The Symmetric Group Representation on Cohomology of the Regular Elements of a Maximal Torus of the Special Linear Group

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1. Some cohomology representations of S_n

Consider the following smooth complex varieties:

$$T(1,n) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \,|\, z_i \neq 0, z_i \neq z_j\},$$

$$\mathbb{P}T(1,n) = \{[z_1 : \cdots : z_n] \in \mathbb{P}^{n-1} \,|\, z_i \neq 0, z_i \neq z_j\},$$

$$ST(1,n) = \{(z_1, \cdots, z_n) \in T(1,n) \,|\, z_1 z_2 \cdots z_n = 1\}.$$

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The character of S_n on $H^i(T(1,n))$ was calculated by Lehrer (1987); this gives a formula for $H^i(\mathbb{P}T(1,n))$ also. These two varieties are hyperplane complements, unlike $ST(1,n)$.

Since μ_n acts on $ST(1,n)$ with quotient $\mathbb{P}T(1,n)$,

$$H^i(ST(1,n)) = \bigoplus_{\chi \in \widehat{\mu_n}} H^i(ST(1,n))_\chi = \bigoplus_{\chi \in \widehat{\mu_n}} H^i(\mathbb{P}T(1,n), \mathcal{L}_\chi).$$

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We have an isomorphism

$$\begin{array}{ccc} T^{(n)}(1,n) & \xrightarrow{\sim} & ST(1,n) \times \mathbb{C}^\times \\ (z_1, \cdots, z_n, z) & \longrightarrow & (z_1 z^{-1}, \cdots, z_n z^{-1}, z), \end{array}$$

so we need only compute the character of $H^i(T^{(n)}(1,n))_\chi$.

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$$\begin{array}{ccccc} T(n,n) & \longrightarrow & T^{(n)}(1,n) & \longrightarrow & T(1,n) \\ (z_1, \cdots, z_n) & \longrightarrow & (z_1^n, \cdots, z_n^n, z_1 \cdots z_n) & \longrightarrow & (z_1^n, \cdots, z_n^n) \end{array}$$

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It follows that $\mathrm{tr}(w, H^i(T^{(n)}(1,n))_\chi)$ equals

$$\sum_{(\zeta_i) \in \mu_n^n} \chi(\zeta_1 \cdots \zeta_n)^{-1} \mathrm{tr}(w(\zeta_1, \cdots, \zeta_n), H^i(T(n,n))).$$

So we are led to consider the action of the wreath product $\mu_n \wr S_n = S_n \ltimes \mu_n^n$ on the hyperplane complement $T(n,n)$.

2. Equivariant Generating Functions

Let $p_\ell(\zeta)$, $\ell \geq 1$, $\zeta \in \mu_n$, be commuting indeterminates.
An element $y \in \mu_n \wr S_m$ has cycle index $p_y = \prod_{\ell,\zeta} p_\ell(\zeta)^{a_\ell(\zeta)}$, where $a_\ell(\zeta)$ is the number of cycles of length ℓ and type ζ.

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$$\mathrm{ch}_{n,m}(M) = \frac{1}{n^m m!} \sum_{y \in \mu_n \wr S_m} \mathrm{tr}(y, M)\, p_y.$$

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The characters of $H^i(T(n,m))$ for all m, i are encoded in

$$P(n,q) = \sum_{m,i} (-1)^i \, \mathrm{ch}_{n,m}(H_c^i(T(n,m)))\, q^{i-m},$$

an element of $\mathbb{C}[q][\![p_\ell(\zeta) \mid \ell \geq 1, \zeta \in \mu_r]\!]$.

Theorem. (Lehrer 1995, in a different form)

$$\log P(n,q) = \sum_{\substack{d,e \geq 1 \\ \zeta \in \mu_n}} \frac{\mu(d)(q^e - 1)}{nde} \log(1 + p_{de}(\zeta^d)).$$

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We want a similar formula for

$$P(\chi,q) = \sum_{m,i} (-1)^i \,\mathrm{ch}_{1,m}(H_c^i(T^{(n)}(1,m))_\chi)\, q^{i-m},$$

an element of $\mathbb{C}[q][\![p_\ell \mid \ell \geq 1]\!]$.

Viewing $T^{(n)}(1,m)$ as a quotient of $T(n,m)$, we see that

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Plugging in Lehrer's result, we get

$$\begin{aligned}\log P(\chi,q) &= \sum_{\substack{d,e,f\geq 1\\ \zeta\in\mu_n}} \frac{-\mu(d)}{ndef}\chi(\zeta)^{df}(q^e-1)(-p_{de})^f\\ &= \sum_{\substack{d,e,f\geq 1\\ r|df}} \frac{-\mu(d)}{def}(q^e-1)(-p_{de})^f,\end{aligned}$$

where r is the order of χ.

3. A Mysterious Fact

Theorem. If r is the order of $\chi \in \widehat{\mu_n}$, then $H^i(ST(1,n))_\chi$ is isomorphic as a representation of S_n to

$$\varepsilon_n \otimes \mathrm{Ind}_{\mu_r \wr S_{n/r}}^{S_n}(\det{}_{n/r} \otimes H^{i-n+n/r}(\mathbb{P}T(r,n/r))).$$

($\varepsilon_n =$ sign char of S_n, $\det{}_{n/r} =$ determinant on $GL_{n/r}$.)

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Extreme Case. If χ is faithful,

$$H^i(ST(1,n))_\chi \cong \begin{cases} \varepsilon_n \otimes \mathrm{Ind}_{\mu_n}^{S_n}(\chi), & \text{if } i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

This representation occurs in many other contexts.