

# Species over a Finite Field

Anthony Henderson  
University of Sydney  
anthonyh@maths.usyd.edu.au

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## Joyal's theory of species

Let  $\mathbf{B}$  be the category of finite sets and bijections.

A **species** is a functor  $A : \mathbf{B} \rightarrow \mathbf{B}$ .

For all  $n \geq 0$ ,  $A[n]$  is a finite set on which  $S_n$  acts.

**E.g.** (writing only the definition on objects  $I$  of  $\mathbf{B}$ )

- $\mathcal{P}(I) = \{\text{subsets of } I\}$
- $\text{Par}(I) = \{\text{set partitions of } I\}$
- $E_{\geq 2}(I) = \{I\}$  if  $|I| \geq 2$ , or  $\emptyset$  otherwise; sim.  $E$ ,  $E_1$
- $\mathcal{T}(I) = \{\text{rooted trees with leaves labelled by } I\}$

(See Bergeron et al., *Combinatorial Species ....*)

The **cycle index series**  $Z_A \in \mathbb{Q}[[x_1, x_2, \dots]]$  is

$$Z_A = \sum_{n \geq 0} \frac{1}{n!} \sum_{w \in S_n} |A[n]^w| x_w,$$

where if  $w$  has cycle-type  $(i^{m_i})$ ,  $x_w = \prod x_i^{m_i}$ .  
Setting  $x_2, x_3, \dots \rightarrow 0$ ,  $x_1 \rightarrow x$ , we get

$$A(x) = \sum_{n \geq 0} \frac{|A[n]|}{n!} x^n.$$

Addition and multiplication of species:

$$(A + B)(I) = A(I) \amalg B(I),$$

$$(A \cdot B)(I) = \coprod_{I=I_1 \amalg I_2} A(I_1) \times B(I_2).$$

Easy to see that

$$Z_{A+B} = Z_A + Z_B, \quad Z_{A \cdot B} = Z_A Z_B.$$

**E.g.**  $\mathcal{P} \cong E \cdot E$ , so

$$Z_{\mathcal{P}} = Z_E^2 = \exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \dots\right)^2.$$

Substitution of species:

$$(A \circ B)(I) = \coprod_{\pi \in \text{Par}(I)} \left( A(\pi) \times \prod_{J \in \pi} B(J) \right).$$

Plethysm of  $f, g \in \mathbb{Q}[[x_1, x_2, \dots]]$  ( $g(0) = 0$ ):

$$(f \circ g)(x_1, x_2, \dots) = f(g(x_1, x_2, \dots), g(x_2, x_4, \dots), \dots).$$

**Thm 1** (Joyal)  $A, B$  species,  $B(\emptyset) = \emptyset$ :

$$Z_{A \circ B} = Z_A \circ Z_B.$$

**E.g.** We have

$$\mathcal{T} \cong E_1 + E_{\geq 2} \circ \mathcal{T},$$

$$\text{so } Z_{\mathcal{T}} = x_1 + \exp\left(\sum_{n \geq 1} \frac{x_n \circ Z_{\mathcal{T}}}{n}\right) - 1 - Z_{\mathcal{T}},$$

$$\text{and } 1 - x + 2\mathcal{T}(x) = \exp(\mathcal{T}(x)).$$

## Species over $\mathbb{F}_q$

Let  $\mathbb{F}_q$  be a finite field with  $q$  elements.

Let  $\mathbf{Var}$  be the category of  $\mathbb{F}_q$ -varieties.

A **species over  $\mathbb{F}_q$**  is a functor  $X : \mathbf{B} \rightarrow \mathbf{Var}$ .

For all  $n \geq 0$ ,  $X[n](\overline{\mathbb{F}_q})$  is a set with commuting actions of  $S_n$  and Frobenius  $F$ . Define

$$Z_X = (Z_X^{(1)}, Z_X^{(2)}, Z_X^{(3)}, \dots), \text{ where}$$

$$Z_X^{(1)} = \sum_{n \geq 0} \frac{1}{n!} \sum_{w \in S_n} |X[n](\overline{\mathbb{F}_q})^{wF}| x_w,$$

and  $Z_X^{(i)}$  is the same thing with  $F$  replaced by  $F^i$ .

Analogous definitions of  $+$ ,  $\cdot$  and  $\circ$ . Plethysm:

$$(f \circ g)^{(i)} = f^{(i)}(g^{(i)}(x_1, x_2, \dots), g^{(2i)}(x_2, x_4, \dots), \dots).$$

**Thm 2**  $X, Y$  species over  $\mathbb{F}_q$ ,  $Y(\emptyset) = \emptyset$ :

$$Z_{X \circ Y} = Z_X \circ Z_Y.$$

For  $X \in \mathbf{Var}$ , define species  $T_X, M_X$  by

$$\begin{aligned} T_X(I) &= X^I = \{I\text{-tuples of points of } X\}, \\ M_X(I) &= \{I\text{-tuples of distinct points of } X\}. \end{aligned}$$

Apart from a topological difference not affecting  $|(-)^{w_F}|$ ,

$$\begin{aligned} T_X &\approx M_X \circ E_{\geq 1}, \\ \text{so } Z_{T_X} &= Z_{M_X} \circ Z_{E_{\geq 1}}, \\ \text{so } Z_{M_X} &= Z_{T_X} \circ \sum_{m \geq 1} \frac{\mu(m)}{m} \log(1 + x_m). \end{aligned}$$

Let  $\mathcal{M}(I)$  be the moduli space of smooth genus 0 curves over  $\mathbb{F}_q$  with marked points labelled by  $I \cup \{\infty\}$ ,  $\overline{\mathcal{M}}(I)$  the same for stable curves. Then

$$\begin{aligned} \overline{\mathcal{M}}(I) &\approx \coprod_{T \in \mathcal{T}(I)} \prod_{v \in \text{Int}(T)} \mathcal{M}(\text{Fibre}(v)), \\ \text{so } \overline{\mathcal{M}} &\approx E_1 + \mathcal{M} \circ \overline{\mathcal{M}}, \\ \text{so } Z_{\overline{\mathcal{M}}} &= p_1 + Z_{\mathcal{M}} \circ Z_{\overline{\mathcal{M}}}. \end{aligned}$$

Since  $Z_{\mathcal{M}} = \frac{Z_{M_{\mathbb{A}^1}} - 1 - p_1}{q-1}$ , this is a recursion for  $Z_{\overline{\mathcal{M}}}$ .