The idea of character sheaves

Fix some notation:

- $F$ is an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime $p$,
- $G$ is a linear algebraic group over $F$,
- $q$ is a power of $p$ and $\mathbb{F}_q = \{ \alpha \in F \mid \alpha^q = \alpha \}$ is the finite subfield of $F$ with $q$ elements,
- $F : G \to G$ is the Frobenius endomorphism resulting from an $\mathbb{F}_q$-structure of $G$ (raising matrix entries to the $q$th power),
- $G^F = G(\mathbb{F}_q)$ is the associated finite group,
- $\text{Rep}(G^F, k)$ is the category of representations of $G^F$ on finite-dimensional vector spaces over a field $k$.

Idea: to define a geometric category $\text{Ch}(G, k)$ that in some sense 'unifies' all the algebraic categories $\text{Rep}(G^F, k)$ as $q, F$ vary.
**Algebra:** every \( V \in \text{Rep}(G^F, k) \) has a character

\[
\chi_V : G^F \to k : g \mapsto \text{tr}(g, V),
\]

which lies in the vector space \( \mathcal{C}(G^F, k) \) of class functions. When \( k \) is algebraically closed of characteristic 0, \( \text{Rep}(G^F, k) \) is a ‘categorification’ of \( \mathcal{C}(G^F, k) \) in the sense that

\[
K_0(\text{Rep}(G^F, k)) \otimes_k k \to \mathcal{C}(G^F, k) : [V] \mapsto \chi_V \text{ is an isomorphism.}
\]

**Geometry:** if \( \mathcal{L} \) is a \( G \)-equivariant (for conjugation) sheaf of finite-dimensional \( k \)-vector spaces on \( G \) and \( \varphi : F^* \mathcal{L} \sim \mathcal{L} \), we define the characteristic function

\[
\chi_{\mathcal{L}, \varphi} : G^F \to k : g \mapsto \text{tr}(\varphi_g, \mathcal{L}_g).
\]

This lies in \( \mathcal{C}(G^F, k) \) also. So we can imagine a category of sheaves on \( G \) giving another ‘categorification’ of \( \mathcal{C}(G^F, k) \).

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**Lusztig’s theory (‘Character sheaves’ I–V, mid-1980s)**

Let \( G \) be **connected** and **reductive**, and take \( k = \overline{\mathbb{Q}_\ell} \) where \( \ell \neq p \). Lusztig defined a set \( \hat{G} \) of irreducible \( G \)-equivariant perverse \( \overline{\mathbb{Q}_\ell} \)-sheaves on \( G \), called **character sheaves**. His definition is geometric and makes no reference to \( q \) or \( F \).

**Theorem (Lusztig)**

[Exclude some small \( p \) if \( G \) has factors of exceptional type.] For any \( F \) as above,

\[
\{ \chi_{A, \varphi} \mid A \in \hat{G} \text{ such that } F^* A \cong A, \ \varphi : F^* A \sim A \text{ normalized} \}
\]

is an orthonormal basis of \( \mathcal{C}(G^F, \overline{\mathbb{Q}_\ell}) \).

Under various further assumptions on \( G \), Lusztig and others showed that this basis is ‘almost’ the basis of irreducible characters. This resulted in an algorithm for computing the character table of \( G^F \).
Remarks:

1. \( \overline{\mathbb{Q}_\ell} \)-sheaves on \( G/\mathbb{F} \) are defined using the étale topology. But Lusztig’s definition of \( \hat{G} \) can be adapted for \( G/\mathbb{C} \) with the usual topology; there one can take \( k = \mathbb{C} \) rather than \( \overline{\mathbb{Q}_\ell} \).

2. Perverse sheaves are not sheaves; the elements of \( \hat{G} \) actually belong to the equivariant derived category \( D_G(G, \mathbb{Q}_\ell) \).

3. Boyarchenko (2013) proved an analogous theorem for \( G \) unipotent, using a definition of \( \hat{G} \) due to him and Drinfeld.

4. We can define \( \text{Ch}(G, \mathbb{Q}_\ell) \) to be the (semisimple abelian) full subcategory of \( D_G(G, \mathbb{Q}_\ell) \) consisting of all direct sums of \( \hat{G} \). Then Lusztig’s theorem says that \( \text{Ch}(G, \mathbb{Q}_\ell) \) ‘categorifies all \( \text{C}(G^F, \mathbb{Q}_\ell) \) at once’, in that sense unifying all \( \text{Rep}(G^F, \mathbb{Q}_\ell) \).

5. One could hope to define a category \( \text{Ch}(G, k) \subset D_G(G, k) \) for general \( k \), perhaps abelian, but not semisimple in general.

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Example \((G = GL_1 = \mathbb{F}^\times)\)

The Frobenius \( F : GL_1 \to GL_1 \) is either \( \alpha \mapsto \alpha^q \) (split case) or \( \alpha \mapsto \alpha^{-q} \) (non-split case), so

\[
GL_1^F = \begin{cases} 
\{ \alpha \in \mathbb{F} | \alpha^{q-1} = 1 \} = \mathbb{F}_q^\times & \text{(split)}, \\
\{ \alpha \in \mathbb{F} | \alpha^{q+1} = 1 \} = \mu_{q+1} & \text{(non-split)}.
\end{cases}
\]

A trivial observation: every irreducible \( V \in \text{Rep}(GL_1^F, \overline{\mathbb{Q}_\ell}) \) is one-dimensional and satisfies \( V \otimes n \cong \overline{\mathbb{Q}_\ell} \) for some \( n \) prime to \( p \). Accordingly, \( \overline{GL_1} \) consists of all rank-one local systems \( \mathcal{L} \) on \( GL_1 \) (i.e. locally constant sheaves of 1-dimensional \( \overline{\mathbb{Q}_\ell} \)-vector spaces) satisfying \( \mathcal{L} \otimes n \cong \overline{\mathbb{Q}_\ell} \) (the constant sheaf) for some \( n \) prime to \( p \).

We have \( F^* \mathcal{L} \cong \mathcal{L} \iff \begin{cases} \mathcal{L} \otimes (q-1) \cong \overline{\mathbb{Q}_\ell} & \text{(split)}, \\
\mathcal{L} \otimes (q+1) \cong \overline{\mathbb{Q}_\ell} & \text{(non-split)}.
\end{cases} \)

The characteristic functions of such \( \mathcal{L} \) are exactly the irreducible characters of \( \mathbb{F}_q^\times \) (split) or \( \mu_{q+1} \) (non-split).
Example \((G = SL_2)\)

Suppose \(F : G \to G\) raises entries to the \(q\)th power. There are two types of \(F\)-stable maximal tori (i.e. subgroups isomorphic to \(GL_1\)):

- **split** \(T\) contained in an \(F\)-stable Borel \(B\),
  
  e.g. \(T = \{(\alpha \ 0 \ \alpha^{-1})\}, \ \ B = \{(\alpha \ \beta \ 0)\},\)

- **non-split** \(T\) not contained in an \(F\)-stable Borel.

Most irreducibles in \(\text{Rep}(G^F, \overline{Q}_\ell)\) have dimension \(q \pm 1\). Those of dimension \(q + 1\) are obtained by *parabolic induction*:

\[
I_{T^F \subset B^F}^G(F)(V) = \text{Ind}_{B^F}^G \text{Res}^T_{B^F}(V)
\]

where \(V \in \text{Rep}(T^F, \overline{Q}_\ell)\) is one-dimensional and \(\text{Res}^T_{B^F}\) is the pullback functor of the projection \(B^F \to T^F\). So we need geometric analogues of these induction and pullback functors.

### Digression on functors

Suppose \(\psi : H \to G\) is a homomorphism of algebraic groups commuting with their Frobenius endomorphisms. We then get a homomorphism \(\psi : H^F \to G^F\) of finite groups, and functors

\[
\text{Res}^{G^F}_{H^F} : \text{Rep}(G^F, k) \to \text{Rep}(H^F, k) \text{ (pullback through } \psi),
\]

\[
\text{Ind}^{G^F}_{H^F} : \text{Rep}(H^F, k) \to \text{Rep}(G^F, k) : V \mapsto k[G^F] \otimes_{k[H^F]} V,
\]

where \(\text{Ind}^{G^F}_{H^F}\) is left adjoint to \(\text{Res}^{G^F}_{H^F}\).

- If \(\psi : H^F \to G^F\) is the inclusion of a subgroup, these are the usual induction and restriction functors, which are biadjoint.

- If \(\psi : H^F \to G^F\) is surjective with kernel \(\Gamma\), then \(\text{Ind}^{G^F}_{H^F}\) is the functor of \(\Gamma\)-coinvariants. The functor of \(\Gamma\)-invariants is right adjoint to \(\text{Res}^{G^F}_{H^F}\).
We want geometric analogues with the right ‘decategorifications’. On characters, $\text{Res}_H^G$ is just pullback through $\psi$, so we define

$$\text{Res}_H^G : \mathcal{D}_G(G, k) \xrightarrow{\text{For}_H^G} \mathcal{D}_H(G, k) \xrightarrow{\psi^*} \mathcal{D}_H(H, k).$$

(This works for equivariant sheaves, without derived categories.) If $\psi$ is injective, for $V \in \text{Rep}(H^F, k)$ and $g \in G^F$ we have

$$\chi_{\text{Ind}_{H^F}^G(V)}(g) = \sum_{[g', h] \in G^F \times H^F} \chi_V(h),$$

so we define

$$\text{Ind}_H^G : \mathcal{D}_H(H, k) \xrightarrow{\sim} \mathcal{D}_G(G \times_H H, k) \xrightarrow{\pi_1} \mathcal{D}_G(G, k),$$

where $\pi : G \times_H H \to G : [g', h] \mapsto g'\psi(h)(g')^{-1}$. (Here $\pi_1$ must be the derived functor.) As expected, $\text{Ind}_H^G$ is left adjoint to $\text{Res}_H^G$.

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**Example ($G = \text{SL}_2$, continued)**

We now have a geometric version of parabolic induction:

$$\text{I}_{T \subset B}^G = \text{Ind}_{B}^G \circ \text{Res}_{B}^T : \mathcal{D}_T(T, \overline{Q}_\ell) \to \mathcal{D}_G(G, \overline{Q}_\ell).$$

For $\mathcal{L} \in \widehat{T}$ we want $\mathcal{I}_{T \subset B}^G(\mathcal{L})$ to be in $\text{Ch}(G, \overline{Q}_\ell)$.

- If $\mathcal{L} \not\cong \overline{Q}_\ell$, then $\mathcal{I}_{T \subset B}^G(\mathcal{L})$ is irreducible, so include it in $\widehat{G}$.
- By definition, $\mathcal{I}_{T \subset B}^G(\overline{Q}_\ell) = \pi_1 \overline{Q}_\ell$ where $\pi : G \times_B B \to G$ is the Grothendieck–Springer map. This decomposes as $\overline{Q}_\ell \oplus \text{St}$, so include both $\overline{Q}_\ell, \text{St}$ in $\widehat{G}$.
- If $p \neq 2$, there is a unique $S \in \widehat{T}$ with $S \not\cong \overline{Q}_\ell, S \otimes^2 \cong \overline{Q}_\ell$.

We have $\mathcal{I}_{T \subset B}^G(S) = X \oplus Y$, so include both $X, Y$ in $\widehat{G}$.

Which of these $A \in \widehat{G}$ satisfy $F^*A \cong A$, and are their characteristic functions equal to the irreducible characters of $G^F = \text{SL}_2(\mathbb{F}_q)$?
Example ($G = SL_2$, continued)

$F^*(I^G_{T \subset B}(L)) \cong I^G_{T \subset B}(L)$ can happen in two ways:

- $T$ split $F$-stable, $F^*L \cong L$. These give the characters of the irreducibles $I^{G^F}_{T^F \subset B^F}(V)$ of dimension $q + 1$.
- $T$ non-split $F$-stable, $F^*L \cong L$. These give the characters of the irreducibles of dimension $q - 1$; note that these are not obtained by parabolic induction in the finite group $G^F$.

$F^*Q_\ell \cong Q_\ell$ and $F^*\text{St} \cong \text{St}$, giving the trivial and Steinberg characters of $G^F$ (the constituents of $\chi_{\text{Ind}_{B^F}^{G^F}(Q_\ell)}$).

If $p = 2$, this completes the list of irreducible characters, so we need no more character sheaves. Henceforth take $p \neq 2$.

$F^*X \cong X$ and $F^*Y \cong Y$, but $\chi_X$ and $\chi_Y$ are not irreducible characters. If the remaining irreducible characters are $\chi_1$ and $\chi_2$ (dimension $\frac{q+1}{2}$) and $\chi_3$ and $\chi_4$ (dimension $\frac{q-1}{2}$), we have

$\chi_X = \frac{\chi_1 + \chi_2 + \chi_3 + \chi_4}{2}, \quad \chi_Y = \frac{\chi_1 + \chi_2 - \chi_3 - \chi_4}{2}.$

Example ($G = SL_2$, $p \neq 2$, continued)

Two more character sheaves are needed. By an easy calculation, any class function orthogonal to the characteristic functions found so far must be supported on $O^F \sqcup (O')^F$, where

$O = \{ u \in G \mid u \text{ unipotent}, u \neq 1 \}$ (the regular unipotent class),
$O' = \{ u \in G \mid -u \in O \}.$

Since the centralizer of $u \in O$ has two connected components, there is a unique nontrivial rank-one $G$-equivariant local system $\mathcal{E}$ on $O$, and a corresponding $\mathcal{E}'$ on $O'$. It turns out that

$\chi_{\mathcal{E}} = \frac{\chi_1 - \chi_2 + \chi_3 - \chi_4}{2}, \quad \chi_{\mathcal{E}'} = \frac{\chi_1 - \chi_2 - \chi_3 + \chi_4}{2}.$

So we regard $\mathcal{E}$ and $\mathcal{E}'$ as sheaves on $G$ (extending by zero) and include them in $\hat{G}$. This completes the definition of $\hat{G}$ for $G = SL_2$. 
The idea of cuspidal character sheaves

As in the $G = SL_2$ example, many character sheaves on $G$ arise as summands of parabolic inductions $I^G_{L \subset P}(A)$ where:

- $P$ is a proper parabolic subgroup of $G$ (e.g. a Borel),
- $L$ is a Levi factor of $P$ (a smaller connected reductive group),
- $A \in \hat{L}$.

A character sheaf not arising in this way is called *cuspidal*: e.g., all character sheaves on $GL_1$ (or any torus), and $\mathcal{E}, \mathcal{E}' \in \hat{SL_2}$ as above.

One way to define and classify character sheaves on $G$ is:

1. define and classify cuspidal character sheaves on all Levi subgroups $L$ of $G$ (including $G$ itself);
2. for each cuspidal $A \in \hat{L}$, classify the summands of $I^G_{L \subset P}(A)$, known as the *induction series* associated to $A$.

As in the $G = SL_2$ example, cuspidal character sheaves are closely related to sheaves on unipotent (more generally, isolated) classes.