1. Nilpotent orbits as moduli spaces

Notation:
- $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$,
- $G$ is the corresponding simply connected group,
- $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone.

Theorem (Jacobson–Morozov, Kostant)
We have a bijection
$$ \{ G\text{-orbits in } \mathcal{N} \} \longleftrightarrow \text{Hom}(\text{SL}(2), G)/\sim $$
where the equivalence relation on the right is $G$-conjugacy. Under this bijection, $\varphi \in \text{Hom}(\text{SL}(2), G)$ corresponds to the $G$-orbit $O_\varphi \subset \mathcal{N}$ containing $d\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Hence we can write
$$ \mathcal{N} = \bigsqcup_{\varphi \in \text{Hom}(\text{SL}(2), G)/\sim} O_\varphi \quad (a \text{ disjoint union}). $$
Let \( X_G = \text{Bun}_G(\mathbb{P}^2; \ell_\infty) \) be the moduli space whose complex points are isomorphism classes of pairs \((\mathcal{F}, \Phi)\) where

- \( \mathcal{F} \) is a principal \( G \)-bundle on \( \mathbb{P}^2 \),
- \( \Phi : \mathcal{F}|_{\ell_\infty} \overset{\sim}{\rightarrow} G \times \ell_\infty \) is a trivialization of \( \mathcal{F} \) on \( \ell_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2 \).

**Facts:** (e.g. see Braverman–Finkelberg–Gaitsgory)

1. Such a pair \((\mathcal{F}, \Phi)\) has trivial automorphism group.
2. \( X_G \) has the structure of an ind-variety.
3. We have an action of \( G \times \text{GL}(2) \) on \( X_G \), where \( G \) changes the trivialization \( \Phi \) and \( \text{GL}(2) \) acts on the base \( \mathbb{P}^2 \) preserving \( \ell_\infty \).
4. If \( \Gamma \) is a subgroup of \( \text{GL}(2) \), we have a disjoint union
   \[
   X_G^\Gamma = \bigsqcup_{\tau \in \text{Hom}(\Gamma, G)/\sim} X_G^{\Gamma, \tau},
   \]
   since if \([((\mathcal{F}, \Phi))] \in X_G^\Gamma\), then \( \Gamma \) acts on the fibre \( \mathcal{F}_0 \) via some \( \tau \in \text{Hom}(\Gamma, G) \), determined up to \( G \)-conjugacy.
5. If \( \Gamma \) is reductive, this disjoint union is a disconnected union.

**Theorem (Kronheimer, reformulated)**

There is a \( G \)-equivariant bijective morphism

\[
\Omega : X_G^{\text{SL}(2)} \longrightarrow \mathcal{N}
\]

inducing isomorphisms \( X_G^{\text{SL}(2), \varphi} \overset{\sim}{\rightarrow} \mathcal{O}_\varphi \) for all \( \varphi \in \text{Hom}(\text{SL}(2), G) \).

The pair \((\mathcal{F}, \Phi)\) corresponding to \( d\varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \in \mathcal{O}_\varphi \) can be obtained by gluing trivial \( G \)-bundles on the open sets \( \{z_0 \neq 0\}, \{z_1 \neq 0\}, \{z_2 \neq 0\} \) in \( \mathbb{P}^2 \) using the following transition functions:

\[
\varphi(\begin{pmatrix} z_0z_1^{-1} & 0 \\ -z_2z_0^{-1} & z_1z_0^{-1} \end{pmatrix}), \quad \varphi(\begin{pmatrix} 0 & z_0z_2^{-1} \\ -z_2z_0^{-1} & z_1z_0^{-1} \end{pmatrix}), \quad \varphi(\begin{pmatrix} 1 & z_0^2z_1z_2^{-1} \\ 0 & 1 \end{pmatrix}).
\]

**Remark**

There is a more general moduli-space interpretation of \( S_{\varphi'} \cap \mathcal{O}_\varphi \) where \( S_{\varphi'} \) is the Slodowy slice determined by \( \varphi' \in \text{Hom}(\text{SL}(2), G) \). For this, one uses the \( \text{SL}(2) \)-action on \( X_G \) given by the embedding \((\varphi', \text{id}) : \text{SL}(2) \hookrightarrow G \times \text{GL}(2) \).
2. Pieces of the affine Grassmannian as moduli spaces

Let \( \text{Gr} = G[t, t^{-1}]/G[t] \) be the affine Grassmannian. Since \( G \) is simply connected, \( \text{Gr} \) is a connected ind-variety.

**Theorem (a case of Bruhat decomposition)**

We have a bijection

\[
\{ G[t]-orbits \text{ in } \text{Gr} \} \longleftrightarrow \text{Hom}(\mathbb{G}_m, G)/\sim
\]

under which \( \lambda \in \text{Hom}(\mathbb{G}_m, G) \) corresponds to the \( G[t] \)-orbit \( \text{Gr}^\lambda \) containing \( t^\lambda G[t]/G[t] \). Hence we can write

\[
\text{Gr} = \bigsqcup_{\lambda \in \text{Hom}(\mathbb{G}_m, G)/\sim} \text{Gr}^\lambda \quad \text{(a disjoint union)}.
\]

Choosing a maximal torus \( T \) and Borel subgroup \( B \) of \( G \), we can identify \( \text{Hom}(\mathbb{G}_m, G)/\sim \) with the set \( \Lambda^+ \) of dominant coweights.

Define

\[
\text{Gr}_0 = \ker(G[t^{-1}] \to G) \quad \text{(first congruence subgroup of } G[t^{-1}]),}
\]

which we identify with the open subset \( G[t^{-1}]G[t]/G[t] \subset \text{Gr} \).

For any \( \lambda \in \Lambda^+ \), let \( \text{Gr}^\lambda_0 = \text{Gr}_0 \cap \text{Gr}^\lambda \), an irreducible variety.

**Theorem (Braverman–Finkelberg)**

Let \( \mathbb{G}_m \) act on \( X_G \) via its identification with the diagonal subgroup of \( \text{SL}(2) \). There is a \( G \)-equivariant bijective morphism

\[
\Psi : X_G^{\mathbb{G}_m} \longrightarrow \text{Gr}_0
\]

inducing isomorphisms \( X_G^{\mathbb{G}_m, \lambda} \cong \text{Gr}^\lambda_0 \) for all \( \lambda \in \Lambda^+ \).

**Remark**

Again there is a more general moduli-space interpretation of \( \text{Gr}^\lambda_\mu = \text{Gr}_\mu \cap \text{Gr}^\lambda \) where \( \text{Gr}_\mu \) is a transverse slice to \( \text{Gr}^\mu \).
3. Nilpotent orbits and the affine Grassmannian

The two bijections $\Omega$ and $\Psi$ fit into a commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{e} & Gr_0 \\
\uparrow & & \uparrow \\
\Omega & & \Psi \\
X_{SL(2)} & \hookrightarrow & X_G^{G_m}
\end{array}
$$

where the bottom embedding is the obvious inclusion and the top is

$$e : N \hookrightarrow Gr_0 : x \mapsto \exp(xt^{-1}).$$

Hence, or by an easy $SL(2)$ computation, we have that

$$e(O_\varphi) \subseteq Gr_0^{G_m} \text{ for all } \varphi \in \Hom(SL(2), G).$$

It is well known that $\varphi$ is determined up to $G$-conjugacy by $\varphi|_{G_m}$, so different nilpotent orbits map into different $Gr^\lambda_0$'s.

Fixing a faithful representation $G \hookrightarrow GL(d)$, we can express elements of $Gr_0$ in the form

$$1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m} \text{ for } x_1, x_2, \cdots, x_m \in \text{Mat}(d).$$

We define $\pi : Gr_0 \rightarrow g$ and the involution $\iota : Gr_0 \rightarrow Gr_0$ by

$$\pi(1 + x_1 t^{-1} + \cdots + x_m t^{-m}) = x_1,$$

$$\iota(1 + x_1 t^{-1} + \cdots + x_m t^{-m}) = (1 - x_1 t^{-1} + \cdots + (-1)^m x_m t^{-m})^{-1}.$$

These are independent of the choice of $G \hookrightarrow GL(d)$. We have

- $\pi \circ e = \text{id}_N$ (that is, $e$ is a section of $\pi$ over $N$),
- $\pi \circ \iota = \pi$ (that is, $\iota$ preserves each fibre of $\pi$),
- $\iota \circ e = e$ (that is, the image of $e$ belongs to $(Gr_0)^\iota$).

It is also easy to see that $\iota(Gr^\lambda_0) = Gr_0^{-w_0^\lambda}$, where $w_0$ is the longest element of the Weyl group.
Example ($G = SL(2)$)

If $G = SL(2)$, then $Gr_0 = \{1\} \sqcup Gr_0^{\alpha^\vee} \sqcup Gr_0^{2\alpha^\vee} \sqcup Gr_0^{3\alpha^\vee} \sqcup \cdots$, where
\[
Gr_0^{m\alpha^\vee} = \{1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m} \mid x_i \in \text{Mat}(2), x_m \neq 0, \det(1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m}) = 1\}.
\]
In this case, $e(N) = \{1\} \sqcup Gr_0^{\alpha^\vee}$. The involution $\iota$ acts nontrivially on $Gr_0^{m\alpha^\vee}$ for all $m \geq 2$. Exercise: $(Gr_0^{m\alpha^\vee})^\iota$ is empty if $m$ is even.

Theorem (Achar–H.)

For $\lambda \in \Lambda^+$, the following are equivalent:
1. $\pi(Gr_0^\lambda) \subset N$,
2. $G$ acts with finitely many orbits on $Gr_0^\lambda$,
3. $\lambda$ is small, i.e. $\lambda \not\geq \tilde{\alpha}^\vee$ where $\tilde{\alpha}$ is the highest root.
If these hold, then $\pi|_{Gr_0^\lambda \cup Gr_0^{-w_0}^\lambda}$ is a quotient map for $\langle \iota \rangle$.
Moreover, if $\phi|_{G_m}$ is small, then $e(O_{\phi}) = (Gr_0^{\phi|_{G_m}})^\iota$.

4. Moduli-space interpretation of the involution $\iota$

Let $N = N_{SL(2)}(G_m) = \langle G_m, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$, and let $\Xi = \text{Hom}(N, G) / \sim$.

Theorem (H.)

Under the Braverman–Finkelberg bijection $\Psi : X_G^{G_m} \rightarrow Gr_0$, the involution $\iota$ of $Gr_0$ corresponds to the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $X_G^{G_m}$.
Hence we have a commutative diagram

\[
\begin{array}{ccc}
N & \overset{e}{\longrightarrow} & (Gr_0)^\iota \\
\downarrow{\Omega} & & \downarrow{\psi} \\
X_G^{SL(2)} & \overset{\Psi|_{X_G^{G_m}}}{\longrightarrow} & X_G^{G_m}
\end{array}
\]

Corresponding to the disconnected union $X_G^N = \bigsqcup_{\xi \in \Xi} X_G^{N,\xi}$, we have a disjoint union $(Gr_0)^\iota = \bigsqcup_{\xi \in \Xi} (Gr_0)^\iota,\xi$ satisfying
\[
(Gr_0)^\iota,\xi \subseteq Gr_0^{\xi|G_m} \text{ and } e(O_{\varphi}) \subseteq (Gr_0)^\iota,\varphi|_{N}.
\]
Thus, for any $\xi \in \Xi$, we have a locally closed subvariety $(\text{Gr}_0)^{\xi}$ of $\text{Gr}$ which is naturally isomorphic to the moduli space $X_G^{\mathbb{N}, \xi}$. There is an “elementary” definition of $(\text{Gr}_0)^{\xi}$:

**Proposition (H.)**

Let $\lambda \in \Lambda_+^+$ be such that $-w_0 \lambda = \lambda$, and let $\gamma \in (\text{Gr}_0^{\lambda})^\nu$. If we write $\gamma = q_1 t^\lambda q_2$ for $q_1, q_2 \in G[t]$, then

$$\sigma := (t^\lambda(q_2|_{t \to 0})(q_1|_{t \to 0})t^\lambda)|_{t \to 0}$$

is a well-defined element of $G$ satisfying

$$\sigma^2 = \lambda(-1) \quad \text{and} \quad \sigma \lambda(z)\sigma^{-1} = \lambda(z)^{-1} \quad \text{for all} \quad z \in \mathbb{G}_m.$$

We have $\gamma \in (\text{Gr}_0)^{\xi}$ where $\xi : \mathbb{N} \to G$ coincides with $\lambda$ on $\mathbb{G}_m$ and sends $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $\sigma$.

**Problems**

For which $\xi \in \Xi$ is $(\text{Gr}_0)^{\xi}$ nonempty? Is it ever disconnected?

5. **The case $G = \text{SL}(r)$**

In the definition of the moduli space $X_{\text{SL}(r)}$, we can replace the principal $\text{SL}(r)$-bundle $\mathcal{F}$ by a rank-$r$ vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ (the trivialization on $\mathcal{L}_\infty$ forces it to have trivial determinant bundle). We have a disconnected union $X_{\text{SL}(r)} = \bigsqcup_{n \geq 0} X^n_{\text{SL}(r)}$ where the invariant $n$ is $\dim H^1(\mathbb{P}^2, \mathcal{E}(-1))$, i.e. the second Chern class of $\mathcal{E}$.

**Theorem (Barth, Atiyah–Hitchin, Drinfeld–Manin, Donaldson)**

The moduli space $X^n_{\text{SL}(r)}$ can be identified with $\Lambda(n, r)^{sc}/\text{GL}(n)$, where $\Lambda(n, r)^{sc}$ is the space of quadruples $(B_1, B_2, i, j)$ of linear maps $B_1, B_2 : \mathbb{C}^n \to \mathbb{C}^n$, $i : \mathbb{C}^r \to \mathbb{C}^n$, $j : \mathbb{C}^n \to \mathbb{C}^r$ satisfying

1. (ADHM equation) $[B_1, B_2] + ij = 0$;
2. (stability) there is no nonzero $B_i$-stable subspace of $\ker j$;
3. (costability) no proper $B_i$-stable subspace of $\mathbb{C}^n$ contains $\text{im} i$.

This is the starting point for Nakajima’s theory of quiver varieties.
In the identification $X_{SL(r)}^n \xrightarrow{\sim} \Lambda(n, r)^{sc}/GL(n)$:

- $[(B_1, B_2, i, j)]$ corresponds to $\mathcal{E} = \ker b/\operatorname{im} a$ where
  
  $$a = \begin{pmatrix} z_0 B_1 - z_1 \text{id}_V \\ z_0 B_2 - z_2 \text{id}_V \\ z_0 j \end{pmatrix} \in \operatorname{Mat}(2n + r, n),$$

  $$b = \begin{pmatrix} -(z_0 B_2 - z_2 \text{id}_V) \\ z_0 B_1 - z_1 \text{id}_V \\ z_0 i \end{pmatrix} \in \operatorname{Mat}(n, 2n + r).$$

- The $GL(2)$-action on $X_{SL(r)}^n$ corresponds to the action
  
  $$(\alpha \beta \gamma \delta) \cdot (B_1, B_2, i, j) = (\alpha B_1 + \gamma B_2, \beta B_1 + \delta B_2, (\alpha \delta - \beta \gamma) i, j).$$

If $\Gamma$ is a subgroup of $GL(2)$, we have a disjoint union

$$X_{SL(r)}^{n, \Gamma} = \bigsqcup_{\rho \in \operatorname{Hom}(\Gamma, GL(n))/\sim} X_{SL(r)}^{n, \Gamma, \rho},$$

where the invariant $\rho$ is the representation of $\Gamma$ on $H^1(\mathbb{P}^2, \mathcal{E}(-1))$.

If $\Gamma$ is reductive, this disjoint union is disconnected.

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**Theorem (Nakajima, and Crawley–Boevey for connectedness)**

Let $\Gamma$ be a reductive subgroup of $SL(2)$, and $\rho \in \operatorname{Hom}(\Gamma, GL(n))$.

1. Under the identification $X_{SL(r)}^n \xrightarrow{\sim} \Lambda(n, r)^{sc}/GL(n)$, the subvariety $X_{SL(r)}^{n, \Gamma, \rho}$ corresponds to $\Lambda(n, r)^{sc, \Gamma, \rho}/Z_{GL(n)}(\rho)$, where $\Lambda(n, r)^{sc, \Gamma, \rho}$ is the $\Gamma$-fixed subvariety of $\Lambda(n, r)^{sc}$ for the $\Gamma$-action defined via $(\rho, \text{id}) : \Gamma \hookrightarrow GL(n) \times GL(2)$.

2. $X_{SL(r)}^{n, \Gamma, \rho} \neq \emptyset$ if and only if there exists $\tau \in \operatorname{Hom}(\Gamma, G)$ such that

   $$(\mathbb{C}^n, \rho) \oplus (\mathbb{C}^n, \rho) \oplus (\mathbb{C}^r, \tau) \cong \left( (\mathbb{C}^n, \rho) \otimes \mathbb{C}^2 \right) \oplus (\mathbb{C}^r, \text{triv})$$

   as representations of $\Gamma$. If so, then $X_{SL(r)}^{n, \Gamma, \rho} \subseteq X_{SL(r)}^{\Gamma, \tau}$.

3. When $X_{SL(r)}^{n, \Gamma, \rho}$ is nonempty, it is connected of dimension

   $$\dim \operatorname{Hom}_\Gamma \left( ((\mathbb{C}^n, \rho), (\mathbb{C}^r, \tau)) \right) + \dim \operatorname{Hom}_\Gamma \left( ((\mathbb{C}^n, \rho), (\mathbb{C}^r, \text{triv})) \right).$$
Corollary (H.)

Let \( G = \text{SL}(r) \), and let \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \Lambda^+ \) be such that

\[-w_0 \lambda = \lambda, \text{ i.e. } \lambda_{r+1-i} = -\lambda_i.\]

Let \( m_0 \) be the number of zero entries in \( \lambda \), i.e. \( \dim(\mathbb{C}^r)^{\lambda(\mathbb{G}_m)} \).

Let \( v_0 \) be the sum of the positive entries in \( \lambda \), and define

\[ v_1 = v_0 - \frac{1}{2}(r - m_0). \]

Suppose that \( \xi \in \text{Hom}(N, \text{SL}(r)) \) restricts to \( \lambda \in \Lambda^+ \), and let \( (m_{0,+}, m_{0,-}) \) be the signature of \( \xi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \) on \( (\mathbb{C}^r)^{\lambda(\mathbb{G}_m)} \).

1. \( (\text{Gr}_0)^{\iota,\xi} \neq \emptyset \) if and only if \( v_1 - m_{0,-} \in 2\mathbb{N} \).

2. When \( (\text{Gr}_0)^{\iota,\xi} \) is nonempty, it is connected of dimension

\[ \langle \lambda, \rho \rangle + \frac{1}{4}(r^2 - (m_{0,+} - m_{0,-})^2). \]

6. Further questions

- Is the closure of \( (\text{Gr}_0)^{\iota,\xi} \) in \( \text{Gr}_0 \) a union of pieces \( (\text{Gr}_0)^{\iota,\xi'} \)?
  (This is known when \( G = \text{SL}(r) \) by Nakajima’s results.)

- If so, what partial order on \( \Xi \) describes the closure ordering?

- What is the intersection cohomology of the closure of \( (\text{Gr}_0)^{\iota,\xi} \)? Does it have a representation-theoretic meaning?

- Braverman–Finkelberg have studied a “double affine” version of \( \text{Gr}_\mu^\lambda \cong X_{\mathbb{G}_m,\lambda} \) defined by replacing \( \mathbb{G}_m \) with a finite cyclic subgroup of \( \text{SL}(2) \) (i.e. type \( A_k \) instead of type \( A_\infty \)).
  What can one say about the analogous “double affine” version of \( (\text{Gr}_0)^{\iota,\xi} \), obtained by replacing \( N \) with a binary dihedral subgroup of \( \text{SL}(2) \) (i.e. type \( D_k \) instead of type \( D_\infty \))?