Fourier transform and Kloosterman sums

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§1. Fourier transform over $\mathbb{F}_q$

Let $\mathbb{F}_q$ be the finite field with q elements. Fix a nontrivial character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}^\times$. We have

$$\sum_{x \in \mathbb{F}_q} \psi(x) = 0.$$

E.g. If $\mathbb{F}_q = \mathbb{Z}/p\mathbb{Z}$, we can take $\psi(a) = \exp(\frac{2\pi i a}{p})$.

Let V be a finite-dimensional vector space over $\mathbb{F}_q$,
V^* its dual, $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{F}_q$ the pairing.

For $f : V \to \overline{\mathbb{Q}}$, its **Fourier transform** $\widehat{f} : V^* \to \overline{\mathbb{Q}}$ is given by

$$\widehat{f}(\alpha) := \sum_{v \in V} \psi(\langle v, \alpha \rangle) f(v).$$

E.g. **(1)** If $f = \delta_{v,0}$, $\widehat{f}$ is constant $= 1$.

(2) If f is constant $= 1$,

$$\widehat{f}(\alpha) = \sum_{v \in V} \psi(\langle v, \alpha \rangle)$$
$$= \begin{cases} 0, & \text{if } \alpha \neq 0, \\ |V| = q^{\dim V}, & \text{if } \alpha = 0. \end{cases}$$

Lemma 1 $\widehat{\widehat{f}}(v) = q^{\dim V} f(-v)$.

Proof

$$\text{LHS} = \sum_{\alpha \in V^*} \psi(\langle v, \alpha \rangle) \sum_{v' \in V} \psi(\langle v', \alpha \rangle) f(v')$$
$$= \sum_{v' \in V} f(v') \sum_{\alpha \in V^*} \psi(\langle v + v', \alpha \rangle)$$
$$= \text{RHS},$$

since only the $v' = -v$ term survives.

§2. Kloosterman sums

Take $V = V^* = \mathbb{F}_q^2$. For $z \in \mathbb{F}_q^\times$, define

$$f_z : \mathbb{F}_q^2 \to \overline{\mathbb{Q}} : (x, y) \mapsto \begin{cases} 1, & \text{if } xy = z, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \widehat{f_z}(x, y) &= \sum_{\substack{x', y' \in \mathbb{F}_q^\times \\ x'y' = z}} \psi(xx' + yy') \\ &= \begin{cases} q - 1, & \text{if } x = y = 0, \\ -1, & \text{if one of } x, y = 0, \\ \mathrm{Kl}_\psi(xyz), & \text{if } x, y \in \mathbb{F}_q^\times. \end{cases} \end{aligned}$$

Here for $\gamma \in \mathbb{F}_q^\times$, we define the **Kloosterman sum**

$$\mathrm{Kl}_\psi(\gamma) := \sum_{\substack{\alpha, \beta \in \mathbb{F}_q^\times \\ \alpha\beta = \gamma}} \psi(\alpha + \beta).$$

Since

$$\overline{\mathsf{Kl}_\psi(\gamma)} = \sum_{\substack{\alpha,\beta\in\mathbb{F}_q^\times \\ \alpha\beta=\gamma}} \psi(-\alpha-\beta) = \mathsf{Kl}_\psi(\gamma),$$

$\mathsf{Kl}_\psi(\gamma)$ is a real algebraic integer.

E.g. If $q = 5$,

$$\begin{aligned}\mathsf{Kl}_\psi(2) &= 2\psi(1+2) + 2\psi(3+4) \\ &= 4\cos(\frac{4\pi}{5}) = -\sqrt{5} - 1, \\ \mathsf{Kl}_\psi(4) &= 2\psi(1+4) + \psi(2+2) + \psi(3+3) \\ &= 2 + 2\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}+3}{2}.\end{aligned}$$

Thm 1 (Weil) *For all* $\gamma \in \mathbb{F}_q^\times$, $|\mathsf{Kl}_\psi(\gamma)| \leqslant 2\sqrt{q}$.

(Follows from the Riemann Hypothesis for curves.)

§3. Springer's results

Let G be a connected reductive algebraic group over $\mathbb{F}_q$, and let $\mathfrak{g}$ be its Lie algebra.

G acts on $\mathfrak{g}$ by the **adjoint** action, and on the dual $\mathfrak{g}^*$ by the **coadjoint** action.

Def Say that $\lambda \in \mathfrak{g}^*$ is **strongly regular** if the stabilizer G_λ is a maximal torus (absolutely, i.e. over $\overline{\mathbb{F}_q}$).

E.g. If $G = GL_n(\mathbb{F}_q)$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F}_q)$, the adjoint action is $g.X = gXg^{-1}$. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via $\langle X, Y \rangle = \mathrm{tr}(XY)$. Then $X \in \mathfrak{g}$ is strongly regular iff it is regular semisimple, i.e. has distinct eigenvalues.

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a strongly regular coadjoint orbit. Let $f_\mathcal{O} : \mathfrak{g}^* \to \overline{\mathbb{Q}}$ be the indicator function of $\mathcal{O}$.

Springer [Inventiones 36 (1976)] considered the Fourier transform $\widehat{f_\mathcal{O}} : \mathfrak{g} \to \overline{\mathbb{Q}}$. By definition,

$$\widehat{f_\mathcal{O}}(X) = \sum_{\lambda \in \mathcal{O}} \psi(\langle X, \lambda \rangle).$$

(Cf. "orbital integrals" of real Lie algebras.) Clearly the functions $\widehat{f_\mathcal{O}}$ are constant on adjoint orbits. In some sense they are a Lie algebra analogue of the generic characters of G.

E.g. If G is a torus, i.e. a product of $\mathbb{F}_{q^d}^\times$'s, the coadjoint action is trivial, and the strongly regular orbits are just the points of $\mathfrak{g}^*$:

$$\widehat{f_{\{\lambda\}}}(X) = \psi(\langle X, \lambda \rangle).$$

E.g. Take $G = GL_2(\mathbb{F}_q)$. Regular semisimple orbits are

$$\mathcal{O}_{\alpha,\beta} := \{X \in \mathfrak{gl}_2(\mathbb{F}_q) \mid \det(X) = \alpha\beta,\ \operatorname{tr}(X) = \alpha + \beta\},$$

where $\alpha \neq \beta$, either both in $\mathbb{F}_q$ or conjugate in $\mathbb{F}_{q^2}$.

If $\alpha, \beta \in \mathbb{F}_q$, then:

$$\begin{aligned}
\widehat{f_{\alpha,\beta}}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) &= \sum_{\substack{a,b,c,d \in \mathbb{F}_q \\ ad-bc=\alpha\beta \\ a+d=\alpha+\beta}} \psi(ax+dy) \\
&= (q-1) \sum_{\substack{a,d \in \mathbb{F}_q \\ a+d=\alpha+\beta}} \psi(ax+dy) + q \sum_{\substack{a,d \in \mathbb{F}_q \\ \{a,d\}=\{\alpha,\beta\}}} \psi(ax+dy) \\
&= \begin{cases} (q^2+q)\psi((\alpha+\beta)x), & \text{if } x = y, \\ q(\psi(\alpha x+\beta y)+\psi(\beta x+\alpha y)), & \text{if } x \neq y. \end{cases}
\end{aligned}$$

$$\widehat{f_{\alpha,\beta}}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \sum_{\substack{a,b,c,d \in \mathbb{F}_q \\ ad-bc=\alpha\beta \\ a+d=\alpha+\beta}} \psi(c) = q \sum_{c \in \mathbb{F}_q^\times} \psi(c) + 2q = q.$$

Thm 2 (Springer) *Let $\mathcal{O}$ be a strongly regular coadjoint orbit. Choose $\lambda \in \mathcal{O}$, and let $T = G_\lambda$.*

(1) *Suppose $N \in \mathfrak{g}$ is nilpotent. Then*

$$\widehat{f_{\mathcal{O}}}(N) = \pm q^{|R^+|} Q_T^G(N),$$

where the **Green function** $Q_T^G : \mathfrak{g}_{\mathrm{nil}} \to \mathbb{Z}$ *depends only on the conjugacy class of T.* (*Under some assumptions on q, this coincides with the Green function on G_{uni} defined by Deligne-Lusztig.*)

(2) *If $X \in \mathfrak{g}$ has Jordan decomposition $X = S + N$,*

$$\widehat{f_{\mathcal{O}}}(X) = \frac{\pm q^{|R^+|}}{|Z_G(S)|} \sum_{\substack{g \in G \\ g.S \in \mathfrak{t}}} \psi(\langle g.S, \lambda \rangle) Q_T^{gZ_G(S)g^{-1}}(g.N).$$

So $\widehat{f_{\mathcal{O}}}$ is "induced" from $\widehat{f_{\{\lambda\}}}$, in a way which is "weighted" by the Green functions.

Note: actually $Z_G(S)$ should be $Z_{G(\overline{\mathbb{F}_q})}(S)^\circ(\mathbb{F}_q)$.

Easiest case is when $T \subset$ Borel B. For $f : \mathfrak{t} \to \overline{\mathbb{Q}}$, define the **parabolic induction** $\mathrm{Ind}_{T\subset B}^{G}(f) : \mathfrak{g} \to \overline{\mathbb{Q}}$ by

$$\mathrm{Ind}_{T\subset B}^{G}(f)(X) = \frac{1}{|B|} \sum_{\substack{g\in G \\ g.X\in\mathfrak{b}}} f((g.X)_{\mathfrak{t}}).$$

Lemma 2 $\widehat{f_{\mathcal{O}}} = q^{|R^+|}\mathrm{Ind}_{T\subset B}^{G}(\widehat{f_{\{\lambda\}}})$.

Proof (Sketch.)

$$\begin{aligned} \mathrm{LHS}(X) &= \frac{1}{|B|} \sum_{\substack{g\in G \\ \mu\in\mathfrak{b}^{\perp}}} \psi(\langle X, g^{-1}.(\lambda+\mu)\rangle) \\ &= \frac{|\mathfrak{b}^{\perp}|}{|B|} \sum_{\substack{g\in G \\ g.X\in\mathfrak{b}}} \psi(\langle g.X, \lambda\rangle) = \mathrm{RHS}(X). \end{aligned}$$

This implies Springer's results in this case.

Lusztig defined a geometric notion of parabolic induction for perverse sheaves on $\mathfrak{t}(\overline{\mathbb{F}_q})$ and $\mathfrak{g}(\overline{\mathbb{F}_q})$, and this can be used to prove Springer's results in general. See Brylinski [Astérisque 140], Lusztig [SLN 1271].

§4. Introducing a $\mathbb{Z}/2\mathbb{Z}$-grading

The next best thing to an adjoint action is the action of G_0 on $\mathfrak{g}_1$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra.

In the reductive case, one can define strongly regular orbits $\mathcal{O} \subset \mathfrak{g}_1^*$, and functions $\widehat{f_{\mathcal{O}}} : \mathfrak{g}_1 \to \overline{\mathbb{Q}}$ as before. They should have something to do with the generic spherical functions of the symmetric space G/G_0.

Question: are there Springer-like results for these?

Lusztig's approach seems best. Grojnowski's work suggests that the parabolic induction will be from maximal quasi-split Levis, where a graded Levi subalgebra $\mathfrak{l}$ is **quasi-split** if $\mathfrak{l}/\mathfrak{z}(\mathfrak{l})$ contains an odd Cartan subalgebra.

So for quasi-split G/G_0, the results will be vacuous.

E.g. $G/G_0 = GL_{2n}(\mathbb{F}_q)/(GL_n(\mathbb{F}_q) \times GL_n(\mathbb{F}_q))$.
Here $\mathfrak{g} = \mathfrak{gl}_{2n}(\mathbb{F}_q)$, with the grading

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \, A, B \in \mathfrak{gl}_n(\mathbb{F}_q) \right\},$$
$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \middle| \, A, B \in \mathfrak{gl}_n(\mathbb{F}_q) \right\}.$$

Identify $\mathfrak{g}_1^*$ with $\mathfrak{g}_1$ via

$$\langle \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & A' \\ B' & 0 \end{smallmatrix}\right) \rangle = \mathrm{tr}(AB' + BA').$$

The regular semisimple G_0-orbits in $\mathfrak{g}_1$ are

$$\sqrt{\mathcal{O}} := \{ \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \in \mathfrak{g}_1 \,|\, AB \in \mathcal{O} \},$$

for $\mathcal{O}$ a regular semisimple class in $GL_n(\mathbb{F}_q)$. We have

$$\widehat{f_{\sqrt{\mathcal{O}}}}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) := \sum_{\substack{A', B' \in GL_n(\mathbb{F}_q) \\ A'B' \in \mathcal{O}}} \psi(\mathrm{tr}(AB' + BA')).$$

If $n = 1$, we recover Kloosterman sums; so this is some sort of "generalized Kloosterman sum".

The maximal quasi-split Levis have the form

$$L \cong GL_2(\mathbb{F}_q) \times \cdots \times GL_2(\mathbb{F}_q)$$

(n copies of the $n = 1$ case), and $\sqrt{\mathcal{O}}$ determines a G_0-conjugacy class $\{L\}$ of these.

Thm 3 *If* $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \in \mathfrak{g}_1$ *is nilpotent,*

$$\widehat{f_{\sqrt{\mathcal{O}}}}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \pm q^{n^2-n} \widetilde{Q}_L^G\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right),$$

where $\widetilde{Q}_L^G : (\mathfrak{g}_1)_{\mathrm{nil}} \to \mathbb{Z}$ *depends only on* $\{L\}$.

I also have a formula for all values of $\widehat{f_{\sqrt{\mathcal{O}}}}$ in terms of the analogous thing for L (i.e. ordinary Kloosterman sums) and Green functions of the $\widetilde{Q}_L^G$ type.

But I don't know how to compute $\widetilde{Q}_L^G$.

However, I can do this for $GL_{2n}(\mathbb{F}_q)/GL_n(\mathbb{F}_{q^2})$.

E.g. Let $\mathcal{O} \subset GL_n(\mathbb{F}_{q^2})$ be the class with distinct eigenvalues $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{F}_q}^\times$, permuted by $\alpha \mapsto \alpha^q$ in cycles of length $\nu_1, \cdots, \nu_s$. Let J_μ be the standard $n \times n$ nilpotent matrix with blocks of sizes $\mu_1, \cdots, \mu_t$. Then

$$\sum_{\substack{A \in GL_n(\mathbb{F}_{q^2}) \\ AA^{[q]} \in \mathcal{O}}} \psi(\mathrm{tr}(J_\mu(A + A^{[q]})))$$

$$= (-1)^{n+s} q^{n^2-n} Q^\mu_\nu(q^2) \prod_{j=1}^{s} (1 + q^{\nu_j}),$$

where $Q^\mu_\nu(\cdot)$ is the **Green polynomial** indexed by μ, ν.