Quiver varieties and zero weight spaces

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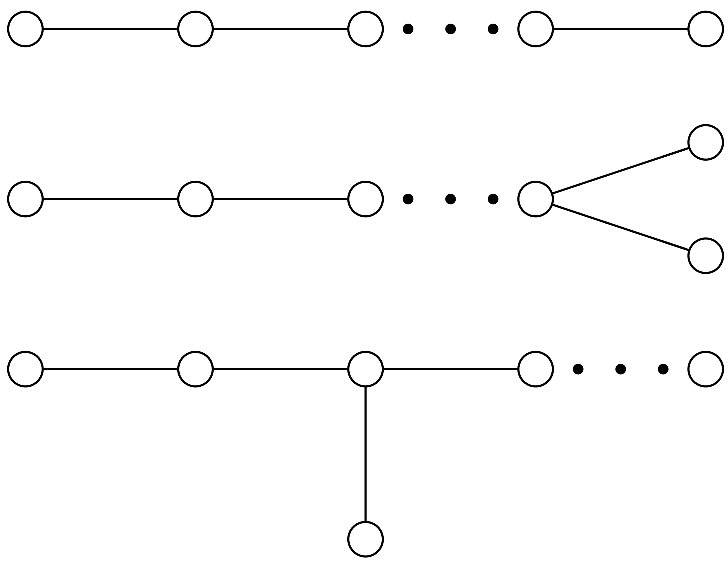
Let $\mathfrak{g}$ be a simple complex Lie algebra of type A_l ($\mathfrak{sl}_{l+1}$), D_l ($\mathfrak{so}_{2l}$), or E_l ($l = 6, 7, 8$).
Its Dynkin diagram is:

A_l :

D_l :

E_l :



The Serre presentation of $\mathfrak{g}$ has generators $\{e_i, h_i, f_i \,|\, 1 \leq i \leq l\}$ and relations:

$$i : [e_i, f_i] = h_i,\ [h_i, e_i] = 2e_i,\ [h_i, f_i] = -2f_i,$$
$$i \not\!\!- j : [x_i, y_j] = 0,\ \text{for } x, y \in \{e, h, f\},$$
$$i - j : [h_i, e_j] = -e_j, [e_i, [e_i, e_j]] = 0,\ \text{etc.}$$

Let $\mathfrak{h} = \mathbb{C}\{h_1, \cdots, h_l\}$ be the Cartan subalgebra. Let $\{\pi_1, \cdots, \pi_l\}$ be the basis of $\mathfrak{h}^*$ dual to $\{h_i\}$, and $\{\alpha_i = 2\pi_i - \sum_{j \text{---} i} \pi_j\}$ the simple roots.

Any finite-dimensional $\mathfrak{g}$-module V is the direct sum of its weight spaces

$$V_\lambda = \{v \in V \mid h_i.v = \lambda(h_i)v \text{ for all } i\},$$

for various $\lambda \in \mathbb{Z}\pi_1 \oplus \cdots \oplus \mathbb{Z}\pi_l$.

Classification of irreducibles: for each $\underline{w} \in \mathbb{N}^l$, there is a unique irreducible $V(\underline{w})$ with highest weight $w_1\pi_1 + \cdots + w_l\pi_l$, whose other weights are of the form

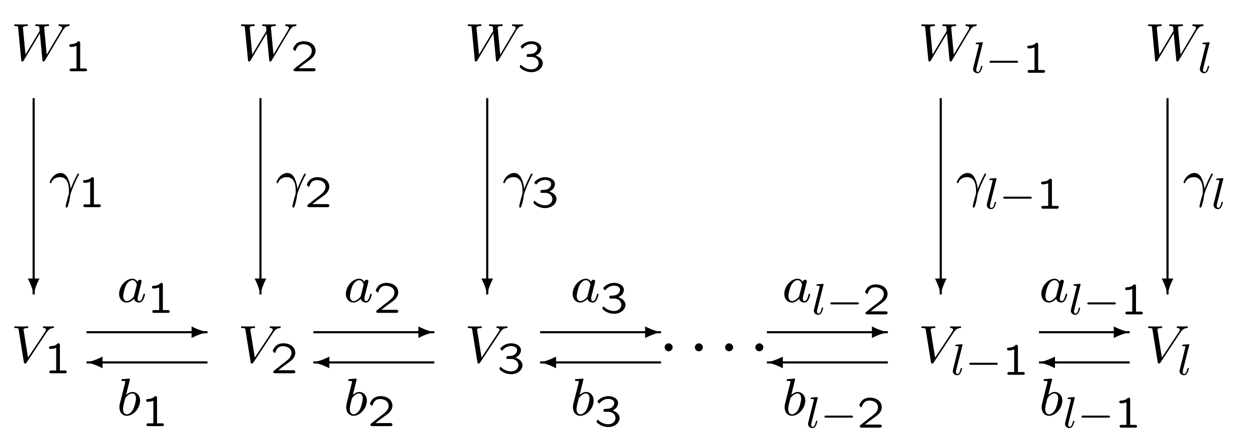
$$\text{“}\underline{w} - \underline{v}\text{”} = w_1\pi_1 + \cdots + w_l\pi_l - v_1\alpha_1 - \cdots - v_l\alpha_l,$$

for various $\underline{v} \in \mathbb{N}^l$.

Nakajima's construction of weight spaces

Nakajima (Duke Math. J.) realized $V(\underline{w})_{\underline{w}-\underline{v}}$ as the top homology of a projective **quiver variety** $L_{\underline{v},\underline{w}}$. Moreover e_i, f_i act via some natural correspondences between these varieties.

To construct $L_{\underline{v},\underline{w}}$: fix v. spaces V_i of dim v_i, W_i of dim w_i, consider diagrams of linear maps



(A_l case)

Impose the following conditions.

1. **Quadratic equations:**

$$\begin{aligned} b_1 a_1 &= 0 \\ b_2 a_2 &= a_1 b_1 \\ &\vdots \\ b_{l-1} a_{l-1} &= a_{l-2} b_{l-2} \\ 0 &= a_{l-1} b_{l-1} \end{aligned}$$

2. **Stability:** $\mathrm{im}\,(\gamma_1) + \cdots + \mathrm{im}\,(\gamma_l)$ generates $V_1 \oplus \cdots \oplus V_l$ under the a's and b's.

Then $G_V = GL(V_1) \times \cdots \times GL(V_l)$ acts on such (a, b, γ) freely, and $L_{\underline{v},\underline{w}}$ is defined to be the quotient variety.

Examples in type A_l

1. Take $\underline{w} = (1, 0, \cdots, 0)$, so $V(\underline{w}) = \mathbb{C}^{l+1}$. Weights are $\underline{w} - \underline{v}$ where $\underline{v} = (1, \cdots, 1, 0, \cdots, 0)$, corresponding to the diagram:

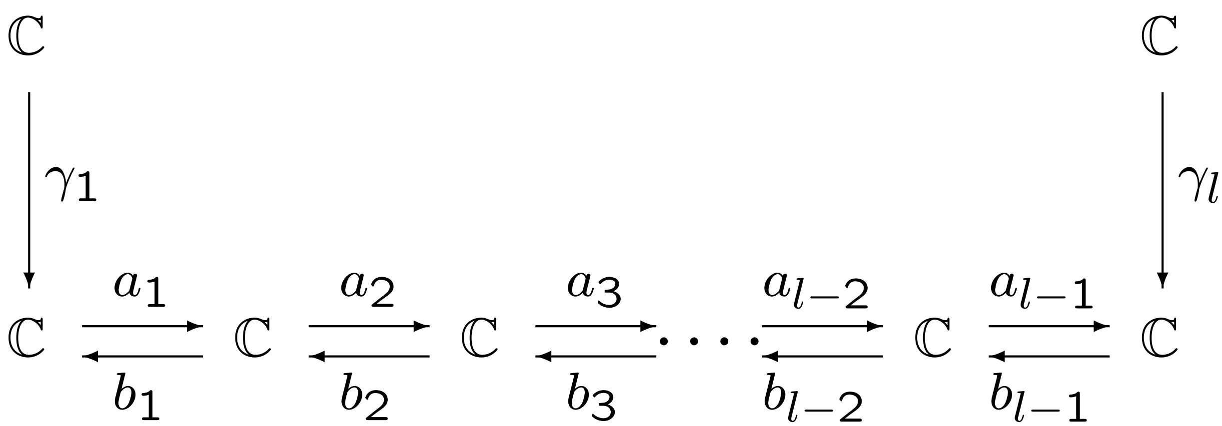
$$\begin{array}{l}
\mathbb{C} \\
\downarrow \gamma \\
\mathbb{C} \underset{b_1}{\overset{a_1}{\rightleftarrows}} \mathbb{C} \underset{b_2}{\overset{a_2}{\rightleftarrows}} \mathbb{C} \underset{b_3}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{i-2}}{\overset{a_{i-2}}{\rightleftarrows}} \mathbb{C} \underset{b_{i-1}}{\overset{a_{i-1}}{\rightleftarrows}} \mathbb{C}
\end{array}$$

Applying quadratic equations and stability:

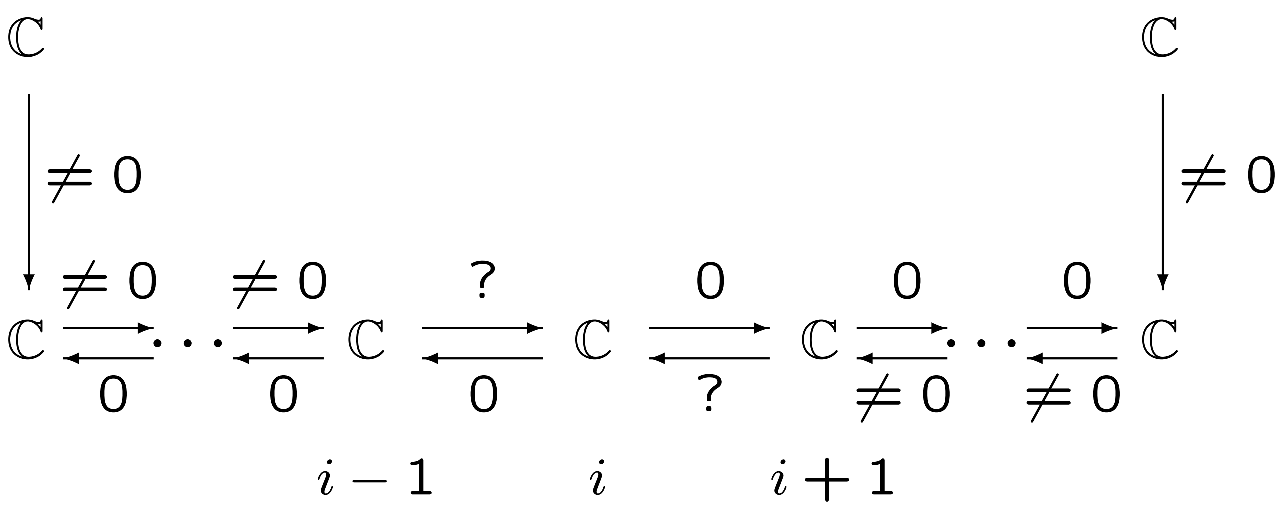
$$\begin{array}{l}
\mathbb{C} \\
\downarrow \neq 0 \\
\mathbb{C} \underset{0}{\overset{\neq 0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{\neq 0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{\neq 0}{\rightleftarrows}} \cdots \underset{0}{\overset{\neq 0}{\rightleftarrows}} \mathbb{C} \underset{0}{\overset{\neq 0}{\rightleftarrows}} \mathbb{C}
\end{array}$$

G_V acts transitively, so $L_{\underline{v},\underline{w}} =$ a point.

2. Take $\underline{w} = (1, 0, \cdots, 0, 1)$, so $V(\underline{w}) = \mathfrak{sl}_{l+1}$. Cartan subalgebra $\rightsquigarrow$ zero weight $\underline{w} - (1, \cdots, 1)$, corresponding to the diagram:



For $1 \leq i \leq l$, have an irreducible component:



where the two ?'s are not both 0. In the quotient each such component is a $\mathbb{P}^1$.

Zero weight spaces

The zero weight space $V(\underline{w})_0$ carries a representation of the Weyl group W. For some values of $\underline{w}$ this is a representation which had previously been realized as the top homology of a **Springer fibre** (Reeder 98). It is natural to conjecture that in these cases, the quiver variety is isomorphic to the Springer fibre.

In type A_l, the relevant $\underline{w}$, $\underline{v}$ satisfy:

$$l+1 = w_1 + 2w_2 + \cdots + lw_l,$$
$$v_i = w_1 + 2w_2 + \cdots + iw_i + \cdots + iw_l - i.$$

The Springer fibre consists of complete flags

$$0 = U_0 \subset U_1 \subset \cdots \subset U_l \subset U_{l+1}$$

in a fixed $(l+1)$-dim U_{l+1} which satisfy

$$e(U_k) \subseteq U_{k-1},\ 1 \le k \le l+1,$$

for a fixed nilpotent $e \in \mathrm{End}(U_{l+1})$ with w_i Jordan blocks of size i.

The isomorphism for type A_l was found by Maffei. Define U_{l+1} and e to be:

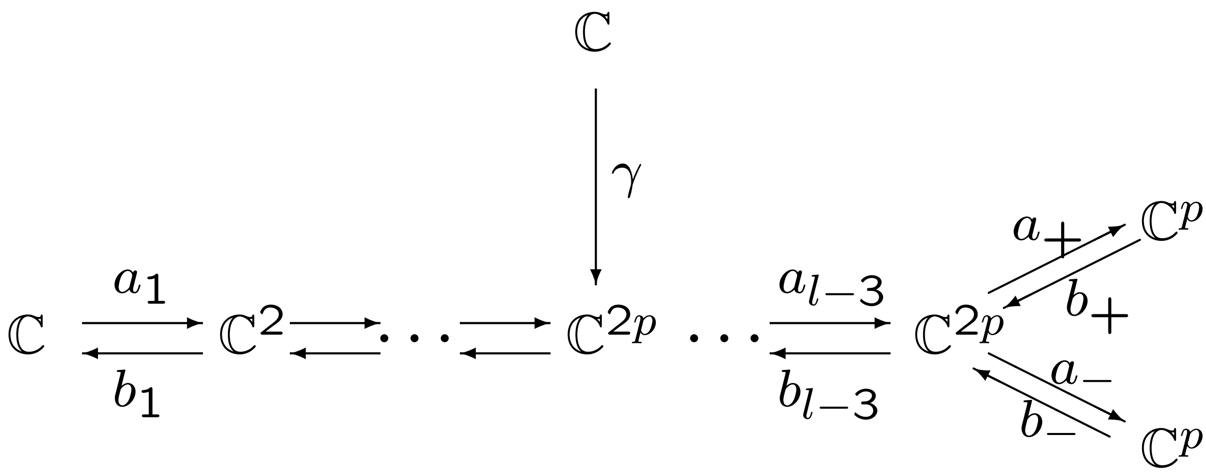
$$\begin{array}{ccccccccc}
W_1^{(1)} & \oplus & W_2^{(1)} & \oplus & W_3^{(1)} & \oplus & \cdots & \oplus & W_l^{(1)} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & \oplus & W_2^{(2)} & \oplus & W_3^{(2)} & \oplus & \cdots & \oplus & W_l^{(2)} \\
 & & & & \uparrow & & & & \uparrow \\
 & & & \oplus & W_3^{(3)} & \oplus & \cdots & \oplus & W_l^{(3)} \\
 & & & & & & \ddots & \vdots & \vdots \\
 & & & & & & & \oplus & W_l^{(l)}
\end{array}$$

The k-subspace associated to (a, b, γ) is the subspace of the first k rows defined by

$$\begin{aligned}
& a_{k-1} \cdots a_1 \Big(\sum_{j=1}^{l} b_1 \cdots b_{j-1} \gamma_j (w_j^{(1)}) \Big) \\
& + a_{k-1} \cdots a_2 \Big(\sum_{j=2}^{l} b_2 \cdots b_{j-1} \gamma_j (w_j^{(2)}) \Big) \\
& + \cdots \\
& + \sum_{j=k}^{l} b_k \cdots b_{j-1} \gamma_j (w_j^{(k)}) = 0.
\end{aligned}$$

Analogue in type D_l

Take $\underline{w} = \pi_{2p}$ for $p \leq \lfloor \frac{l-2}{2} \rfloor$, so $V(\underline{w}) = \Lambda^{2p}(\mathbb{C}^{2l})$. 0-wt space $\rightsquigarrow \underline{v} = (1, 2, \cdots, 2p, 2p, \cdots, 2p, p, p)$.



The Springer fibre consists of isotropic flags

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{l-1}$$

in a $2l$-dim orthogonal space U_{2l} which satisfy

$$e(U_k) \subseteq U_{k-1},\ 1 \leq k \leq l-1,\ e(U_{l-1}^{\perp}) \subseteq U_{l-1},$$

for a fixed nilpotent $e \in \mathfrak{so}(U_{2l})$ with Jordan blocks of sizes $2l - 2p - 1$ and $2p + 1$.

We have a morphism from $L_{\underline{v},\underline{w}}$ to the Springer fibre which we conjecture to be an isomorphism (if $p = 1$ this is easy to see).

If the blocks of e have coords $v^{(1)}, \cdots, v^{(2l-2p-1)}$ and $w^{(1)}, \cdots, w^{(2p+1)}$, the $(l-1)$-subspace associated to (a, b, γ) is defined by

$$
\begin{aligned}
&v^{(l)} = v^{(l+1)} = \cdots = v^{(2l-2p-1)} = w^{(2p+1)} = 0,\\
&a_+ b_- a_- b_+ \cdots a_+ b_- a_- a_{l-3} \cdots a_{2p} \gamma (v^{(l-2p)} - w^{(1)})\\
&- a_+ b_- \cdots a_- b_+ a_+ a_{l-3} \cdots a_{2p} \gamma (v^{(l-2p+1)} + w^{(2)})\\
&+ \cdots\\
&- a_+ b_- a_- b_+ a_+ a_{l-3} \cdots a_{2p} \gamma (v^{(l-3)} + w^{(2p-2)})\\
&+ a_+ b_- a_- a_{l-3} \cdots a_{2p} \gamma (v^{(l-2)} - w^{(2p-1)})\\
&- a_+ a_{l-3} \cdots a_{2p} \gamma (v^{(l-1)} + w^{(2p)}) = 0,
\end{aligned}
$$

and a similar equation in V_-.