\mathbf{m}'}] = IC_{\mathbf{m},\mathbf{m}'}(1),$$

so

$$[L_{\mathbf{m}}] = \sum_{\mathbf{m}'} IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'}(1)\,[M_{\mathbf{m}'}]$$

in the Grothendieck group of $U_v(\widehat{\mathfrak{sl}_r})$-modules.

Similar things hold for the **affine Hecke algebra** $\widehat{\mathcal{H}_d}$.

Zelevinsky's theorem

To a multisegment $\mathbf{m}$ attach a permutation $w_{\mathbf{m}} \in S_d$ by:

1. Split $\{1, \cdots, d\}$ into blocks of sizes $d_1, \cdots, d_n$.
2. The k_i segments starting with i determine the images of the first k_i elements of the ith block (highest values not yet used).
3. The rest of the ith block gets mapped into the $(i-1)$th block (reversed order).

Thm 1 (Zelevinsky 1985) *We have*

$$IC_{\mathbf{m},\mathbf{m}'} = P_{w_{\mathbf{m}},w_{\mathbf{m}'}},\ IC^{\langle -1\rangle}_{\mathbf{m},\mathbf{m}'} = \sum_{x \in S_{(d_i)} w_{\mathbf{m}'} S_{(d_i)}} \varepsilon(x w_{\mathbf{m}})\, P_{x w_0^{(d)}, w_{\mathbf{m}} w_0^{(d)}},$$

where $P_{-,-}$ means Kazhdan-Lusztig polynomial for S_d.

The idea of the proof is to embed the nilpotent orbits in Schubert varieties.

Applying a result of Billey and Warrington

Prop 1 (BW 2003) *Let* $y \leq w \in S_d$*,* $i \in \{1, \cdots, d\}$ *s.t.*

$$\boxed{y(i) = w(i),\ |\{j < i \,|\, y(j) > y(i)\}| = |\{j < i \,|\, w(j) > w(i)\}|}\,.$$

Then the $i \mapsto y(i)$ *string can be cancelled from both* y *and* w *without changing* $P_{y,w}$*.*

My observation:

Prop 2 *Let* $\mathbf{m}, \mathbf{m}'$ *be such that* $\mathcal{O}_{\mathbf{m}} \subseteq \overline{\mathcal{O}_{\mathbf{m}'}}$*. In the defn of* $w_{\mathbf{m}}$*, all Step 3 strings are cancellable for* $w_{\mathbf{m}}$ *and* $w_{\mathbf{m}'}$*.*

Let k be the number of segments of $\mathbf{m}$, and define $w^{(k)}_{\mathbf{m}}, w^{(k)}_{\mathbf{m}'} \in S_k$ by cancelling these strings from $w_{\mathbf{m}}, w_{\mathbf{m}'}$.

Cor 1 (Zelevinsky simplified) *We have*

$$IC_{\mathbf{m},\mathbf{m}'} = P_{w^{(k)}_{\mathbf{m}}, w^{(k)}_{\mathbf{m}'}},\ \ IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = \sum_{x \in S_{(\cdots)} w^{(k)}_{\mathbf{m}'} S_{(\cdots)}} \varepsilon(x w^{(k)}_{\mathbf{m}})\, P_{x w^{(k)}_0, w^{(k)}_{\mathbf{m}} w^{(k)}_0},$$

where $P_{-,-}$ *means Kazhdan-Lusztig polynomial for* S_k*.*

The $q = 1$ specialization of this result was already known (Suzuki 1998, Arakawa 1999).

A special case

Suppose we have integers

$$s_1 < s_2 < \cdots < s_k,\ \ t_1 < t_2 < \cdots < t_k,$$

and $w, w' \in S_k$ such that $t_{w(i)}, t_{w'(i)} \geq s_i - 1$. Let

$$\mathbf{m} = \sum_{i=1}^{k} [s_i, t_{w(i)}],\ \ \mathbf{m}' = \sum_{i=1}^{k} [s_i, t_{w'(i)}].$$

(Note that no segments start or end the same.) Then

$$IC_{\mathbf{m},\mathbf{m}'} = P_{w,w'},\ \ IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = \varepsilon(ww')\, P_{w'w_0^{(k)}, ww_0^{(k)}}.$$

In particular: if $\mathbf{m}$ has no inclusions of segments,

$$IC_{\mathbf{m},\mathbf{m}'} = P_{1,w'},\ \ IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = \varepsilon(w'),$$

and so

$$[L_{\mathbf{m}}] = \sum_{\substack{w' \in S_k \\ t_{w'(i)} \geq s_i - 1}} \varepsilon(w')\, [M_{[s_1, t_{w'(1)}] + \cdots + [s_k, t_{w'(k)}]}].$$

Such $L_{\mathbf{m}}$ are called **tame** (Yangian context) or **calibrated** (affine Hecke algebra context).

The case of cyclic quivers

Much of the above still works for the cyclic quiver with n vertices, relevant for $U_\zeta(\widehat{\mathfrak{sl}_r})$ where $\zeta^n = 1$.

Suppose we have integers

$$\cdots < s_0 < s_1 < s_2 < \cdots\ ,\ s_{i+k} = s_i + n,$$
$$\cdots < t_0 < t_1 < t_2 < \cdots\ ,\ t_{i+k} = t_i + n,$$

and $w, w' \in \widetilde{S_k}$ such that $t_{w(i)}, t_{w'(i)} \geq s_i - 1$. Let

$$\mathbf{m} = \sum_{i=1}^{k} [s_i, t_{w(i)}],\ \ \mathbf{m}' = \sum_{i=1}^{k} [s_i, t_{w'(i)}].$$

Then

$$IC_{\mathbf{m},\mathbf{m}'} = P_{w,w'},\ \ IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = \varepsilon(ww')\, Q_{w,w'}.$$

In particular: if $\mathbf{m}$ has no inclusions of segments,

$$IC_{\mathbf{m},\mathbf{m}'} = P_{1,w'},\ \ IC^{\langle -1 \rangle}_{\mathbf{m},\mathbf{m}'} = \varepsilon(w'),$$

and so

$$[L_{\mathbf{m}}] = \sum_{\substack{w' \in \widetilde{S}_k \\ t_{w'(i)} \geq s_i - 1}} \varepsilon(w')\, [M_{[s_1, t_{w'(1)}] + \cdots + [s_k, t_{w'(k)}]}].$$