Equalities of Kazhdan-Lusztig polynomials of type A (math.RT/0501054)

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Kazhdan-Lusztig polys: algebra/combinatorics

Let (W, S) be a Coxeter group, $\mathcal{H}$ its Iwahori-Hecke algebra with basis $\{T_w \,|\, w \in W\}$. The Kazhdan-Lusztig polynomials $P_{y,w} \in \mathbb{Z}[q]$ satisfy:

(1) $P_{y,w} = 0$ unless $y \leq w$ (Bruhat order).

(2) $P_{w,w} = 1$, $\deg(P_{y,w}) < \frac{\ell(w)-\ell(y)}{2}$ for $y < w$.

(3) $C_w := \sum_{y \leq w} P_{y,w} T_y$. If $sw < w$, $C_w - (T_s + 1)C_{sw}$ is a linear combination of $\{C_x \,|\, x < w\}$ with coefficients which are stable under $q^i \mapsto q^{\ell(w)-\ell(x)-i}$.

Given C_x for $x < w$, (3) specifies C_w , since by (1),(2),

$$C_w = T_w + \sum_{x<w} a_x C_x, \ \deg(a_x) < \frac{\ell(w) - \ell(x)}{2}.$$

In particular, if $(T_s + 1)C_{sw}$ also has this form, it must equal C_w, e.g. $C_s = T_s + 1$, $C_{ss'} = T_{ss'} + T_s + T_{s'} + 1$.

E.g. $W = S_4$. In subscripts, write s_i as i. Having calculated $C_{132} = (T_1+1)(T_3+1)(T_2+1)$, we get

$$\begin{aligned}(T_2+1)C_{132} &= T_{2132} + T_{121} + T_{132} + T_{213} + T_{232} \\ &\quad + T_{21} + T_{12} + T_{13} + T_{32} + T_{23} \\ &\quad + T_1 + (q+1)T_2 + T_3 + (q+1)1 \\ &= T_{2132} + C_{121} + C_{132} + C_{213} + C_{232} \\ &\quad - C_{21} - C_{12} - C_{13} - C_{32} - C_{23} \\ &\quad + C_1 + (q+1)C_2 + C_3 - 1,\end{aligned}$$

so this equals C_{2132}. Next we find that

$$\begin{aligned}(T_1+1)C_{2132} &= T_{12132} + T_{2132} + T_{1213} + T_{1232} + \cdots \\ &= T_{12132} + C_{2132} + C_{1213} + C_{1232} \\ &\quad + (q-1)C_{121} + (q-1)C_{132} \\ &\quad - C_{213} - C_{232} - C_{123} \\ &\quad + C_{21} + C_{12} + C_{13} + C_{32} + C_{23} \\ &\quad - C_1 - (q+1)C_2 - C_3 + 1,\end{aligned}$$

so $C_{12132} = (T_1+1)C_{2132} - qC_{121} - qC_{132}$. Final result:

$$P_{y,w} = \begin{cases} q+1, & \text{if } w = s_2s_1s_3s_2 \text{ and } y \le s_2 \\ & \text{or } w = s_1s_3s_2s_1s_3 \text{ and } y \le s_1s_3, \\ 1, & \text{otherwise.}\end{cases}$$

Kazhdan-Lusztig polys: geometry/topology

Now suppose W is the Weyl group of a reductive group G over $\mathbb{C}$. The flag variety $\mathcal{B} = G/B$ is the union of B-orbits $\mathcal{B}_w = BwB/B$ for $w \in W$, whose closures are the Schubert varieties $\overline{\mathcal{B}_w} = \bigcup_{y \leq w} \mathcal{B}_y$.

Kazhdan-Lusztig: in terms of local intersection cohomology, $\boxed{P_{y,w} = \sum_i \dim IH^{2i}_{yB/B}(\overline{\mathcal{B}_w})\, q^i.}$

Now $yU^-B/B \cap \overline{\mathcal{B}_w}$ is an open neighbourhood of yB/B in $\overline{\mathcal{B}_w}$, isomorphic to $\mathbb{C}^{\ell(y)} \times U_{y,w}$ where

$$U_{y,w} = \{g \in {}^yU^- \cap U^- \mid gyB/B \in \overline{\mathcal{B}_w}\}.$$

So also $\boxed{P_{y,w} = \sum_i \dim IH^{2i}_1(U_{y,w})\, q^i.}$

If $W = S_n$, $U_{y,w}$ is the set of $(g_{i,j})_{i,j=1}^{n}$ such that:

(1) $g_{i,i} = 1$;

(2) $g_{i,j} = 0$ if $i < j$ or $y^{-1}(i) < y^{-1}(j)$;

(3) $\mathrm{rk}\,(g_{i,y(j)})_{i \geq a, j \leq b} \leq |\{j \leq b \,|\, w(j) \geq a\}|$.

E.g. $W = S_5$, $y = \underline{14325}$, $w = \underline{45312}$. (1) and (2) give

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ g_{2,1} & 1 & 0 & 0 & 0 \\ g_{3,1} & 0 & 1 & 0 & 0 \\ g_{4,1} & 0 & 0 & 1 & 0 \\ g_{5,1} & g_{5,2} & g_{5,3} & g_{5,4} & 1 \end{pmatrix},$$

then (3) says $g_{5,1} = g_{2,1}g_{5,2} + g_{3,1}g_{5,3} + g_{4,1}g_{5,4} = 0$.

Since this $U_{y,w}$ is a cone over a nonsingular variety,

$$IH_1^{2i}(U_{y,w}) = \begin{cases} H^{2i}(U_{y,w} \setminus \{1\}), & \text{if } 2i < \dim U_{y,w}, \\ 0, & \text{otherwise,} \end{cases}$$

which leads to $P_{y,w} = q^2 + 1$.

Kazhdan-Lusztig polys: representation theory

Let $\widehat{\mathcal{H}}_d$ be the affine Hecke algebra over $\mathbb{C}$ associated to GL_d, with generators $T_1, \cdots, T_{d-1}, X_1^{\pm 1}, \cdots, X_d^{\pm 1}$, and generic param t. (Relations include $T_i X_i T_i = t X_{i+1}$.)

Let $\lambda, \mu \in \mathbb{Z}^n$ be such that $\lambda_i \geq \mu_i$, $\forall i$, and $d = \sum_i \lambda_i - \mu_i$. Let $\widehat{\mathcal{H}}_{\lambda/\mu}$ be the corresponding parabolic subalgebra of $\widehat{\mathcal{H}}_d$, and define the **standard module**

$$M_{\lambda/\mu} := \widehat{\mathcal{H}}_d \otimes_{\widehat{\mathcal{H}}_{\lambda/\mu}} \mathbb{C}_{\lambda/\mu},$$

where $\mathbb{C}_{\lambda/\mu}$ is the one-dimensional representation on which

$$X_1 = t^{\mu_1}, X_2 = t^{\mu_1+1}, \cdots, X_{\lambda_1-\mu_1} = t^{\lambda_1-1},$$
$$X_{\lambda_1-\mu_1+1} = t^{\mu_2-1}, \cdots, X_{\lambda_1-\mu_1+\lambda_2-\mu_2} = t^{\lambda_2-2},$$
$$X_{\lambda_1-\mu_1+\lambda_2-\mu_2+1} = t^{\mu_3-2}, \cdots, X_{\lambda_1-\mu_1+\lambda_2-\mu_2+\lambda_3-\mu_3} = t^{\lambda_3-3},$$
$$\cdots\cdots$$

Zelevinsky: $M_{\lambda/\mu}$ has a unique simple quotient $L_{\lambda/\mu}$.

Let $W = S_n$, which acts on $\mathbb{Z}^n$ by the dot action:

$$(w \cdot \lambda)_i = \lambda_{w^{-1}(i)} - w^{-1}(i) + i.$$

A fundamental domain is

$$D_n = \{\lambda \in \mathbb{Z}^n \,|\, \lambda_1 - 1 \geq \lambda_2 - 2 \geq \cdots \geq \lambda_n - n\}.$$

For $\lambda, \mu \in D_n$, let

$$S_n[\lambda,\mu] = \{w \in S_n \,|\, \lambda_i \geq (w \cdot \mu)_i,\ \forall i\},$$
$$S_n[\lambda,\mu]^\circ = \{w \in S_n[\lambda,\mu] \,|\, w \text{ max length in } W_\lambda w W_\mu\},$$

where W_λ and W_μ are the stabilizers.

Suzuki: For $y \in S_n[\lambda,\mu]^\circ$,

$$[M_{\lambda/y\cdot\mu}] = \sum_{w \in S_n[\lambda,\mu]^\circ} P_{y,w}(1)\,[L_{\lambda/w\cdot\mu}]$$

in the Grothendieck group of $\widehat{\mathcal{H}}_d$-modules. A refinement using a Jantzen-like filtration involves $P_{y,w}$ itself.

Suzuki's result is deduced from the analogous statement for Verma modules of $\mathfrak{gl}_n$ (K-L Conjecture): for $\mu \in D_n$,

$$[M(y \cdot \mu)] = \sum_{w \in S_n[\mu]^\circ} P_{y,w}(1)\,[L(w \cdot \mu)].$$

For $\lambda \in D_n$, he defines an exact functor F_λ from $\mathcal{O}(\mathfrak{gl}_n)$ to $\{\widehat{\mathcal{H}}_d^{\text{degen}}\text{-mod}\}$ with the following properties.

(1) For all $\mu \in \mathbb{Z}^n$,

$$F_\lambda(M(\mu)) = \begin{cases} M_{\lambda/\mu}, & \text{if } \lambda_i \geq \mu_i,\ \forall i,\ d = \sum \lambda_i - \mu_i \\ 0, & \text{otherwise.} \end{cases}$$

(2) For all such $\mu \in D_n$ and $w \in S_n[\mu]^\circ$,

$$F_\lambda(L(w \cdot \mu)) = \begin{cases} L_{\lambda/w\cdot\mu}, & \text{if } w \in S_n[\lambda, \mu]^\circ \\ 0, & \text{otherwise.} \end{cases}$$

As a vector space, $\boxed{F_\lambda(V) = \text{Hom}_{\mathfrak{gl}_n}(M(\lambda), V \otimes (\mathbb{C}^n)^{\otimes d}).}$

Orellana-Ram: similar $F_\lambda : \mathcal{O}(U_t(\mathfrak{gl}_n)) \to \{\widehat{\mathcal{H}}_d\text{-mod}\}$.

Cancellation for the symmetric group

Still $W = S_n$. For any $1 \leq i \leq n$, define

$$\mathrm{inv}_i(w) = |\{i' < i \,|\, w(i') > w(i)\}|.$$

If $y \leq w$, we say i is **cancellable** for $[y, w]$ if

$$y(i) = w(i) \text{ and } \mathrm{inv}_i(y) = \mathrm{inv}_i(w).$$

Billey-Warrington: If so, then $\boxed{P_{y,w} = P_{y^{\hat{i}},w^{\hat{i}}}}$ where $y^{\hat{i}}, w^{\hat{i}} \in S_{n-1}$ are obtained by 'deleting the ith string'.

E.g. In S_6, take $y = \underline{154326}$, $w = \underline{456312}$. Then

$$y(2) = w(2) = 5,\ \mathrm{inv}_2(y) = \mathrm{inv}_2(w) = 0,$$

so 2 is cancellable for $[y, w]$ (whereas 4 isn't). Hence

$$P_{\underline{154326},\underline{456312}} = P_{\underline{14325},\underline{45312}} = q^2 + 1.$$

Geometrically: if i is cancellable for $[y,w]$, then all matrices in $U_{y,w}$ have $w(i)$th row and column zero except for the 1 in the $(w(i),w(i))$-position. Deleting this row and column gives an isomorphism $U_{y,w} \xrightarrow{\sim} U_{y^{\hat{i}},w^{\hat{i}}}$.

E.g. $U_{\underline{154326},\underline{456312}}$ is the set of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ g_{2,1} & 1 & 0 & 0 & 0 & 0 \\ g_{3,1} & 0 & 1 & 0 & 0 & 0 \\ g_{4,1} & 0 & 0 & 1 & 0 & 0 \\ g_{5,1} & 0 & 0 & 0 & 1 & 0 \\ g_{6,1} & g_{6,2} & g_{6,3} & g_{6,4} & g_{6,5} & 1 \end{pmatrix}$$

such that $\operatorname{rk}\left(\begin{smallmatrix} g_{5,1} \\ g_{6,1} \end{smallmatrix}\right) = 0$, $\operatorname{rk}\left(g_{6,1} \;\; g_{6,5}\right) = 0$, and

$$\operatorname{rk}\begin{pmatrix} g_{2,1} & 1 & 0 & 0 & 0 \\ g_{3,1} & 0 & 1 & 0 & 0 \\ g_{4,1} & 0 & 0 & 1 & 0 \\ g_{5,1} & 0 & 0 & 0 & 1 \\ g_{6,1} & g_{6,2} & g_{6,3} & g_{6,4} & g_{6,5} \end{pmatrix} \leq 4.$$

Deleting the fifth row and column, these become the conditions for $U_{\underline{14325},\underline{45312}}$ seen before.

Representation-theoretically: it is cancellation that makes Suzuki's equations consistent.

E.g. Let $W = S_6$, and set

$$\lambda = (3,3,4,5,5,5), \mu = (2,2,2,3,4,4) \in D_6,$$
$$y = \underline{154326} \in S_6[\lambda,\mu]^\circ.$$

The elements of $S_6[\lambda,\mu]^\circ$ which are $> y$ are:

$w_1 = \underline{156432}, w_2 = \underline{453216}, w_3 = \underline{456312}, w_4 = \underline{456321}$.

So $[M_{\lambda/y\cdot\mu}] = [L_{\lambda/y\cdot\mu}] + \sum_{i=1}^4 P_{y,w_i}(1)\,[L_{\lambda/w_i\cdot\mu}]$.

But note that $y\cdot\mu = (2,1,2,3,5,4)$ has a common component with λ, which does not contribute. So $M_{\lambda/y\cdot\mu}$ can also be written as $M_{\widetilde{\lambda}/\widetilde{y}\cdot\widetilde{\mu}}$ where

$$\widetilde{\lambda} = (3,3,4,5,4), \widetilde{\mu} = (2,1,2,3,3) \in D_5,$$
$$\widetilde{y} = \underline{14325} \in S_5[\widetilde{\lambda},\widetilde{\mu}]^\circ.$$

The equation derived from this is the same, because 2 is cancellable for $[y, w_4]$, $\widetilde{y} = y^{\hat{2}}$, and the elements of $S_5[\widetilde{\lambda},\widetilde{\mu}]^\circ$ which are $> y^{\hat{2}}$ are exactly $w_1^{\hat{2}}, w_2^{\hat{2}}, w_3^{\hat{2}}, w_4^{\hat{2}}$.

Cancellation for the affine symmetric group

Let $\widehat{S_n}$ be the group of permutations w of $\mathbb{Z}$ such that $w(i+n) = w(i) + n$. Its subgroup

$$\widetilde{S_n} = \{w \in \widehat{S_n} \mid \sum_{i=1}^{n} (w(i) - i) = 0\}$$

has Coxeter generators $s_0, s_1, \cdots, s_{n-1}$, where

$$s_i(j) = \begin{cases} j+1, & \text{if } j \equiv i \bmod n \\ j-1, & \text{if } j \equiv i+1 \bmod n \\ j, & \text{otherwise.} \end{cases}$$

$\widehat{S_n} = \langle \tau \rangle \ltimes \widetilde{S_n}$, where $\tau : i \mapsto i+1$. For $y, w \in \widetilde{S_n}$, define

$$P_{\tau^a y, \tau^b w} = \begin{cases} P_{y,w}, & \text{if } a = b \\ 0, & \text{otherwise.} \end{cases}$$

If i is cancellable for $[y, w]$, Billey-Warrington's argument still applies, so $\boxed{P_{y,w} = P_{y^{\hat{i}}, w^{\hat{i}}}.}$ (To get $w^{\hat{i}}$ from w, remove the jth string whenever $j \equiv i \bmod n$; this gives an element of $\widehat{S_{n-1}}$, defined up to left/right mult by τ.)

The geometric proof generalizes also.

Let $(\widehat{\mathcal{H}}_d)_\zeta$ be the specialization of $\widehat{\mathcal{H}}_d$ at a primitive eth root of unity ζ. For $\lambda, \mu \in \mathbb{Z}^n$ such that $\lambda_i \geq \mu_i$, $\forall i$, and $d = \sum_i \lambda_i - \mu_i$, define an $(\widehat{\mathcal{H}}_d)_\zeta$-module $M_{\lambda/\mu}$ as before. It has a quotient $L_{\lambda/\mu}$ which is either simple or zero.

Extend the dot action to $\widetilde{S_n}$ by

$$(s_0 \cdot \lambda)_i = \begin{cases} \lambda_n - n + 1 + e, & \text{if } i = 1, \\ \lambda_i, & \text{if } 2 \leq i \leq n-1, \\ \lambda_1 - 1 + n - e, & \text{if } i = n. \end{cases}$$

A fundamental domain for $\widetilde{S_n}$ on $\mathbb{Z}^n$ is

$$\widetilde{D_n} := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 - 1 \geq \lambda_2 - 2 \geq \cdots \geq \lambda_n - n \geq \lambda_1 - e - 1\}.$$

For $\lambda, \mu \in \widetilde{D_n}$, define

$$\widetilde{S_n}[\lambda, \mu] = \{w \in \widetilde{S_n} \mid \lambda_i \geq (w \cdot \mu)_i, \forall i\},$$
$$\widetilde{S_n}[\lambda, \mu]^\circ = \{w \in \widetilde{S_n}[\lambda, \mu] \mid w \text{ max length in } \widetilde{W_\lambda} w \widetilde{W_\mu}\},$$

where $\widetilde{W_\lambda}$ and $\widetilde{W_\mu}$ are the stabilizers in $\widetilde{S_n}$.

Theorem: For $y \in \widetilde{S_n}[\lambda, \mu]^\circ$,

$$[M_{\lambda/y\cdot\mu}] = \sum_{w \in \widetilde{S_n}[\lambda,\mu]^\circ} P_{y,w}(1)\,[L_{\lambda/w\cdot\mu}]$$

in the Grothendieck group of $(\widehat{\mathcal{H}}_d)_\zeta$-modules. (Some of the terms on the RHS are zero.)

This was previously known for one particular way of writing each standard module as $M_{\lambda/y\cdot\mu}$, by work of Lusztig and Ginzburg. Cancellation shows it in general.

This formulation suggests the question: is there an exact functor $F_\lambda : \mathcal{O}(U_t(\widehat{\mathfrak{gl}_n})) \to \{(\widehat{\mathcal{H}}_d)_\zeta\text{-mod}\}$, with similar properties to Suzuki's?