

Character sheaves and modular generalized Springer correspondence

Part 2: The generalized Springer correspondence

Anthony Henderson
(joint with Pramod Achar, Daniel Juteau, Simon Riche)

University of Sydney

January 2015



Simplifying the context

To get a ‘toy model’ of character sheaves on G :

1. Instead of G -equivariant perverse sheaves on G , consider G -equivariant perverse sheaves on the **unipotent variety** \mathcal{U}_G . This is simpler because there are only finitely many G -orbits, but still highly relevant e.g. for cuspidal character sheaves.
2. Assume p is large enough so that there is a G -equivariant isomorphism $\mathcal{U}_G \xrightarrow{\sim} \mathcal{N}_G$ where \mathcal{N}_G is the **nilpotent cone** in the Lie algebra \mathfrak{g} ; then we can use Fourier transform on \mathfrak{g} .
3. The behaviour for large p is no different from considering G over \mathbb{C} with the usual topology rather than étale topology.

This setting (first with simplification 1 only, later with 2 and 3) was studied by Lusztig in the case of $\overline{\mathbb{Q}}_\ell$ -sheaves: one of his main results here was the ‘generalized Springer correspondence’.

Aim: to prove an analogue in the **modular** case where $\text{char}(k) = \ell$, as a first step towards understanding modular character sheaves.



The new set-up

New notation:

- ▶ G is a connected reductive algebraic group over \mathbb{C} ,
- ▶ \mathfrak{g} is its Lie algebra, on which G has the adjoint action,
- ▶ $\mathcal{N}_G = \{x \in \mathfrak{g} \mid x \text{ nilpotent}\}$ is the nilpotent cone, on which G has finitely many orbits,
- ▶ k is a sufficiently large field of characteristic $\ell \geq 0$,
- ▶ $\mathcal{D}_G(\mathcal{N}_G, k)$ is the constructible equivariant derived category.

For any $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ and G -orbit \mathcal{O} in \mathcal{N}_G , the restrictions $\mathcal{H}^i A|_{\mathcal{O}}$ are G -equivariant local systems (i.e. G -equivariant sheaves of finite-dimensional k -vector spaces) on \mathcal{O} , so they correspond to finite-dimensional representations over k of the finite group

$$A_G(x) = G_x / G_x^\circ, \text{ where } G_x \text{ is the stabilizer in } G \text{ of } x \in \mathcal{O}.$$

Let $\mathfrak{N}_{G,k}$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ where \mathcal{O} is a G -orbit in \mathcal{N}_G and \mathcal{E} is an **irreducible** G -equivariant local system on \mathcal{O} .



Example ($G = GL_n$)

When $G = GL_n$, $\mathfrak{g} = \text{Mat}_n$ and $\mathcal{N}_G = \{x \in \text{Mat}_n \mid x^n = 0\}$. By the Jordan form theorem, we have a bijection

$$G \backslash \mathcal{N}_G \longleftrightarrow \mathcal{P}_n = \{\text{partitions } \lambda \text{ of } n\},$$

where $x \in \mathcal{O}_\lambda$ means that x has Jordan blocks of sizes $\lambda_1, \lambda_2, \dots$. In this case $A_G(x) = 1$ for all x , so $\mathfrak{N}_{G,k} \longleftrightarrow \mathcal{P}_n$ for all fields k .

Example (G of type G_2)

The five nilpotent orbits, in order of decreasing dimension, are:

$$G_2 \text{ (regular)}, G_2(a_1) \text{ (subregular)}, \widetilde{A}_1, A_1, 0.$$

These Bala–Carter labels record the type of the smallest Levi subalgebra meeting the orbit (where \widetilde{A}_1 means the short-root A_1). We have $A_G(x) = 1$ for all x except $A_G(x) = S_3$ for $x \in G_2(a_1)$, so $|\mathfrak{N}_{G,k}| = 7$ usually, $|\mathfrak{N}_{G,k}| = 6$ if $\text{char}(k) \in \{2, 3\}$.

There is an anti-autoequivalence D of $\mathcal{D}_G(\mathcal{N}_G, k)$, *Verdier duality*. We study the abelian subcategory $\text{Perv}_G(\mathcal{N}_G, k)$ of G -equivariant perverse k -sheaves on \mathcal{N}_G , where $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ is *perverse* if

$$\mathcal{H}^i A|_{\mathcal{O}} = \mathcal{H}^i(DA)|_{\mathcal{O}} = 0 \text{ whenever } i > -\dim \mathcal{O}.$$

The simple objects in $\text{Perv}_G(\mathcal{N}_G, k)$ are in bijection with $\mathfrak{N}_{G,k}$:

$$\text{IC}(\mathcal{O}, \mathcal{E}) = \begin{array}{l} \text{'intermediate extension' of } \mathcal{E}[\dim \mathcal{O}] \text{ to } \overline{\mathcal{O}}, \\ \text{extended by zero to the whole of } \mathcal{N}_G. \end{array}$$

Example ($G = GL_2$, cf. Juteau–Mautner–Williamson)

The two orbits are $\mathcal{O}_{(1,1)} = \{0\}$ and $\mathcal{O}_{(2)} = \mathcal{N}_G \setminus \{0\}$. We have

$$\begin{aligned} \text{IC}(\mathcal{O}_{(1,1)}, \underline{k}) &= \underline{k}_0 \text{ (skyscraper sheaf),} \\ \text{IC}(\mathcal{O}_{(2)}, \underline{k}) &= \underline{k}_{\mathcal{N}_G}[2] \text{ if } \ell \neq 2. \end{aligned}$$

The $\ell = 2$ case is different, because then $H^1(\mathcal{O}_{(2)}, k) \neq 0$.

Cuspidal pairs and induction series

Let P be a parabolic subgroup of G and L a Levi factor of P . We have a geometric parabolic induction functor

$$\mathbf{I}_{LCP}^G = \text{Ind}_P^G \circ \text{Res}_P^L : \mathcal{D}_L(\mathcal{N}_L, k) \rightarrow \mathcal{D}_G(\mathcal{N}_G, k),$$

defined in the same way as for character sheaves:

$$\begin{aligned} \text{Res}_P^L : \mathcal{D}_L(\mathcal{N}_L, k) &\xrightarrow{\sim} \mathcal{D}_P(\mathcal{N}_L, k) \xrightarrow{(\cdot)^*} \mathcal{D}_P(\mathcal{N}_P, k), \\ \text{Ind}_P^G : \mathcal{D}_P(\mathcal{N}_P, k) &\xrightarrow{\sim} \mathcal{D}_G(G \times_P \mathcal{N}_P, k) \xrightarrow{(\cdot)!} \mathcal{D}_G(\mathcal{N}_G, k). \end{aligned}$$

Lemma (Lusztig when $\ell = 0$, [AHR] when $\ell > 0$)

\mathbf{I}_{LCP}^G commutes with D and maps $\text{Perv}_L(\mathcal{N}_L, k)$ to $\text{Perv}_G(\mathcal{N}_G, k)$. It has left adjoint $\mathbf{R}_{LCP}^G = \text{Ind}_P^L \circ \text{Res}_P^G$ and right adjoint $\mathbf{R}_{LCP^-}^G$ where P^- denotes the opposite parabolic with the same Levi L .

We say that a pair $(\mathcal{O}, \mathcal{E}) \in \mathfrak{M}_{G,k}$, or the corresponding $\text{IC}(\mathcal{O}, \mathcal{E})$, is *cuspidal* if the following equivalent conditions hold:

1. $\mathbf{R}_{LCP}^G(\text{IC}(\mathcal{O}, \mathcal{E})) = 0$ for all $L \subset P \subsetneq G$;
2. $\text{IC}(\mathcal{O}, \mathcal{E})$ is not a quotient of $\mathbf{I}_{LCP}^G(A)$ for any $L \subset P \subsetneq G$ and any $A \in \text{Perv}_L(\mathcal{N}_L, k)$;
3. $\text{IC}(\mathcal{O}, \mathcal{E})$ is not a subobject of $\mathbf{I}_{LCP}^G(A)$ for any $L \subset P \subsetneq G$ and any $A \in \text{Perv}_L(\mathcal{N}_L, k)$.

Remark

When $\ell = 0$, the Decomposition Theorem of [BBD] implies that if $A \in \text{Perv}_L(\mathcal{N}_L, k)$ is simple, then $\mathbf{I}_{LCP}^G(A)$ is semisimple, so one can replace ‘quotient’/‘subobject’ with ‘summand’. Semisimplicity can fail if $\ell > 0$, and cuspiduals **can** occur as constituents of $\mathbf{I}_{LCP}^G(A)$. This is analogous to modular representations of $G(\mathbb{F}_q)$ when $\ell \neq p$.

Lemma (Lusztig – same proof works for $\ell > 0$)

If $(\mathcal{O}, \mathcal{E})$ is cuspidal, \mathcal{O} is distinguished, i.e. meets no proper Levi.

Let $\mathfrak{M}_{G,k}$ be the set of *cuspidal data* $(L, \mathcal{O}_L, \mathcal{E}_L)$ where L is a Levi subgroup of G (take only one representative of each G -conjugacy class, allowing $L = G$) and $(\mathcal{O}_L, \mathcal{E}_L)$ is a cuspidal pair for L .

Proposition (Lusztig when $\ell = 0$, [AHJR] when $\ell > 0$)

For any $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$, $\mathbf{I}_{LCP}^G(\text{IC}(\mathcal{O}_L, \mathcal{E}_L))$ is independent of the parabolic P , and its head and socle are isomorphic.

The *induction series* associated to $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$ is the set of simple quotients (equivalently, subobjects) of $\mathbf{I}_{LCP}^G(\text{IC}(\mathcal{O}_L, \mathcal{E}_L))$.

Lemma (Lusztig – same proof works for $\ell > 0$)

Any simple object $\text{IC}(\mathcal{O}, \mathcal{E})$ in $\text{Perv}_G(\mathcal{N}_G, k)$ belongs to the induction series associated to some $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$ as above. (If $\text{IC}(\mathcal{O}, \mathcal{E})$ is cuspidal, then $(L, \mathcal{O}_L, \mathcal{E}_L) = (G, \mathcal{O}, \mathcal{E})$.)

The **(modular) generalized Springer correspondence** is:

Theorem (Lusztig when $\ell = 0$, [AHJR] when $\ell > 0$)

1. Induction series associated to different cuspidal data are disjoint: in other words, a given $\text{IC}(\mathcal{O}, \mathcal{E})$ belongs to the induction series associated to a **unique** $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}$.
2. The induction series associated to $(L, \mathcal{O}_L, \mathcal{E}_L)$ is canonically in bijection with the set of irreducible k -reps of $N_G(L)/L$.
3. Hence we have a bijection

$$\mathfrak{N}_{G,k} \longleftrightarrow \bigsqcup_{(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,k}} \text{Irr}(N_G(L)/L, k).$$

The proof will be discussed in the next lecture.

Remark

The analogue of 1 holds for modular representations of $G(\mathbb{F}_q)$ also; for the analogue of 2 one needs a q -deformed group algebra.

Background: the Springer correspondence

- ▶ In the mid-1970s, Springer gave a geometric construction of the irreducible $\overline{\mathbb{Q}}_\ell$ -reps of the Weyl group $W = N_G(T)/T$.
- ▶ As reformulated by Lusztig and Borho–Macpherson, this comes from an action of W on the semisimple perverse sheaf

$$\text{Spr} = \mathbf{I}_{T \subset B}^G(\overline{\mathbb{Q}}_{\ell_0}) = \mu_! \overline{\mathbb{Q}}_\ell[\dim \mathcal{N}_G] \in \text{Perv}_G(\mathcal{N}_G, \overline{\mathbb{Q}}_\ell),$$

where $\mu : G \times_B \mathcal{N}_B \rightarrow \mathcal{N}_G$ is the *Springer resolution* of \mathcal{N}_G . The *Springer correspondence* is the resulting bijection

$$\{\text{simple summands of Spr}\} \longleftrightarrow \text{Irr}(W, \overline{\mathbb{Q}}_\ell)$$

$$\text{IC}(\mathcal{O}, \mathcal{E}) \mapsto \text{Hom}_{\text{Perv}_G(\mathcal{N}_G, \overline{\mathbb{Q}}_\ell)}(\text{Spr}, \text{IC}(\mathcal{O}, \mathcal{E})).$$

- ▶ Lusztig then found that this was the $(L, \mathcal{O}_L, \mathcal{E}_L) = (T, 0, \overline{\mathbb{Q}}_\ell)$ case of the generalized Springer correspondence, thus accounting for the $\text{IC}(\mathcal{O}, \mathcal{E})$'s that are not summands of Spr.
- ▶ Juteau (2007) showed that the Springer correspondence holds with k instead of $\overline{\mathbb{Q}}_\ell$ and 'quotients' instead of 'summands'

Example ($G = GL_n$, $W = S_n$)

- ▶ When $\ell = 0$, $|\text{Irr}(S_n, k)| = |\mathcal{P}_n|$, so every $\text{IC}(\mathcal{O}_\lambda, \underline{k})$ is a summand of Spr , i.e. the Springer correspondence for GL_n is already ‘generalized’. In particular, GL_n does not have a cuspidal pair unless $n = 1$.
- ▶ When $\ell > 0$, James constructed the irreps D^λ of S_n over k , labelled by λ that are ℓ -regular (no part occurs $\geq \ell$ times). Under Juteau’s correspondence, D^λ maps to $\text{IC}(\mathcal{O}_{\lambda^\dagger}, \underline{k})$ where λ^\dagger is the transpose partition; so these are the simple quotients of Spr . (All simples occur as constituents of Spr .)
The only distinguished orbit in \mathcal{N}_G is $\mathcal{O}_{(n)}$; we will see that

$$(\mathcal{O}_{(n)}, \underline{k}) \text{ is cuspidal} \iff n \text{ is a power of } \ell.$$

So $\mathfrak{M}_{G,k}$ is essentially the set of Levis of the form $\prod_{i \geq 0} GL_{\ell^i}^{m_i}$, where m_i are nonnegative integers such that $\sum_{i \geq 0} m_i \ell^i = n$.

Example ($G = GL_n$, $\ell > 0$ continued)

For $L = \prod_{i \geq 0} GL_{\ell^i}^{m_i}$ such a Levi subgroup of GL_n , we have

$$N_G(L)/L \cong \prod_{i \geq 0} S_{m_i},$$

$$\text{Irr}(N_G(L)/L, k) \leftrightarrow \prod_{i \geq 0} \{\ell\text{-regular } \lambda^{(i)} \vdash m_i\}.$$

Under our correspondence, the collection $(\lambda^{(i)})$ maps to $\text{IC}(\mathcal{O}_\lambda, \underline{k})$ where $\lambda = \sum_{i \geq 0} \ell^i (\lambda^{(i)})^\dagger$. Note that $\text{IC}(\mathcal{O}_{(n)}, \underline{k})$ occurs in the series of $L = \prod_{i \geq 0} GL_{\ell^i}^{b_i}$ where $\sum_{i \geq 0} b_i \ell^i = n$ and all $b_i < \ell$.

Remark

The above combinatorial correspondence is a simplified version of what appears in the analogous theory of induction series for modular representations of $GL_n(\mathbb{F}_q)$ (Dipper–Du).

Example ($G = G_2$, W dihedral of order 12)

- ▶ When $\ell = 0$, $|\text{Irr}(W, k)| = 6 < 7 = |\mathfrak{R}_{G,k}|$. The non-Springer pair is $(G_2(a_1), \mathcal{E}_{\text{sign}})$, which must be cuspidal because the other proper Levi subgroups are both isomorphic to GL_2 .
- ▶ When $\ell = 2$, $|\text{Irr}(W, k)| = 2$, and only $\text{IC}(0, \underline{k})$ and $\text{IC}(\widetilde{A}_1, \underline{k})$ belong to Juteau's correspondence. The other series are:

$$(L \text{ of type } A_1, \mathcal{O}_{(2)}, \underline{k}), |N_G(L)/L| = 2 : \quad \text{IC}(A_1, \underline{k}),$$

$$(L \text{ of type } \widetilde{A}_1, \mathcal{O}_{(2)}, \underline{k}), |N_G(L)/L| = 2 : \quad \text{IC}(G_2(a_1), \mathcal{E}_{\text{refln}}),$$

leaving 2 cuspidal pairs, $(G_2(a_1), \underline{k})$ and (G_2, \underline{k}) .

The above GL_n and G_2 examples illustrate:

Theorem ([AHJR])

When $\ell > 0$, $\text{IC}(\mathcal{O}_{\text{reg}}, \underline{k})$ belongs to the induction series associated to $(L, \mathcal{O}_{L,\text{reg}}, \underline{k})$ where L is minimal such that $\ell \nmid |W/W_L|$.