Involutions on the affine Grassmannian and moduli spaces of principal bundles

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1. Nilpotent orbits as moduli spaces

Notation:

- $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$,
- G is the corresponding simply connected group,
- $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone.

Theorem (Jacobson–Morozov, Kostant)

We have a bijection

$$\{G\text{-orbits in } \mathcal{N}\} \longleftrightarrow \mathrm{Hom}(\mathrm{SL}(2), G)/\sim$$

where the equivalence relation on the right is G-conjugacy. Under this bijection, $\varphi \in \mathrm{Hom}(\mathrm{SL}(2), G)$ corresponds to the G-orbit $\mathcal{O}_\varphi \subset \mathcal{N}$ containing $d\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Hence we can write

$$\mathcal{N} = \bigsqcup_{\varphi \in \mathrm{Hom}(\mathrm{SL}(2), G)/\sim} \mathcal{O}_\varphi \qquad \text{(a disjoint union).}$$

Let $X_G = \mathrm{Bun}_G(\mathbb{P}^2; \ell_\infty)$ be the moduli space whose complex points are isomorphism classes of pairs $(\mathcal{F}, \Phi)$ where

- $\mathcal{F}$ is a principal G-bundle on $\mathbb{P}^2$,
- $\Phi : \mathcal{F}|_{\ell_\infty} \xrightarrow{\sim} G \times \ell_\infty$ is a trivialization of $\mathcal{F}$ on $\ell_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$.

Facts: (e.g. see Braverman–Finkelberg–Gaitsgory)

1. Such a pair $(\mathcal{F}, \Phi)$ has trivial automorphism group.
2. X_G has the structure of an ind-variety.
3. We have an action of $G \times \mathrm{GL}(2)$ on X_G, where G changes the trivialization Φ and $\mathrm{GL}(2)$ acts on the base $\mathbb{P}^2$ preserving ℓ_∞.
4. If Γ is a subgroup of $\mathrm{GL}(2)$, we have a disjoint union

$$X_G^\Gamma = \bigsqcup_{\tau \in \mathrm{Hom}(\Gamma, G)/\sim} X_G^{\Gamma,\tau},$$

since if $[(\mathcal{F}, \Phi)] \in X_G^\Gamma$, then Γ acts on the fibre $\mathcal{F}_0$ via some $\tau \in \mathrm{Hom}(\Gamma, G)$, determined up to G-conjugacy.

5. If Γ is reductive, this disjoint union is a disconnected union.

Theorem (Kronheimer, reformulated)

There is a G-equivariant bijective morphism

$$\Omega : X_G^{\mathrm{SL}(2)} \longrightarrow \mathcal{N}$$

inducing isomorphisms $X_G^{\mathrm{SL}(2),\varphi} \xrightarrow{\sim} \mathcal{O}_\varphi$ for all $\varphi \in \mathrm{Hom}(\mathrm{SL}(2), G)$. The pair $(\mathcal{F}, \Phi)$ corresponding to $d\varphi(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}) \in \mathcal{O}_\varphi$ can be obtained by gluing trivial G-bundles on the open sets $\{z_0 \neq 0\}$, $\{z_1 \neq 0\}$, $\{z_2 \neq 0\}$ in $\mathbb{P}^2$ using the following transition functions:

$$\varphi\begin{pmatrix} z_0 z_1^{-1} & 0 \\ -z_2 z_0^{-1} & z_1 z_0^{-1} \end{pmatrix}, \ \varphi\begin{pmatrix} 0 & z_0 z_2^{-1} \\ -z_2 z_0^{-1} & z_1 z_0^{-1} \end{pmatrix}, \ \varphi\begin{pmatrix} 1 & z_0^2 z_1^{-1} z_2^{-1} \\ 0 & 1 \end{pmatrix}.$$

Remark

There is a more general moduli-space interpretation of $\mathcal{S}_{\varphi'} \cap \mathcal{O}_\varphi$ where $\mathcal{S}_{\varphi'}$ is the Slodowy slice determined by $\varphi' \in \mathrm{Hom}(\mathrm{SL}(2), G)$. For this, one uses the $\mathrm{SL}(2)$-action on X_G given by the embedding $(\varphi', \mathrm{id}) : \mathrm{SL}(2) \hookrightarrow G \times \mathrm{GL}(2)$.

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2. Pieces of the affine Grassmannian as moduli spaces

Let $\mathrm{Gr} = G[t, t^{-1}]/G[t]$ be the affine Grassmannian. Since G is simply connected, Gr is a connected ind-variety.

Theorem (a case of Bruhat decomposition)

We have a bijection

$$\{G[t]\text{-orbits in } \mathrm{Gr}\} \longleftrightarrow \mathrm{Hom}(\mathbb{G}_m, G)/\sim$$

under which $\lambda \in \mathrm{Hom}(\mathbb{G}_m, G)$ *corresponds to the* $G[t]$*-orbit* Gr^λ *containing* $t^\lambda G[t]/G[t]$*. Hence we can write*

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \mathrm{Hom}(\mathbb{G}_m, G)/\sim} \mathrm{Gr}^\lambda \qquad (\textit{a disjoint union}).$$

Choosing a maximal torus T and Borel subgroup B of G, we can identify $\mathrm{Hom}(\mathbb{G}_m, G)/\sim$ with the set Λ^+ of dominant coweights.

Define

$$\mathrm{Gr}_0 = \ker(G[t^{-1}] \to G) \quad (\text{first congruence subgroup of } G[t^{-1}]),$$

which we identify with the open subset $G[t^{-1}]G[t]/G[t] \subset \mathrm{Gr}$. For any $\lambda \in \Lambda^+$, let $\mathrm{Gr}_0^\lambda = \mathrm{Gr}_0 \cap \mathrm{Gr}^\lambda$, an irreducible variety.

Theorem (Braverman–Finkelberg)

Let $\mathbb{G}_m$ *act on* X_G *via its identification with the diagonal subgroup of* $\mathrm{SL}(2)$*. There is a* G*-equivariant bijective morphism*

$$\Psi : X_G^{\mathbb{G}_m} \longrightarrow \mathrm{Gr}_0$$

inducing isomorphisms $X_G^{\mathbb{G}_m,\lambda} \xrightarrow{\sim} \mathrm{Gr}_0^\lambda$ *for all* $\lambda \in \Lambda^+$*.*

Remark

Again there is a more general moduli-space interpretation of $\mathrm{Gr}_\mu^\lambda = \mathrm{Gr}_\mu \cap \mathrm{Gr}^\lambda$ where Gr_μ is a transverse slice to Gr^μ.

3. Nilpotent orbits and the affine Grassmannian

The two bijections Ω and Ψ fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{N} & \stackrel{e}{\hookrightarrow} & \mathsf{Gr}_0 \\ \Omega \uparrow & & \uparrow \Psi \\ X_G^{\mathrm{SL}(2)} & \hookrightarrow & X_G^{\mathbb{G}_m} \end{array}$$

where the bottom embedding is the obvious inclusion and the top is

$$e : \mathcal{N} \hookrightarrow \mathsf{Gr}_0 : x \mapsto \exp(xt^{-1}).$$

Hence, or by an easy $\mathrm{SL}(2)$ computation, we have that

$$e(\mathcal{O}_\varphi) \subseteq \mathsf{Gr}_0^{\varphi|_{\mathbb{G}_m}} \text{ for all } \varphi \in \mathrm{Hom}(\mathrm{SL}(2), G).$$

It is well known that φ is determined up to G-conjugacy by $\varphi|_{\mathbb{G}_m}$, so different nilpotent orbits map into different Gr_0^λ's.

Fixing a faithful representation $G \hookrightarrow \mathrm{GL}(d)$, we can express elements of Gr_0 in the form

$$1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m} \text{ for } x_1, x_2, \cdots, x_m \in \mathrm{Mat}(d).$$

We define $\pi : \mathsf{Gr}_0 \to \mathfrak{g}$ and the involution $\iota : \mathsf{Gr}_0 \to \mathsf{Gr}_0$ by

$$\pi(1 + x_1 t^{-1} + \cdots + x_m t^{-m}) = x_1,$$
$$\iota(1 + x_1 t^{-1} + \cdots + x_m t^{-m}) = (1 - x_1 t^{-1} + \cdots + (-1)^m x_m t^{-m})^{-1}.$$

These are independent of the choice of $G \hookrightarrow \mathrm{GL}(d)$. We have

- $\pi \circ e = \mathrm{id}_{\mathcal{N}}$ (that is, e is a section of π over $\mathcal{N}$),
- $\pi \circ \iota = \pi$ (that is, ι preserves each fibre of π),
- $\iota \circ e = e$ (that is, the image of e belongs to $(\mathsf{Gr}_0)^\iota$).

It is also easy to see that $\iota(\mathsf{Gr}_0^\lambda) = \mathsf{Gr}_0^{-w_0\lambda}$, where w_0 is the longest element of the Weyl group.

Example ($G = \mathrm{SL}(2)$)

If $G = \mathrm{SL}(2)$, then $\mathrm{Gr}_0 = \{1\} \sqcup \mathrm{Gr}_0^{\alpha^\vee} \sqcup \mathrm{Gr}_0^{2\alpha^\vee} \sqcup \mathrm{Gr}_0^{3\alpha^\vee} \sqcup \cdots$, where

$$\mathrm{Gr}_0^{m\alpha^\vee} = \{1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m} \mid x_i \in \mathrm{Mat}(2),\ x_m \neq 0,\ \det(1 + x_1 t^{-1} + x_2 t^{-2} + \cdots + x_m t^{-m}) = 1\}.$$

In this case, $e(\mathcal{N}) = \{1\} \sqcup \mathrm{Gr}_0^{\alpha^\vee}$. The involution ι acts nontrivially on $\mathrm{Gr}_0^{m\alpha^\vee}$ for all $m \geq 2$. Exercise: $(\mathrm{Gr}_0^{m\alpha^\vee})^\iota$ is empty if m is even.

Theorem (Achar–H.)

For $\lambda \in \Lambda^+$, the following are equivalent:

1. $\pi(\mathrm{Gr}_0^\lambda) \subset \mathcal{N}$,
2. *G acts with finitely many orbits on Gr_0^λ,*
3. *λ is small, i.e. $\lambda \not\geq 2\widetilde{\alpha}^\vee$ where $\widetilde{\alpha}$ is the highest root.*

If these hold, then $\pi|_{\mathrm{Gr}_0^\lambda \cup \mathrm{Gr}_0^{-w_0\lambda}}$ is a quotient map for $\langle \iota \rangle$.

Moreover, if $\varphi|_{\mathbb{G}_m}$ is small, then $e(\mathcal{O}_\varphi) = (\mathrm{Gr}_0^{\varphi|_{\mathbb{G}_m}})^\iota$.

4. Moduli-space interpretation of the involution ι

Let $N = N_{\mathrm{SL}(2)}(\mathbb{G}_m) = \langle \mathbb{G}_m, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$, and let $\Xi = \mathrm{Hom}(N, G)/\sim$.

Theorem (H.)

Under the Braverman–Finkelberg bijection $\Psi : X_G^{\mathbb{G}_m} \to \mathrm{Gr}_0$, the involution ι of Gr_0 corresponds to the action of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $X_G^{\mathbb{G}_m}$. Hence we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{N} & \overset{e}{\hookrightarrow} & (\mathrm{Gr}_0)^\iota & \hookrightarrow & \mathrm{Gr}_0 \\
\uparrow \Omega & & \uparrow \Psi|_{X_G^N} & & \uparrow \Psi \\
X_G^{\mathrm{SL}(2)} & \hookrightarrow & X_G^N & \hookrightarrow & X_G^{\mathbb{G}_m}
\end{array}$$

Corresponding to the disconnected union $X_G^N = \bigsqcup_{\xi \in \Xi} X_G^{N,\xi}$, we have a disjoint union $(\mathrm{Gr}_0)^\iota = \bigsqcup_{\xi \in \Xi} (\mathrm{Gr}_0)^{\iota,\xi}$ satisfying

$$(\mathrm{Gr}_0)^{\iota,\xi} \subseteq \mathrm{Gr}_0^{\xi|_{\mathbb{G}_m}} \text{ and } e(\mathcal{O}_\varphi) \subseteq (\mathrm{Gr}_0)^{\iota,\varphi|_N}.$$

Thus, for any $\xi \in \Xi$, we have a locally closed subvariety $(\mathrm{Gr}_0)^{\iota,\xi}$ of Gr which is naturally isomorphic to the moduli space $X_G^{N,\xi}$.
There is an "elementary" definition of $(\mathrm{Gr}_0)^{\iota,\xi}$:

Proposition (H.)

Let $\lambda \in \Lambda^+$ be such that $-w_0\lambda = \lambda$, and let $\gamma \in (\mathrm{Gr}_0^\lambda)^\iota$. If we write $\gamma = q_1 t^\lambda q_2$ for $q_1, q_2 \in G[t]$, then

$$\sigma := \left(t^\lambda (q_2|_{t\mapsto 0})(q_1|_{t\mapsto 0}) t^\lambda\right)|_{t\mapsto 0}$$

is a well-defined element of G satisfying

$$\sigma^2 = \lambda(-1) \ \text{ and } \ \sigma\lambda(z)\sigma^{-1} = \lambda(z)^{-1} \ \text{ for all } \ z \in \mathbb{G}_m.$$

We have $\gamma \in (\mathrm{Gr}_0)^{\iota,\xi}$ where $\xi : N \to G$ coincides with λ on $\mathbb{G}_m$ and sends $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ to σ.

Problems

For which $\xi \in \Xi$ is $(\mathrm{Gr}_0)^{\iota,\xi}$ nonempty? Is it ever disconnected?

5. The case $G = \mathrm{SL}(r)$

In the definition of the moduli space $X_{\mathrm{SL}(r)}$, we can replace the principal $\mathrm{SL}(r)$-bundle $\mathcal{F}$ by a rank-r vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ (the trivialization on ℓ_∞ forces it to have trivial determinant bundle).
We have a disconnected union $X_{\mathrm{SL}(r)} = \bigsqcup_{n\geq 0} X_{\mathrm{SL}(r)}^n$ where the invariant n is $\dim H^1(\mathbb{P}^2, \mathcal{E}(-1))$, i.e. the second Chern class of $\mathcal{E}$.

Theorem (Barth, Atiyah–Hitchin, Drinfeld–Manin, Donaldson)

The moduli space $X_{\mathrm{SL}(r)}^n$ can be identified with $\Lambda(n,r)^{sc}/\mathrm{GL}(n)$, where $\Lambda(n,r)^{sc}$ is the space of quadruples $(B_1, B_2, \mathbf{i}, \mathbf{j})$ of linear maps $B_1, B_2 : \mathbb{C}^n \to \mathbb{C}^n$, $\mathbf{i} : \mathbb{C}^r \to \mathbb{C}^n$, $\mathbf{j} : \mathbb{C}^n \to \mathbb{C}^r$ satisfying

1. *(ADHM equation)* $[B_1, B_2] + \mathbf{ij} = 0$;
2. *(stability) there is no nonzero B_i-stable subspace of* $\ker \mathbf{j}$;
3. *(costability) no proper B_i-stable subspace of $\mathbb{C}^n$ contains* $\operatorname{im} \mathbf{i}$.

This is the starting point for Nakajima's theory of quiver varieties.

In the identification $X^n_{\mathrm{SL}(r)} \xrightarrow{\sim} \Lambda(n,r)^{sc}/\mathrm{GL}(n)$:

- $[(B_1, B_2, \mathbf{i}, \mathbf{j})]$ corresponds to $\mathcal{E} = \ker b/\operatorname{im} a$ where

$$a = \begin{pmatrix} z_0 B_1 - z_1 \mathrm{id}_V \\ z_0 B_2 - z_2 \mathrm{id}_V \\ z_0 \mathbf{j} \end{pmatrix} \in \mathrm{Mat}(2n+r, n),$$

$$b = \begin{pmatrix} -(z_0 B_2 - z_2 \mathrm{id}_V) & z_0 B_1 - z_1 \mathrm{id}_V & z_0 \mathbf{i} \end{pmatrix} \in \mathrm{Mat}(n, 2n+r).$$

- The $\mathrm{GL}(2)$-action on $X^n_{\mathrm{SL}(r)}$ corresponds to the action

$$\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \cdot (B_1, B_2, \mathbf{i}, \mathbf{j}) = (\alpha B_1 + \gamma B_2, \beta B_1 + \delta B_2, (\alpha\delta - \beta\gamma)\mathbf{i}, \mathbf{j}).$$

If Γ is a subgroup of $\mathrm{GL}(2)$, we have a disjoint union

$$X^{n,\Gamma}_{\mathrm{SL}(r)} = \bigsqcup_{\rho \in \mathsf{Hom}(\Gamma, \mathrm{GL}(n))/\sim} X^{n,\Gamma,\rho}_{\mathrm{SL}(r)}$$

where the invariant ρ is the representation of Γ on $H^1(\mathbb{P}^2, \mathcal{E}(-1))$. If Γ is reductive, this disjoint union is disconnected.

Theorem (Nakajima, and Crawley–Boevey for connectedness)

Let Γ be a reductive subgroup of $\mathrm{SL}(2)$, and $\rho \in \mathsf{Hom}(\Gamma, \mathrm{GL}(n))$.

1. *Under the identification $X^n_{\mathrm{SL}(r)} \xrightarrow{\sim} \Lambda(n,r)^{sc}/\mathrm{GL}(n)$, the subvariety $X^{n,\Gamma,\rho}_{\mathrm{SL}(r)}$ corresponds to $\Lambda(n,r)^{sc,\Gamma,\rho}/Z_{\mathrm{GL}(n)}(\rho)$, where $\Lambda(n,r)^{sc,\Gamma,\rho}$ is the Γ-fixed subvariety of $\Lambda(n,r)^{sc}$ for the Γ-action defined via $(\rho, \mathrm{id}) : \Gamma \hookrightarrow \mathrm{GL}(n) \times \mathrm{GL}(2)$.*
2. *$X^{n,\Gamma,\rho}_{\mathrm{SL}(r)} \neq \emptyset$ if and only if there exists $\tau \in \mathsf{Hom}(\Gamma, G)$ such that*

$$(\mathbb{C}^n, \rho) \oplus (\mathbb{C}^n, \rho) \oplus (\mathbb{C}^r, \tau) \cong \left((\mathbb{C}^n, \rho) \otimes \mathbb{C}^2\right) \oplus (\mathbb{C}^r, \mathrm{triv})$$

   as representations of Γ. If so, then $X^{n,\Gamma,\rho}_{\mathrm{SL}(r)} \subseteq X^{\Gamma,\tau}_{\mathrm{SL}(r)}$.
3. *When $X^{n,\Gamma,\rho}_{\mathrm{SL}(r)}$ is nonempty, it is connected of dimension*

$$\dim \mathsf{Hom}_\Gamma\left((\mathbb{C}^n, \rho), (\mathbb{C}^r, \tau)\right) + \dim \mathsf{Hom}_\Gamma\left((\mathbb{C}^n, \rho), (\mathbb{C}^r, \mathrm{triv})\right).$$

Corollary (H.)

Let $G = \mathrm{SL}(r)$, and let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \Lambda^+$ be such that

$$-w_0\lambda = \lambda, \text{ i.e. } \lambda_{r+1-i} = -\lambda_i.$$

Let m_0 be the number of zero entries in λ, i.e. $\dim(\mathbb{C}^r)^{\lambda(\mathbb{G}_m)}$. Let v_0 be the sum of the positive entries in λ, and define

$$v_1 = v_0 - \frac{1}{2}(r - m_0).$$

Suppose that $\xi \in \mathrm{Hom}(N, \mathrm{SL}(r))$ restricts to $\lambda \in \Lambda^+$, and let $(m_{0,+}, m_{0,-})$ be the signature of $\xi(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ on $(\mathbb{C}^r)^{\lambda(\mathbb{G}_m)}$.

1. $(\mathrm{Gr}_0)^{\iota,\xi} \neq \emptyset$ *if and only if* $v_1 - m_{0,-} \in 2\mathbb{N}$.
2. *When $(\mathrm{Gr}_0)^{\iota,\xi}$ is nonempty, it is connected of dimension*

$$\langle \lambda, \rho \rangle + \frac{1}{4}(r^2 - (m_{0,+} - m_{0,-})^2).$$

6. Further questions

- Is the closure of $(\mathrm{Gr}_0)^{\iota,\xi}$ in Gr_0 a union of pieces $(\mathrm{Gr}_0)^{\iota,\xi'}$? (This is known when $G = \mathrm{SL}(r)$ by Nakajima's results.)
- If so, what partial order on Ξ describes the closure ordering?
- What is the intersection cohomology of the closure of $(\mathrm{Gr}_0)^{\iota,\xi}$? Does it have a representation-theoretic meaning?
- Braverman–Finkelberg have studied a "double affine" version of $\mathrm{Gr}^\lambda_\mu \cong X_G^{\mathbb{G}_m,\lambda}$ defined by replacing $\mathbb{G}_m$ with a finite cyclic subgroup of $\mathrm{SL}(2)$ (i.e. type A_k instead of type A_∞). What can one say about the analogous "double affine" version of $(\mathrm{Gr}_0)^{\iota,\xi}$, obtained by replacing N with a binary dihedral subgroup of $\mathrm{SL}(2)$ (i.e. type D_k instead of type D_∞)?