Exotic nilpotent cones and Springer representations of Weyl groups

Anthony Henderson

University of Sydney

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Outline

The Springer correspondence

Kato's exotic Springer correspondence

Geometry of the exotic nilpotent cone

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We consider algebraic groups over $\mathbb{C}$. The nilpotent cone of GL_n is

$$\mathcal{N}(\mathfrak{gl}_n) = \{x \in \mathrm{Mat}_n \mid x \text{ nilpotent, i.e. all eigenvalues } 0\}.$$

GL_n acts by conjugation: $g.x = gxg^{-1}$.

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Theorem (Jordan canonical form)

The GL_n-orbits in $\mathcal{N}(\mathfrak{gl}_n)$ are in bijection with $\mathcal{P}_n$, the set of partitions of n. For $\lambda \in \mathcal{P}_n$, the orbit $\mathcal{O}_\lambda$ consists of those $x \in \mathcal{N}(\mathfrak{gl}_n)$ whose Jordan form has blocks of sizes $\lambda_1, \lambda_2, \cdots$.

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Since $\mathcal{P}_n$ also labels irreducible representations V_λ of S_n, we have

$$GL_n \setminus \mathcal{N}(\mathfrak{gl}_n) \longleftrightarrow \{\text{irreducible representations of } S_n\}.$$

This bijection can be defined using a certain action of S_n on the cohomology of Springer fibres: we have $H^{\text{top}}(\mathcal{B}_x) \cong V_\lambda$ if $x \in \mathcal{O}_\lambda$.

For G a connected reductive group, and B a Borel subgroup, define

$$\pi : G \times_B \mathfrak{b} \to \mathfrak{g} : (g, x) \mapsto g.x.$$

Then the Springer fibre $\mathcal{B}_x = \pi^{-1}(x)$ can be identified with

$$\begin{aligned}
&\text{(in general)} && \{\text{Borel subalgebras } \mathfrak{b}' \subset \mathfrak{g} \,|\, x \in \mathfrak{b}'\} \\
&(G = GL_n) && \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \,|\, x(V_i) \subseteq V_i\} \\
&(G = Sp_{2n}) && \{0 = V_0 \subset V_1 \subset \cdots \subset V_{2n} = \mathbb{C}^{2n} \,| \\
& && \qquad \langle V_i, V_{2n-i}\rangle = 0,\ x(V_i) \subseteq V_i\}.
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There are two important restrictions of π:

- $\pi_{\mathrm{nil}} : G \times_B \mathfrak{n} \to \mathcal{N}(\mathfrak{g})$ is the Springer resolution of the nilpotent cone. (Note that if x is nilpotent, then the condition $x(V_i) \subseteq V_i$ can be rewritten $x(V_i) \subseteq V_{i-1}$.)
- $\pi_{\mathrm{rs}} : G \times_B \mathfrak{b}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ is a Galois covering of the regular semisimple set with Galois group W, the Weyl group of G. So for $x \in \mathfrak{g}_{\mathrm{rs}}$, W acts simply transitively on $\mathcal{B}_x$.

Lusztig observed that since π is a small map, the derived push-forward complex $R\pi_*\mathbb{C}$ is the intersection cohomology extension of the local system $(\pi_{\mathrm{rs}})_*\mathbb{C}$ on $\mathfrak{g}_{\mathrm{rs}}$. Hence W acts on the complex $R\pi_*\mathbb{C}$, and therefore on $H^i(\mathcal{B}_x)$ for any $x \in \mathfrak{g}$.

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Theorem (Borho–MacPherson)

The endomorphism algebra of $R(\pi_{\mathrm{nil}})_*\mathbb{C}$ in the derived category on $\mathcal{N}(\mathfrak{g})$ is isomorphic to the group algebra of W. Hence we have a Springer correspondence between the irreps of W and the simple constituents of $R(\pi_{\mathrm{nil}})_*\mathbb{C}$, which are intersection cohomology complexes $IC(\overline{\mathcal{O}}, \mathcal{E})$ where $\mathcal{O}$ is a G-orbit in $\mathcal{N}(\mathfrak{g})$ and $\mathcal{E}$ is a G-equivariant simple local system on $\mathcal{O}$.

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In general, not all such $IC(\overline{\mathcal{O}}, \mathcal{E})$ occur in $R(\pi_{\mathrm{nil}})_*\mathbb{C}$. But the trivial local systems always do occur, so we get an injective map

$$G \setminus \mathcal{N}(\mathfrak{g}) \hookrightarrow \{\text{irreps of } W\} : G.x \mapsto H^{\mathrm{top}}(\mathcal{B}_x)^{G_x}.$$

The reason this is bijective in the case of GL_n is that all stabilizers are connected, so there are no non-trivial equivariant local systems.

Consider the situation in types B_n and C_n.

Theorem (Gerstenhaber)

Nilpotent orbits of SO_{2n+1} and Sp_{2n} are classified by Jordan form:

$$SO_{2n+1} \setminus \mathcal{N}(\mathfrak{so}_{2n+1}) \leftrightarrow \left\{ \begin{array}{c} \textit{partitions of } 2n+1 \textit{ in which every} \\ \textit{even part occurs with even multiplicity} \end{array} \right\}$$

$$Sp_{2n} \setminus \mathcal{N}(\mathfrak{sp}_{2n}) \leftrightarrow \left\{ \begin{array}{c} \textit{partitions of } 2n \textit{ in which every} \\ \textit{odd part occurs with even multiplicity} \end{array} \right\}$$

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Example

Take $n = 3$. The possible Jordan types for $\mathcal{N}(\mathfrak{so}_7)$ are:

$$(7),\ (51^2),\ (3^2 1),\ (32^2),\ (31^4),\ (2^2 1^3),\ (1^7),$$

and those for $\mathcal{N}(\mathfrak{sp}_6)$ are:

$$(6), (42),\ (41^2),\ (3^2),\ (2^3),\ (2^2 1^2),\ (21^4),\ (1^6).$$

The Springer correspondence gives a new parametrization of these nilpotent orbits in terms of irreps of the common Weyl group

$$W(B_n) = W(C_n) = \{\pm 1\} \wr S_n.$$

These irreps are labelled by the set $\mathcal{Q}_n$ of bipartitions $(\mu; \nu)$ of n.

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Theorem (Lusztig, Shoji)

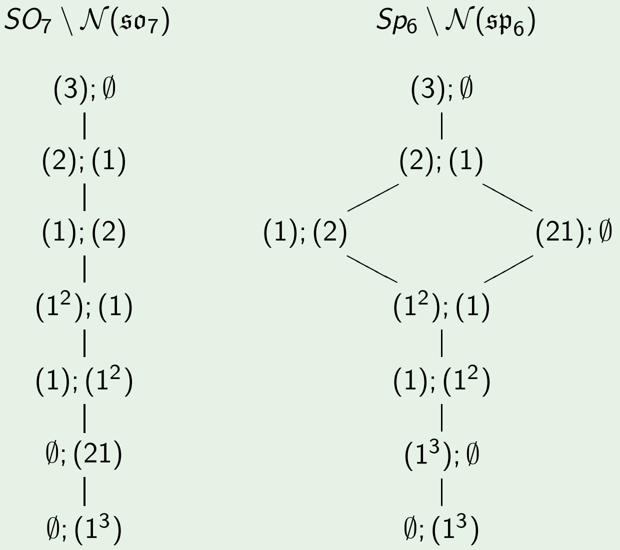
The Springer parametrizations by bipartitions

$$SO_{2n+1} \setminus \mathcal{N}(\mathfrak{so}_{2n+1}) \longleftrightarrow \{(\mu;\nu) \in \mathcal{Q}_n \,|\, \mu_i \geq \nu_i - 2,\ \nu_i \geq \mu_{i+1}\}$$
$$Sp_{2n} \setminus \mathcal{N}(\mathfrak{sp}_{2n}) \longleftrightarrow \{(\mu;\nu) \in \mathcal{Q}_n \,|\, \mu_i \geq \nu_i - 1,\ \nu_i \geq \mu_{i+1} - 1\}$$

are obtained from the previous by taking 2-quotients of partitions, where the conventions are such that the open orbit corresponds to $((n);\emptyset)$ *and the zero orbit corresponds to* $(\emptyset;(1^n))$.

Write $\mathcal{Q}_n^B$ and $\mathcal{Q}_n^C$ for these subsets of $\mathcal{Q}_n$, and $\mathcal{O}^B_{\mu;\nu}$ or $\mathcal{O}^C_{\mu;\nu}$ for the orbit corresponding to $(\mu;\nu) \in \mathcal{Q}_n^B$ or $(\mu;\nu) \in \mathcal{Q}_n^C$ respectively.

Example ($n = 3$, ordering by orbit closure inclusion)



Kato's exotic Springer correspondence

The Sp_{2n}-invariant complement of $\mathfrak{sp}_{2n}$ in $\mathfrak{gl}_{2n}$ is

$$S = \{x \in \mathrm{Mat}_{2n} \mid \langle xv, w\rangle = \langle v, xw\rangle,\ \forall v, w \in \mathbb{C}^{2n}\}.$$

Let $\mathcal{N}(S) = S \cap \mathcal{N}(\mathfrak{gl}_{2n})$ be its Hilbert nullcone.

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Theorem (Kostant, Sekiguchi)

The Sp_{2n}-orbits in $\mathcal{N}(S)$ are classified by Jordan form:

$$Sp_{2n} \setminus \mathcal{N}(S) \longleftrightarrow \left\{ \begin{array}{c} \textit{partitions of } 2n \textit{ in which every} \\ \textit{part occurs with even multiplicity} \end{array} \right\} \longleftrightarrow \mathcal{P}_n.$$

In terms of real groups, $Sp_{2n} \setminus \mathcal{N}(S) \longleftrightarrow GL_n(\mathbb{H}) \setminus \mathcal{N}(\mathfrak{gl}_n(\mathbb{H}))$.

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The natural resolution of $\mathcal{N}(S)$, as in Hesselink's theory, has fibres

$$\{V_0 \subset V_1 \subset \cdots \subset V_n \,|\, \dim V_i = 2i,\ \langle V_i, V_{n-i}\rangle = 0,\ x(V_i) \subseteq V_{i-1}\}.$$

Note that this resolution uses partial flags.

Syu Kato's exotic nilpotent cone of type C_n is

$$\mathfrak{N} := \mathcal{N}(\mathbb{C}^{2n} \oplus S) = \{(v,x) \mid v \in \mathbb{C}^{2n}, x \in \mathcal{N}(S)\}.$$

The natural resolution ψ of $\mathfrak{N}$ has the following fibre over (v,x):

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with complete flags as in the Springer resolution of $\mathcal{N}(\mathfrak{sp}_{2n})$.

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Theorem (Kato, Duke Math. J. 2009)

The endomorphism algebra of $R\psi_*\mathbb{C}$ in the derived category on $\mathfrak{N}$ is isomorphic to the group algebra of $W(C_n)$. Moreover, all stabilizers in Sp_{2n} of points in $\mathfrak{N}$ are connected. Hence we have an exotic Springer correspondence $Sp_{2n} \setminus \mathfrak{N} \longleftrightarrow \{\text{irreps of } W(C_n)\}$.

Note that this exotic Springer correspondence is 'cleaner' than the usual Springer correspondence in type C: it is more like type A. Write $\mathbb{O}_{\mu;\nu}$ for the orbit in $\mathfrak{N}$ corresponding to $(\mu; \nu) \in \mathcal{Q}_n$.

Geometry of the exotic nilpotent cone

Theorem (Achar–H., Advances in Math. 2008)

For $(\rho;\sigma), (\mu;\nu) \in \mathcal{Q}_n$,

$$\mathbb{O}_{\rho;\sigma} \subseteq \overline{\mathbb{O}_{\mu;\nu}} \iff \begin{array}{rcl} \rho_1 & \leq & \mu_1, \\ \rho_1 + \sigma_1 & \leq & \mu_1 + \nu_1, \\ \rho_1 + \sigma_1 + \rho_2 & \leq & \mu_1 + \nu_1 + \mu_2, \\ \vdots \quad \vdots & \vdots & \vdots \quad \vdots \end{array}$$

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$$\mathbb{O}_{\rho;\sigma}\subseteq\overline{\mathbb{O}_{\mu;\nu}}\quad\Longleftrightarrow\quad\begin{array}{rcl}\rho_1&\leq&\mu_1,\\ \rho_1+\sigma_1&\leq&\mu_1+\nu_1,\\ \rho_1+\sigma_1+\rho_2&\leq&\mu_1+\nu_1+\mu_2,\\ \vdots\quad\vdots&\vdots&\vdots\quad\vdots\end{array}$$

This partial order on $\mathcal{Q}_n$ arose in Shoji's study of 'limit symbols', in which he also defined type-B/C analogues of Kostka polynomials. We conjecture that these give the intersection cohomology:

$$\sum_i \dim IH^{2i}_{\mathbb{O}_{\rho;\sigma}}(\overline{\mathbb{O}_{\mu;\nu}})\,q^i \stackrel{?}{=} q^{b(\rho;\sigma)-b(\mu;\nu)}K_{(\mu;\nu),(\rho;\sigma)}(q^{-1}).$$

We proved the special case $\mu=\emptyset$. An obstacle to the general proof is that we do not know a resolution of every orbit closure.

The poset $\mathcal{Q}_n$ contains the posets $\mathcal{Q}_n^B$ and $\mathcal{Q}_n^C$, corresponding to nilpotent orbits in $\mathcal{N}(\mathfrak{so}_{2n+1})$ and $\mathcal{N}(\mathfrak{sp}_{2n})$ respectively. We can define two partitions of $\mathfrak{N}$, into type-B pieces and type-C pieces:

$$\mathfrak{N} = \bigcup_{(\mu;\nu)\in\mathcal{Q}_n^B} \mathbb{T}^B_{\mu;\nu} \quad \text{where} \quad \mathbb{T}^B_{\mu;\nu} = \overline{\mathbb{O}_{\mu;\nu}} \setminus \bigcup_{\substack{(\tau;\upsilon)\in\mathcal{Q}_n^B \\ (\tau;\upsilon)<(\mu;\nu)}} \overline{\mathbb{O}_{\tau;\upsilon}},$$

$$\mathfrak{N} = \bigcup_{(\mu;\nu)\in\mathcal{Q}_n^C} \mathbb{T}^C_{\mu;\nu} \quad \text{where} \quad \mathbb{T}^C_{\mu;\nu} = \overline{\mathbb{O}_{\mu;\nu}} \setminus \bigcup_{\substack{(\tau;\upsilon)\in\mathcal{Q}_n^C \\ (\tau;\upsilon)<(\mu;\nu)}} \overline{\mathbb{O}_{\tau;\upsilon}}.$$

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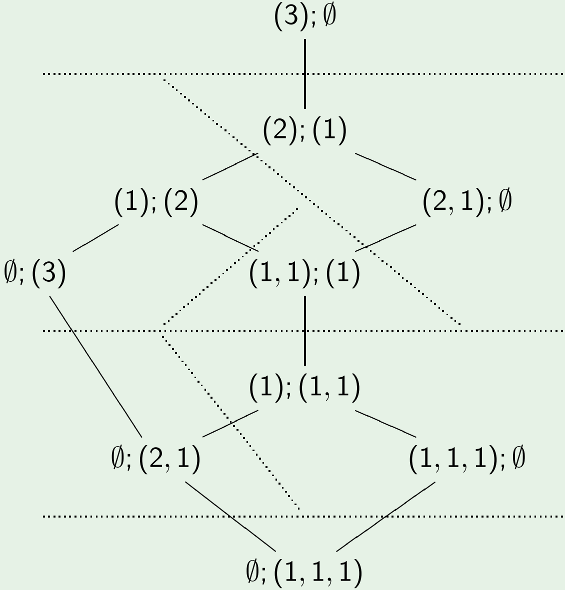
Theorem (Achar–H.–Sommers, arXiv:1001.4283)

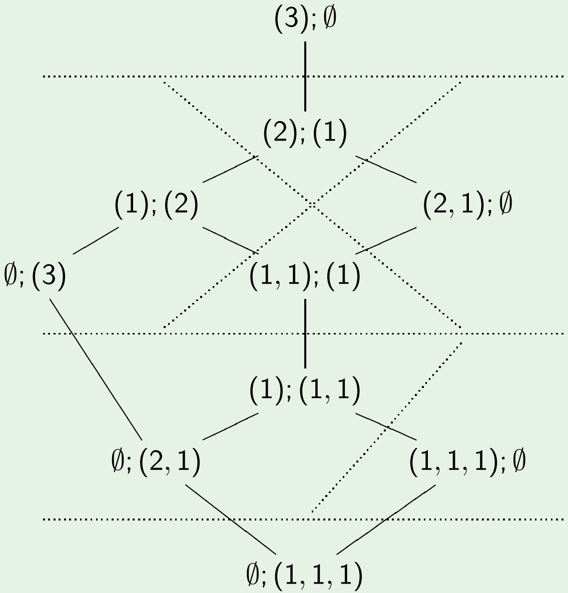
1. *For every $(\mu;\nu)\in\mathcal{Q}_n^B$, the piece $\mathbb{T}^B_{\mu;\nu}\subset\mathfrak{N}$ is smooth and has the same equivariant cohomology as $\mathcal{O}^B_{\mu;\nu}\subset\mathcal{N}(\mathfrak{so}_{2n+1})$.*
2. *For every $(\mu;\nu)\in\mathcal{Q}_n^C$, the piece $\mathbb{T}^C_{\mu;\nu}\subset\mathfrak{N}$ is smooth and has the same equivariant cohomology as $\mathcal{O}^C_{\mu;\nu}\subset\mathcal{N}(\mathfrak{sp}_{2n})$.*

For "equivariant cohomology", one can put "number of $\mathbb{F}_q$-points".

Example ($n = 3$, type-B pieces)



Example ($n = 3$, type-C pieces)



Why is the nullcone $\mathfrak{N} = \mathcal{N}(\mathbb{C}^{2n} \oplus S) = \mathcal{N}(\mathbb{C}^{2n} \oplus \Lambda^2(\mathbb{C}^{2n}))$ special, and why does it seem just as close to type B as to type C? A clue is the fact that the nonzero weights of $\mathbb{C}^{2n} \oplus \Lambda^2(\mathbb{C}^{2n})$ form a root system of type B_n, and they each have multiplicity 1.

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Proposition

Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. There is a bijective isogeny $SO_{2n+1}(\mathbb{F}) \to Sp_{2n}(\mathbb{F})$, and short exact sequences:

$$0 \longrightarrow \mathbb{F}^{2n} \longrightarrow \mathfrak{so}_{2n+1}(\mathbb{F}) \longrightarrow \Lambda^2(\mathbb{F}^{2n}) \longrightarrow 0$$

$$0 \longrightarrow \Lambda^2(\mathbb{F}^{2n}) \longrightarrow \mathfrak{sp}_{2n}(\mathbb{F}) \longrightarrow (\mathbb{F}^{2n})^{(1)} \longrightarrow 0$$

So $\mathbb{F}^{2n} \oplus \Lambda^2(\mathbb{F}^{2n})$ is an 'exotic adjoint representation', obtained by degenerating $\mathfrak{so}_{2n+1}(\mathbb{F})$ or $\mathfrak{sp}_{2n}(\mathbb{F})$. Similarly, $\mathfrak{N}(\mathbb{F})$ can be viewed as a degeneration of $\mathcal{N}(\mathfrak{so}_{2n+1}(\mathbb{F}))$ or of $\mathcal{N}(\mathfrak{sp}_{2n}(\mathbb{F}))$. This implies the desired equalities of the numbers of $\mathbb{F}_q$-points when $q = 2^s$.

To prove smoothness of the type-B and type-C pieces of $\mathfrak{N}$ in any characteristic, we gave resolutions of $\overline{\mathbb{O}_{\mu;\nu}}$ for $(\mu;\nu) \in \mathcal{Q}_{\boldsymbol{n}}^B \cup \mathcal{Q}_{\boldsymbol{n}}^C$. These were inspired by the Jacobson–Morozov resolutions of $\overline{\mathcal{O}_{\mu;\nu}^B}$ and $\overline{\mathcal{O}_{\mu;\nu}^C}$, and are in the spirit of Hesselink's general theory.

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Let $\mathcal{Q}_n^\circ = \mathcal{Q}_n^B \cap \mathcal{Q}_n^C$ be the set labelling Lusztig's special irreps. All cones have a partition into special pieces indexed by $(\mu;\nu) \in \mathcal{Q}_n^\circ$:

$$\mathcal{N}(\mathfrak{so}_{2n+1}) = \bigcup \mathcal{S}_{\mu;\nu}^B,\ \mathcal{N}(\mathfrak{sp}_{2n}) = \bigcup \mathcal{S}_{\mu;\nu}^C,\ \mathfrak{N} = \bigcup \mathbb{S}_{\mu;\nu}.$$

It follows immediately from our result that $\mathcal{S}_{\mu;\nu}^B$, $\mathcal{S}_{\mu;\nu}^C$, and $\mathbb{S}_{\mu;\nu}$ have the same equivariant cohomology (or number of $\mathbb{F}_q$-points). Lusztig had already proved this for $\mathcal{S}_{\mu;\nu}^B$ and $\mathcal{S}_{\mu;\nu}^C$.

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Theorem (Kraft–Procesi)

In characteristic $\neq 2$, $\mathcal{S}_{\mu;\nu}^B$ and $\mathcal{S}_{\mu;\nu}^C$ are rationally smooth.

So it is natural to conjecture that $\mathbb{S}_{\mu;\nu}$ is rationally smooth (or possibly smooth?) in any characteristic.