

The toric variety of the symmetric group

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Outline

The toric variety \mathcal{T}_n

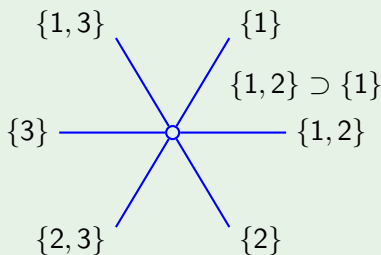
The cohomology of \mathcal{T}_n

The cohomology of the real locus $\mathcal{T}_n(\mathbb{R})$

The toric variety \mathcal{T}_n

The **Coxeter arrangement** of S_n is the hyperplane arrangement defining the reflection representation of S_n on $\mathbb{R}^n/\mathbb{R}(1, 1, \dots, 1)$. Its rays are in bijection with nonempty proper subsets $K \subset [n]$, its m -dimensional facets with chains $[n] \supset K_1 \supset K_2 \supset \dots \supset K_m \supset \emptyset$. So its chambers are in bijection with S_n itself.

Example ($n = 3$)



Every facet is the \mathbb{R}_+ -span of a subset of $\mathbb{Z}^n/\mathbb{Z}(1, 1, \dots, 1)$. Using an abstract construction of gluing affine pieces, one constructs the **Coxeter toric variety** \mathcal{T}_n equipped with an action of the torus

$$T_n = (\mathbb{C}^\times)^n / \{(a, a, \dots, a) \mid a \in \mathbb{C}^\times\}.$$

The T_n -orbits in \mathcal{T}_n are in bijection with the facets, and we have

$$\mathcal{O}_{K_1, K_2, \dots, K_m} \cong T_{[n] \setminus K_1} \times T_{K_1 \setminus K_2} \times \dots \times T_{K_m} \quad (\text{codim } m).$$

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Remarks

1. As the reflection group of the arrangement, S_n acts on \mathcal{T}_n , compatibly with its action on T_n by permuting coordinates.
2. By general theory, \mathcal{T}_n is nonsingular and projective.
3. One could similarly define \mathcal{T}_W for any Weyl group W .

There is a more concrete way to define \mathcal{T}_n . For nonempty $J \subseteq [n]$, let $\mathbb{P}^J = \{[a_j^J]_{j \in J} \mid a_j^J \in \mathbb{C} \text{ not all } 0\}$ (homogeneous coordinates).

Proposition (De Concini–Procesi 1985)

There is an isomorphism between \mathcal{T}_n and the variety

$$Y_n = \{([a_j^J]) \in \prod_{J \subseteq [n]} \mathbb{P}^J \mid (a_i^J)_{i \in I} \propto (a_i^I)_{i \in I} \text{ for all } I \subset J\}$$

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which respects the actions of S_n and T_n on each.

Example ($n = 3$)

$Y_3 = \{([a_1 : a_2 : a_3], [b_1 : b_2], [c_1 : c_3], [d_2 : d_3]) \mid$
 $a_1 b_2 = a_2 b_1, a_1 c_3 = a_3 c_1, a_2 d_3 = a_3 d_2\}$
is the blow-up of \mathbb{P}^2 at the points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

Y_n can be obtained from \mathbb{P}^{n-1} by a sequence of blow-ups:

- ▶ blow up \mathbb{P}^{n-1} at the points $[\cdots : 0 : 1 : 0 : \cdots]$ to give Z_n^1 ;
- ▶ blow up Z_n^1 along the preimages of the lines $[\cdots : 0 : * : 0 : \cdots : 0 : * : 0 : \cdots]$ to give Z_n^2 ;
- ▶ blow up Z_n^2 along the preimages of the planes $[\cdots : 0 : * : 0 : \cdots : 0 : * : 0 : \cdots : 0 : * : 0 : \cdots]$ to give Z_n^3 ;

and so on until $Z_n^{n-2} = Y_n$. In the step giving Z_n^3 , the preimage of each plane is a copy of Y_3 , and so forth; so the family of varieties Y_n has some recursive structure.

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The variety Y_n is the (compact) **De Concini–Procesi model** of the arrangement of coordinate subspaces of \mathbb{C}^n .

Remark

The De Concini–Procesi model of the braid arrangement we started with is something different, though analogous: the moduli space $\overline{\mathcal{M}}_{0,n+1}$ of stable rational curves with marked points.

The cohomology of \mathcal{T}_n

As with any nonsingular projective toric variety, $H_{\text{odd}}(\mathcal{T}_n) = 0$.
 $H^{\text{even}}(\mathcal{T}_n, \mathbb{Q})$ is the quotient of the Stanley–Reisner algebra

$$\mathbb{Q}\langle e_J, \emptyset \neq J \subset [n] \mid e_J e_K = 0 \text{ if } J \not\subseteq K, K \not\subseteq J \rangle$$

by the linear relations which say that $\sum_{J \ni i} e_J$ is independent of i .

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by the linear relations which say that $\sum_{J \ni i} e_J$ is independent of i .
One can derive a generating function for the Betti numbers:

$$1 + \sum_{n \geq 1} \sum_j \dim H^{2j}(\mathcal{T}_n, \mathbb{Q}) t^j \frac{z^n}{n!} = \frac{(1-t)\exp(z)}{\exp(tz) - t \exp(z)},$$

and deduce that $\sum \dim H^{2j}(\mathcal{T}_n, \mathbb{Q}) t^j$ is the Eulerian polynomial

$$A_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{exc}(\sigma)}, \text{ where } \text{exc}(\sigma) = |\{i \in [n] \mid \sigma(i) > i\}|.$$

For a fuller statement, regard $H^{2j}(\mathcal{T}_n, \mathbb{Q})$ as a representation of S_n and define the **Eulerian symmetric function** $Q_{n,j} = \text{ch } H^{2j}(\mathcal{T}_n, \mathbb{Q})$.

Theorem (Procesi–Stanley 1985)

Let $H(z)$ denote the power series $\sum_{n \geq 0} h_n z^n$. Then

$$1 + \sum_{n \geq 1} \sum_j Q_{n,j} t^j z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)}.$$

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Define a **marked set** to be a finite set B such that $|B| \geq 2$ together with an integer k (the **mark**) such that $1 \leq k < |B|$.

Theorem (Stembridge 1992)

$H^{2j}(\mathcal{T}_n, \mathbb{Q})$ is isomorphic to $\pi_{n,j}$, the permutation representation of S_n on the set of sequences (B_1, B_2, \dots, B_m) of disjoint marked subsets of $[n]$ whose marks add up to j .

Remarks

1. By Poincaré duality of \mathcal{T}_n and the hard Lefschetz theorem, the sequence $Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-2}, Q_{n,n-1}$ is palindromic and **Schur-unimodal** in the sense that $Q_{n,j} - Q_{n,j-1}$ is an \mathbb{N} -linear combination of Schur functions for $1 \leq j \leq \frac{n-1}{2}$.
2. Stembridge also gave a combinatorial formula for the multiplicity of each irreducible representation in $\pi_{n,j}$.

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2. Stembridge also gave a combinatorial formula for the multiplicity of each irreducible representation in $\pi_{n,j}$.

Let $F_{B,n}$ be the fundamental quasisymmetric function of degree n associated to $B \subseteq [n-1]$. Shareshian and Wachs found a way to attach to each $\sigma \in S_n$ a subset $\text{DEX}(\sigma) \subseteq [n-1]$ so that

$$Q_{n,j} = \sum_{\substack{\sigma \in S_n \\ \text{exc}(\sigma)=j}} F_{\text{DEX}(\sigma),n}.$$

For any partition $\lambda \vdash n$, they define $Q_{\lambda,j}$ by restricting the sum on the right-hand side to σ of cycle-type λ .

Theorem (Shareshian–Wachs 2010)

1. $Q_{\lambda,j}$ is a symmetric function, not just quasisymmetric.
2. $Q_{\lambda,j} = Q_{\lambda, n - m_1(\lambda) - j}$.

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Theorem (H.–Wachs)

The sequence $Q_{\lambda,0}, \dots, Q_{\lambda, n-m_1(\lambda)}$ is Schur-unimodal.

In proving this we construct representations $V_{\lambda,j}$ of S_n such that $Q_{\lambda,j} = \text{ch } V_{\lambda,j}$. These are similar to Stembridge's $\pi_{n,j}$ but are not permutation representations. Shareshian and Wachs observed that

$$\text{Res}_{S_n}^{S_{n+1}}(V_{(n+1),j+1}) \cong \pi_{n,j}.$$

Therefore the representation of S_n on $H^{2j}(\mathcal{T}_n, \mathbb{Q})$ extends to S_{n+1} . Is there a geometric way to see this (as with $\overline{\mathcal{M}}_{0,n+1}$)?

The cohomology of the real locus $\mathcal{T}_n(\mathbb{R})$

From general theory,

$$\sum_j \dim H^j(\mathcal{T}_n(\mathbb{R}), \mathbb{F}_2) t^j = \sum_j \dim H^{2j}(\mathcal{T}_n, \mathbb{Q}) t^j = A_n(t).$$

But the integral and rational cohomology of $\mathcal{T}_n(\mathbb{R})$ are completely different from the \mathbb{F}_2 -cohomology. There is no general theory of such cohomology of real toric varieties, but there is a theory, due to Rains, for real De Concini–Procesi models. So we use $Y_n \cong \mathcal{T}_n$.

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$$H^\bullet(\mathbb{P}^{m-1}(\mathbb{R}), \mathbb{Q}) \cong \begin{cases} H^\bullet(\mathbb{S}^{m-1}, \mathbb{Q}), & \text{if } m \text{ is even,} \\ H^\bullet(\text{point}, \mathbb{Q}), & \text{if } m \text{ is odd.} \end{cases}$$

So in the blow-up construction of Y_n , only the steps from Z_n^{j-1} to Z_n^j when $n - j$ is **even** (blowing up along subvarieties of even codimension) have an effect on rational cohomology.

For such a step, there is a long exact sequence

$$H^\bullet(Z_n^{j-1}(\mathbb{R}), \mathbb{Q}) \longrightarrow H^\bullet(Z_n^j(\mathbb{R}), \mathbb{Q}) \longrightarrow H^{\bullet-(n-j)+1}(Y_j(\mathbb{R}), \mathbb{Q})^{\oplus \binom{n}{j}}$$

so $H^\bullet(Y_n(\mathbb{R}), \mathbb{Q})$ depends recursively on $H^\bullet(Y_j(\mathbb{R}), \mathbb{Q})$.

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so $H^\bullet(Y_n(\mathbb{R}), \mathbb{Q})$ depends recursively on $H^\bullet(Y_j(\mathbb{R}), \mathbb{Q})$.

As a special case of Rains' theorem, the solution of this recursion is given by the homology of the poset of even-size subsets of $[n]$.

Theorem (H.)

Set $G(z) = \sum_{n \text{ even}} e_n z^n$, and $H(z)$ as before. Then

$$1 + \sum_{n \geq 1} \sum_j \text{ch } H^j(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) (-t)^j z^n = H(z) G(t^{1/2} z)^{-1}.$$

As a consequence, $\dim H^j(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) = \binom{n}{2j} A_{2j}$ where A_{2j} is the coefficient of $\frac{z^{2j}}{(2j)!}$ in $\sec(z)$.

The degree-1 cohomology $H^1(\mathcal{T}_n(\mathbb{R}), \mathbb{Q})$ has dimension $\binom{n}{2}$, and a basis of it is easy to construct from the $\binom{n}{2}$ projections $Y_n \rightarrow \mathbb{P}^1$. However, these elements do not generate the cohomology ring (in contrast to the situation for $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$).

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Proposition (H.)

Assume $n \geq 4$. Let C be the span in $H^2(\mathcal{T}_n(\mathbb{R}), \mathbb{Q})$ of all products of two elements of $H^1(\mathcal{T}_n(\mathbb{R}), \mathbb{Q})$. Then

$$C \cong \text{Ind}_{S_4 \times S_{n-4}}^{S_n} (V_{(2,1,1)} \boxtimes \text{triv}).$$

In particular, $\dim C = 3\binom{n}{4} < 5\binom{n}{4} = \dim H^2(\mathcal{T}_n(\mathbb{R}), \mathbb{Q})$.

It is easy to deduce that the representation of S_n on C cannot be extended to S_{n+1} . So there appears to be no geometric version of the Shareshian–Wachs observation.