A. Henderson

Geometric Representation Theory Week 2 Lecture 1

Last week: $\left\{\begin{matrix}\text{irreducible}\\ \text{fin.-dim.}\\ \text{reps of } GL_n\end{matrix}\right\}\Big/_{\text{iso}} \longleftrightarrow \left\{\begin{matrix}\text{dominant}\\ \text{integral}\\ \text{weights } \lambda\end{matrix}\right\}$ $\quad \lambda\in\mathbb{Z}^n$, $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$

$$V_\lambda \longleftarrow\!\!\!\shortmid\ \ \lambda$$

This week: Lusztig's construction of a canonical basis for V_λ using a geometric construction of $U_q(\underline{n}^-)$.

Today: algebraic approach to V_λ via universal enveloping algebras.
Ref: Humphreys, 'Introduction to Lie Algebras & Representation Theory

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Recall: V_λ contains a unique (up to scalar) highest-weight vector v_λ of weight λ.

This means:

Ⓐ GL_n version:
$$\begin{pmatrix}\alpha_1 & & 0\\ & \alpha_2 & \\ 0 & & \ddots\ \alpha_n\end{pmatrix} v_\lambda = \alpha_1^{\lambda_1}\cdots\alpha_n^{\lambda_n} v_\lambda$$
$$\alpha_i\in\mathbb{C}^\times$$
$$\begin{pmatrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{pmatrix} v_\lambda = v_\lambda$$

$\Updownarrow$

Ⓑ $\mathfrak{gl}_n$ version (i.e. using the Lie algebra repn $\mathfrak{gl}_n\to\mathfrak{gl}(V_\lambda)$):
$$\begin{pmatrix}a_1 & & 0\\ & a_2 & \\ 0 & & \ddots\ a_n\end{pmatrix} v_\lambda = (\lambda_1 a_1+\cdots+\lambda_n a_n)v_\lambda$$
$$\begin{pmatrix}0 & & *\\ & \ddots & \\ 0 & & 0\end{pmatrix} v_\lambda = 0$$

__Prop__ V_λ is spanned by the elements

$$f_{i_1}f_{i_2}\cdots f_{i_r}v_\lambda, \qquad r\geqslant 0,\ i_1,\ldots,i_r\in\{1,\ldots,n-1\}$$

where $f_i = \begin{array}{cc} & \overset{i}{\downarrow}\\ {\scriptstyle i+1}\rightarrow & \begin{pmatrix}0 & & \\ & \ddots & \\ & 1 & \ddots \\ & & & 0\end{pmatrix}\end{array}$. (Pf: show using bracket relations that the span of these elements is a sub-representation)

NB: $f_{i_1}f_{i_2}\cdots f_{i_r}v_\lambda$ means $f_{i_1}(f_{i_2}(\cdots(f_{i_r}v_\lambda)\cdots))$, nothing to do with the product of the $n\times n$ matrices $f_{i_1}, f_{i_2},\ldots,f_{i_r}$

Recall: f_i decreases weights by $\alpha_i = (0, \cdots, 0, \underset{i}{1}, \underset{i+1}{-1, 0}, \cdots, 0)$.

So $f_{i_1} f_{i_2} \cdots f_{i_r} v_\lambda$ is a weight vector of weight $\lambda - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_r}$,

and so $V_\lambda = \bigoplus_{\substack{\mu \in \mathbb{Z}^n \\ \lambda - \mu \in \mathbb{N}\{\alpha_1, \cdots, \alpha_{n-1}\}}} V_\lambda(\mu)$ (weight space decomposition)

where $V_\lambda(\mu)$ is spanned by $\{ f_{i_1} \cdots f_{i_r} v_\lambda \mid \alpha_{i_1} + \cdots + \alpha_{i_r} = \lambda - \mu \}$.

Example $n = 2$, so only $f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, call it f.

v_λ has weight (λ_1, λ_2)

$f v_\lambda$ — — — $(\lambda_1 - 1, \lambda_2 + 1)$

$f^2 v_\lambda$ — — — $(\lambda_1 - 2, \lambda_2 + 2)$

$\vdots$

$f^{\lambda_1 - \lambda_2} v_\lambda$ — — — (λ_2, λ_1)

(Exercise:) $f^{\lambda_1 - \lambda_2 + 1} v_\lambda = 0$.

So in this case all the weight spaces are one-dimensional.

Corollary for general n,

$$(\dagger) \qquad \boxed{f_i^{\lambda_i - \lambda_{i+1} + 1} v_\lambda = 0} \qquad \forall i \in \{1, \cdots, n-1\}$$

What other linear dependence relations are there on the spanning set of V_λ?

The lower-triangular Lie algebra $\underline{n}^- = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\}$ is generated by $f_1, \cdots, f_{n-1}$:

(for $i < j$, $e_{ji} = \pm [\cdots[[f_i, f_{i+1}], f_{i+2}], \cdots, f_{j-1}]$)

subject to the following Serre relations (and only these and their consequences):

$$(*) \qquad \begin{aligned} [f_i, f_j] &= 0 \quad \text{if } |i-j| \geq 2 \quad (i \neq j \text{ not adjacent}) \\ [f_i, [f_i, f_j]] &= 0 \quad \text{if } |i-j| = 1 \quad (i, j \text{ adjacent}) \end{aligned}$$

Just as: representations of a finite group G $=$ modules over the group algebra $\mathbb{C}G$,

representations of a Lie algebra $\underline{g}$ $=$ modules over the universal enveloping algebra $U(\underline{g}$

$U(\underline{g})$ is the associative unital algebra with the same presentation as $\underline{g}$, where $[\cdot,\cdot]$ is interpreted as commutator.

e.g. $U(\underline{n}^-)$ is generated by $f_1, \cdots, f_{n-1}$ subject to $(*)$, i.e.

$$(*)' \quad \begin{array}{ll} f_i f_j = f_j f_i & \text{if } |i-j| \geq 2 \\ f_i^2 f_j + f_j f_i^2 = 2 f_i f_j f_i & \text{if } |i-j| = 1 \end{array}$$

NB: this is not the same as the associative algebra of lower-triangular matrices (e.g. $\begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix}^2 = 0$ in matrices, $f_1^2 \neq 0$ in $U(\underline{n}^-)$).

<u>Example</u> $n=2$: $U(\underline{n}^-) = \mathbb{C}[f]$ (ring of polys in one variable)

$n=3$: $U(\underline{n}^-) = \mathbb{C}\langle f_1, f_2 \mid [f_1,[f_1,f_2]] = 0 = [f_2,[f_1,f_2]] \rangle$
i.e. $[f_1, f_2]$ is central.

<u>Poincaré-Birkhoff-Witt theorem:</u> if a Lie algebra $\underline{g}$ has basis $\{x_1, x_2, \cdots, x_d\}$, then $U(\underline{g})$ has basis

$$\{ x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} \mid a_1, \cdots, a_d \in \mathbb{N} \}.$$

NB: if you reorder $\{x_1, \cdots, x_d\}$, you get a <u>different</u> basis of $U(\underline{g})$ in general.

<u>Example</u> $n=3$: since $\{f_1, f_2, [f_1,f_2]\}$ is a basis of $\underline{n}^-$,

$$\{ [f_1,f_2]^a f_2^b f_1^c \mid a,b,c \in \mathbb{N} \} \text{ is a basis of } U(\underline{n}^-).$$

But so is $\{ [f_1,f_2]^a f_1^b f_2^c \mid a,b,c \in \mathbb{N} \}$, and this is a <u>different</u> basis.

(<u>Exercise:</u>) $$f_1^{(a)} f_2^{(b)} = \sum_{0 \leq k \leq \min\{a,b\}} [f_1,f_2]^{(k)} f_2^{(b-k)} f_1^{(a-k)}.$$

where $X^{(a)} := \dfrac{X^a}{a!}$

Algebraic construction of V_λ:

We have an isomorphism of $U(\underline{n}^-)$-modules

$$V_\lambda \cong U(\underline{n}^-) \,/\, \underbrace{U(\underline{n}^-)\{ f_i^{\lambda_i-\lambda_{i+1}+1}}_{\text{(left ideal generated by)}} \mid 1 \le i \le n-1 \}$$

$$x v_\lambda \longleftarrow\!\shortmid\; x \quad \text{(well-defined by (†))}.$$

That is, the linear dependence relations on the spanning set $\{f_{i_1} \cdots f_{i_r} v_\lambda\}$ of V_λ are all implied by (†) and (∗).

Example $n=3$, $\lambda=(1,1,0)$:

$$V_{(1,1,0)} \cong U(\underline{n}^-) \,/\, J_1 + J_2$$

where $J_1 = U(\underline{n}^-) f_1 = \operatorname{span}\{ [f_1,f_2]^a f_2^b f_1^c \mid c \geq 1 \}$,
$J_2 = U(\underline{n}^-) f_2^2 = \operatorname{span}\{ [f_1,f_2]^{a'} f_1^{b'} f_2^{c'} \mid c' \geq 2 \}$.

Since neither basis is adapted to both J_1 and J_2, it takes some thought to verify that

(Exercise:) $V_{(1,1,0)}$ has basis $\{ v_{(1,1,0)}, f_2 v_{(1,1,0)}, f_1 f_2 v_{(1,1,0)} \}$.

General defn: if V is a vector space, say that a basis B of V is adapted to the subspace $U \subseteq V$ if $U = \operatorname{span}\{ b \mid b \in B' \}$ for some subset B' of B. Then V/U has basis $\{ b + U \mid b \in B \setminus B' \}$

Lusztig defined a canonical basis $\mathbf{B}$ of $U(\underline{n}^-)$ which is adapted to all left ideals of the form $\overline{U(\underline{n}^-)\{ f_i^{m_i} \mid 1 \le i \le n-1 \}}$ for all $(m_1, \dots, m_{n-1}) \in \mathbb{N}^{n-1}$.
So $\mathbf{B}$ simultaneously induces bases in all the V_λ's.

Example $n=2$: canonical basis of $U(\underline{n}^-) = \mathbb{C}[f]$ is $\{ f^{(a)} \mid a \in \mathbb{N} \}$.

$$n=3: \quad \mathbf{B} = \{ f_1^{(a)} f_2^{(b)} f_1^{(c)} \mid a,b,c \in \mathbb{N},\ b \geq a+c \} \cup \{ f_2^{(a')} f_1^{(b')} f_2^{(c')} \mid a',b',c' \in \mathbb{N},\ b' \geq a'+c' \}$$

the sole intersection being that

$$f_1^{(a)} f_2^{(a+c)} f_1^{(c)} = f_2^{(c)} f_1^{(a+c)} f_2^{(c)} \quad \forall a,c \in \mathbb{N}.$$

Geometric Representation Theory Week 2 Lecture 2

Yesterday: fin.dim. irred. reps of GL_n arise as quotients of $U(\underline{n}^-)$.

Today: $U_q(\underline{n}^-)$ and Lusztig's geometric construction of it.
(& tomorrow)
Ref: Lusztig, 'Canonical bases arising from quantized enveloping algebras', JAMS 1990.
Schiffmann, 'Lectures on canonical and crystal bases', arXiv: 0910.4460

The quantized universal enveloping algebra $U_q(\underline{n}^-)$ is the associative unital algebra with generators $f_1, \cdots, f_{n-1}$ and relations

$$(*)_q \qquad \begin{aligned} f_if_j &= f_jf_i && \text{if } |i-j|\geq 2 \\ f_i^2f_j + f_jf_i^2 &= [2]_q f_if_jf_i && \text{if } |i-j| = 1 \end{aligned}$$

where $[2]_q = q+q^{-1}$. (In general, $[m]_q = \underbrace{q^{m-1}+q^{m-3}+\cdots+q^{1-m}}_{m \text{ terms}}$.)
"quantum integers"

Here q is a formal variable, i.e. $U_q(\underline{n}^-)$ is an algebra over $\mathbb{C}(q)$.

To recover $U(\underline{n}^-)$, we want to set $q \mapsto 1$ so that $[2]_q$ becomes 2. This doesn't make sense over $\mathbb{C}(q)$ $\left(\frac{1}{q-1} \mapsto ??\right)$ but we can let $U_q^{\mathbb{Z}}(\underline{n}^-)$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q(\underline{n}^-)$ generated by all $f_i^{(a)} := \dfrac{f_i^a}{[a]_q[a-1]_q\cdots[1]_q}$, and then:

$$U(\underline{n}^-) \cong \mathbb{C} \underset{\mathbb{Z}[q,q^{-1}]}{\otimes} U_q^{\mathbb{Z}}(\underline{n}^-).$$

q acting as identity on $\mathbb{C}$

Since the relations $(*)_q$ are homogeneous in each f_i independently, $U_q(\underline{n}^-)$ is $\mathbb{N}^{n-1}$-graded:

$$U_q(\underline{n}^-) = \bigoplus_{\underline{d}\in\mathbb{N}^{n-1}} U_q(\underline{n}^-)_{\underline{d}}$$

where $U_q(\underline{n}^-)_{\underline{d}} = \text{span}\{\underbrace{f_{i_1}f_{i_2}\cdots f_{i_d}}_{\text{call this } f_{(i_1,i_2,\ldots,i_d)}} \mid (i_1,\cdots,i_d)\in I_{\underline{d}}\}$

and $I_{\underline{d}} = \{$ sequences $(i_1,i_2,\cdots,i_d)$ in which each i appears d_i times $\}$
if $\underline{d} = (d_1,\cdots,d_{n-1})$, $d = d_1+\cdots+d_{n-1}$.

Example $n=3$, $\underline{d} = (2,2)$ so $d = 4$.

$I_{\underline{d}} = \{(1,1,2,2), (1,2,1,2), (1,2,2,1), (2,1,1,2), (2,1,2,1), (2,2,1,1)\}$

Spanning set of $U_q(\underline{n}^-)_{(2,2)}$:

$$\underset{f_{(1,1,2,2)}}{\overset{\|}{f_1^2 f_2^2}}, \quad \underset{\text{etc.}}{\overset{\|}{f_1 f_2 f_1 f_2}}, \; f_1 f_2^2 f_1, \; f_2 f_1^2 f_2, \; f_2 f_1 f_2 f_1, \; f_2^2 f_1^2.$$

Exercise: show using the relation $(*)_q$ that
$$f_1 f_2^2 f_1 = f_2 f_1^2 f_2,$$ and
$U_q(\underline{n}^-)_{(2,2)}$ has basis $\{f_1^2 f_2^2, f_1 f_2^2 f_1, f_2^2 f_1^2\}$.

Prop $\dim U_q(\underline{n}^-)_{\underline{d}} = \dim U(\underline{n}^-)_{\underline{d}}$ for all $\underline{d} \in \mathbb{N}^{n-1}$.

We can calculate the right-hand side using the PBW theorem:

$$\underline{n}^- \text{ has basis } \{ \underbrace{[\cdots[f_i, f_{i+1}], \cdots, f_j]}_{} \mid 1 \leq i \leq j \leq n-1 \}$$

this has degree $(0,\cdots,0,\overset{i}{1},\cdots,\overset{j}{1},0,\cdots,0)$ - call such an $(n-1)$-tuple a "segment".

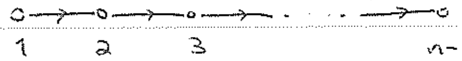
So $\dim U(\underline{n}^-)_{\underline{d}} = \#$ of ways to write $\underline{d}$ as a sum of segments.

Example $n=3$. The segments are $(1,0)$, $(0,1)$, $(1,1)$, so the ways to write (d_1, d_2) as a sum of segments are:

$$r(1,1) + (d_1 - r)(1,0) + (d_2 - r)(0,1)$$

$$\text{for } 0 \leq r \leq \min\{d_1, d_2\}$$

So $\dim U_q(\underline{n}^-)_{(d_1,d_2)} = \dim U(\underline{n}^-)_{(d_1,d_2)} = \min\{d_1, d_2\} + 1$.

Another problem with the same answer: classifying representations of the quiver 

of dimension $\underline{d} = (d_1, \dots, d_{n-1})$.
(Of course, the relevance of this quiver is really that it is an orientation of the Dynkin diagram of $\underline{sl}_n$.)

Fix $\underline{d} \in \mathbb{N}^{n-1}$. Let $V^{(1)}, V^{(2)}, \dots, V^{(n-1)}$ be fixed vector spaces of dimensions $d_1, d_2, \dots, d_{n-1}$.
Let $E_{\underline{d}} = \mathrm{Hom}(V^{(1)}, V^{(2)}) \oplus \mathrm{Hom}(V^{(2)}, V^{(3)}) \oplus \cdots \oplus \mathrm{Hom}(V^{(n-2)}, V^{(n-}$
So an element $x \in E_{\underline{d}}$ consists of linear maps

$$V^{(1)} \xrightarrow{x^{(1)}} V^{(2)} \xrightarrow{x^{(2)}} V^{(3)} \longrightarrow \cdots \xrightarrow{x^{(n-2)}} V^{(n-1)},$$

i.e. it is a representation of the above quiver.

A group naturally acting on $E_{\underline{d}}$ is

$$G_{\underline{d}} = GL(V^{(1)}) \times GL(V^{(2)}) \times \cdots \times GL(V^{(n-1)}) :$$

for $g = (g^{(1)}, g^{(2)}, \dots, g^{(n-1)}) \in G_{\underline{d}}$, $x = (x^{(1)}, \dots, x^{(n-2)}) \in E_{\underline{d}}$,

$$g \cdot x = \left(g^{(2)} x^{(1)} (g^{(1)})^{-1}, g^{(3)} x^{(2)} (g^{(2)})^{-1}, \dots, g^{(n-1)} x^{(n-2)} (g^{(n-2)})^{-1} \right).$$

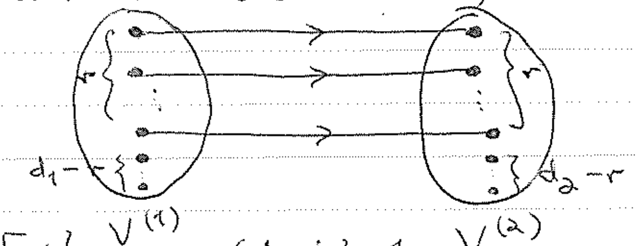
Two points $x, x' \in E_{\underline{d}}$ belong to the same $G_{\underline{d}}$-orbit iff they are equivalent as representations of the quive (i.e. they are the same up to change of basis of the vector spaces).

<u>Example</u> $n=3$: $E_{\underline{d}} = \mathrm{Hom}(V^{(1)}, V^{(2)}) \cong \mathrm{Mat}_{d_2 \times d_1}(\mathbb{C})$
$G_{\underline{d}} = GL(V^{(1)}) \times GL(V^{(2)}) \cong GL_{d_1}(\mathbb{C}) \times GL_{d_2}(\mathbb{C}$

$$g \cdot x = g^{(2)} x (g^{(1)})^{-1}.$$

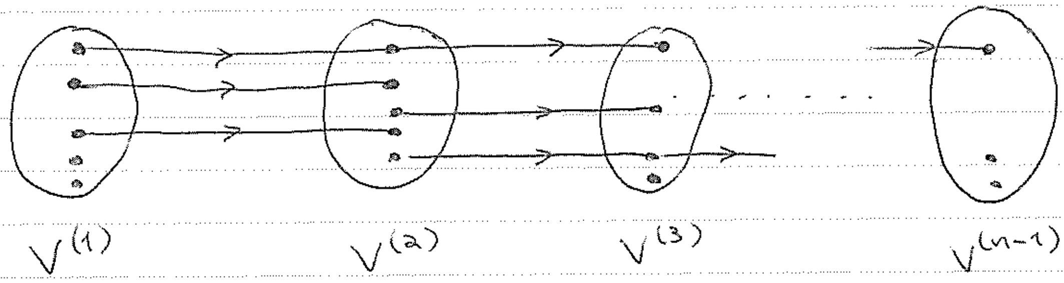
Linear algebra: after left & right multiplying by invertible matrices, any $x \in \mathrm{Mat}_{d_2 \times d_1}(\mathbb{C})$ can be put in the form $\begin{pmatrix} \left.\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}\right\} r & 0 \\ 0 & 0 \end{pmatrix}$ where $r = \mathrm{rank}(x) \in \{0, 1, \dots, \min\{d_1, d_2\}\}$

Equivalently, we can find bases of $V^{(1)}, V^{(2)}$ such that x has the form



So $\#\{G_{\underline{d}}\text{-orbits in } E_{\underline{d}}\} = \min\{d_1, d_2\} + 1.$

Exercise For general n and $\underline{d}$, the $G_{\underline{d}}$-orbits in $E_{\underline{d}}$ are in bijection with the ways to write $\underline{d}$ as a sum of segments, where a sum of segments determines a $G_{\underline{d}}$-orbit via a picture like:



Corollary $\dim U_q(\underline{n}^-)_{\underline{d}} = \#\{G_{\underline{d}}\text{-orbits in } E_{\underline{d}}\}$.

$= \dim \mathcal{X}_{\underline{d}}$,

where $\mathcal{X}_{\underline{d}}$ is the vector space of $G_{\underline{d}}$-invariant functions $E_{\underline{d}} \longrightarrow \mathbb{C}(q)$. Let $\mathcal{X} = \bigoplus_{\underline{d}\in\mathbb{N}^{n-1}} \mathcal{X}_{\underline{d}}$.

Plan: define an algebra structure on $\mathcal{X}$ and an isomorphism of ($\mathbb{N}^{n-1}$-graded) algebras

$$\Psi : U_q(\underline{n}^-) \xrightarrow{\sim} \mathcal{X}.$$

$\mathcal{X}_{\underline{d}}$ has a basis $\{1_{\mathcal{O}} \mid \mathcal{O} \text{ a } G_{\underline{d}}\text{-orbit in } E_{\underline{d}}\}$ where $1_{\mathcal{O}}$ is the indicator function of the orbit $\mathcal{O}$, so to define the multiplication map $\mathcal{X}_{\underline{d}} \times \mathcal{X}_{\underline{d}'} \longrightarrow \mathcal{X}_{\underline{d}+\underline{d}'}$ it suffices to define $1_{\mathcal{O}}1_{\mathcal{O}'}$ for $\mathcal{O}$ a $G_{\underline{d}}$-orbit in $E_{\underline{d}}$, $\mathcal{O}'$ a $G_{\underline{d}'}$-orbit in $E_{\underline{d}'}$.

First approximation: for $x \in E_{\underline{d}+\underline{d}'}$,

$(1_{\mathcal{O}}1_{\mathcal{O}'})(x) = \#$ of subrepresentations of the representation x which are of type $\mathcal{O}$ and such that the corresponding quotient representation is of type $\mathcal{O}'$.

Here a subrepresentation of

$$\begin{array}{ccccc} V^{(1)} & \xrightarrow{x^{(1)}} & V^{(2)} & \longrightarrow \cdots \xrightarrow{x^{(n-2)}} & V^{(n-1)} \\ \cup & & \cup & & \cup \\ W^{(1)} & & W^{(2)} & & W^{(n-1)} \end{array}$$

is a collection of subspaces such that $x^{(i)}(W^{(i)}) \subseteq W^{(i+1)}$; this is itself a representation of the quiver, $W^{(1)} \xrightarrow{x^{(1)}|_{W^{(1)}}} W^{(2)} \longrightarrow W^{(3)} \longrightarrow \cdots \longrightarrow W^{(n-1)}$, and gives rise to a quotient representation $V^{(1)}/W^{(1)} \longrightarrow V^{(2)}/W^{(2)} \longrightarrow \cdots$.

But "# of subrepresentations" would usually be infinite, so this is not right.

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Geometric Representation Theory Week 2 Lecture 3

Recall: $\mathcal{X}_{\underline{d}} = \{ G_{\underline{d}}\text{-invariant functions } E_{\underline{d}} \longrightarrow \mathbb{Q}(q) \}$

$\mathcal{X} = \bigoplus_{\underline{d} \in \mathbb{N}^{n-1}} \mathcal{X}_{\underline{d}}$, we want to make this a graded algebra isomorphic to $U_q(\mathfrak{n}^-)$, so we are trying to define $1_{\mathcal{O}} 1_{\mathcal{O}'} \in \mathcal{X}_{\underline{d}+\underline{d}'}$ for $\mathcal{O}$ a $G_{\underline{d}}$-orbit in $E_{\underline{d}}$, $\mathcal{O}'$ a $G_{\underline{d}'}$-orbit in $E_{\underline{d}'}$.

Temporary change of notation: let q^2 be the cardinality of a finite field F (rather than q being a formal variable). Define $G_{\underline{d}}(F)$, $E_{\underline{d}}(F)$ in the same way as $G_{\underline{d}}, E_{\underline{d}}$, using F-vector spaces. The classification of $G_{\underline{d}}(F)$-orbits in $E_{\underline{d}}(F)$ is identical to that over $\mathbb{C}$.

For $\mathcal{O}$ a $G_{\underline{d}}(F)$-orbit in $E_{\underline{d}}(F)$, $\mathcal{O}'$ a $G_{\underline{d}'}(F)$-orbit in $E_{\underline{d}'}(F)$, and $x \in E_{\underline{d}+\underline{d}'}(F)$, i.e.

$$\begin{tikzcd} F^{d_1+d_1'} \arrow[r, "x^{(1)}"] & F^{d_2+d_2'} \arrow[r, "x^{(2)}"] & \cdots \arrow[r] & F^{d_{n-1}+d_{n-1}'} \\ V^{(1)} \arrow[u, phantom, sloped, "\cong"] & V^{(2)} \arrow[u, phantom, sloped, "\cong"] & & V^{(n-1)} \arrow[u, phantom, sloped, "\cong"] \end{tikzcd}$$

define $g_{\mathcal{O},\mathcal{O}'}(x) = \#$ of subrepresentations of x of type $\mathcal{O}$ with quotient of type $\mathcal{O}'$.

now makes sense, since we're over a finite field

Recall a subrepresentation is a collection of subspaces $W^{(i)} \subseteq V^{(i)}$ such that $x^{(i)}(W^{(i)}) \subseteq W^{(i+1)}$; then

$$W^{(1)} \xrightarrow{x^{(1)}|_{W^{(1)}}} W^{(2)} \xrightarrow{x^{(2)}|_{W^{(2)}}} \cdots$$ is itself a repn

and so is $$V^{(1)}/W^{(1)} \xrightarrow{x^{(1)}|_{V^{(1)}/W^{(1)}}} V^{(2)}/W^{(2)} \longrightarrow \cdots$$

Say the subrep is of type $\mathcal{O}$ if $\dim W^{(i)} = d_i$, and after identifying $W^{(i)}$ with the fixed d_i-dim vector space in the defn of $E_{\underline{d}}(F)$,

$$W^{(1)} \longrightarrow W^{(2)} \longrightarrow \cdots$$

belongs to the orbit $\mathcal{O}$. Similarly for the quotient.

<u>Example</u> $n=3$, $\underline{d} = \underline{d}' = (1,1)$. There are only 2 orbits in $E_{(1,1)}(F)$, zero and nonzero.

The point $x \in E_{(2,2)}(F)$ is a map $V^{(1)} \cong F^2 \longrightarrow V^{(2)} \cong F^2$, and

(e.g.) $$g_{\text{zero},\text{nonzero}}(x) = \#\{ (W^{(1)} \text{ 1-dim'l} \subseteq V^{(1)}, W^{(2)} \text{ 1-dim'l} \subseteq V^{(2)}) \mid W^{(1)} \subseteq \ker(x), W^{(2)} \not\supseteq \operatorname{im}(x) \}$$

$$= \begin{cases} 0 & \text{if } x = 0 \\ q^2 & \text{if } \operatorname{rank}(x) = 1 \\ 0 & \text{if } \operatorname{rank}(x) = 2 \end{cases}$$, because $|\mathbb{P}^1(F)| = q^2 + 1$.

It is clear that $g_{\mathcal{O},\mathcal{O}'}(x)$ only depends on the $G_{\underline{d}+\underline{d}'}^{(F)}$-orbit of x.

<u>Theorem</u> (Ringel) Fixing "$\mathcal{O}, \mathcal{O}'$ and the orbit of x" but letting F vary, $g_{\mathcal{O},\mathcal{O}'}(x)$ is a polynomial function of q^2 with integer coefficients.

So we have a well-defined element $g_{\mathcal{O},\mathcal{O}'}(x) \in \mathbb{C}(q)$ in our actual setting where q is a formal variable and $G_{\underline{d}}$, $E_{\underline{d}}$ etc are defined over $\mathbb{C}$.

One more piece of notation: define a $\mathbb{Z}$-bilinear form $(\cdot,\cdot): \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \longrightarrow \mathbb{Z}$ by

$$(\underline{d}, \underline{d}') = d_1 d_1' + \cdots + d_{n-1} d_{n-1}' + d_2 d_1' + \cdots + d_{n-1} d_{n-2}'.$$

Finally we can define our multiplication map $\mathcal{X}_{\underline{d}} \times \mathcal{X}_{\underline{d}'} \longrightarrow \mathcal{X}_{\underline{d}+\underline{d}'}$ by

$$(1_{\mathcal{O}} 1_{\mathcal{O}'})(\underset{\substack{\cap \\ E_{\underline{d}+\underline{d}'}}}{x}) = q^{-(\underline{d},\underline{d}')} g_{\mathcal{O},\mathcal{O}'}(x),$$

extended bilinearly over $\mathbb{C}(q)$.

<u>Prop</u> This makes $\mathcal{X} = \bigoplus_{\underline{d} \in \mathbb{N}^{n-1}} \mathcal{X}_{\underline{d}}$ an associative algebra.

<u>Proof</u> (sketch): It's enough to show $(1_{\mathcal{O}} 1_{\mathcal{O}'}) 1_{\mathcal{O}''} = 1_{\mathcal{O}} (1_{\mathcal{O}'} 1_{\mathcal{O}''})$ for $\mathcal{O}$ a $G_{\underline{d}}$-orbit in $E_{\underline{d}}$, similarly for $\mathcal{O}'$ and $\underline{d}'$, $\mathcal{O}''$ and $\underline{d}''$. Unravelling the definitions, for $x \in E_{\underline{d}+\underline{d}'+\underline{d}''}$,

$$(1_{\mathcal{O}} 1_{\mathcal{O}'}) 1_{\mathcal{O}''}(x) = 1_{\mathcal{O}} (1_{\mathcal{O}'} 1_{\mathcal{O}''})(x)$$

$$= q^{-(\underline{d},\underline{d}')-(\underline{d},\underline{d}'')-(\underline{d}',\underline{d}'')} g_{\mathcal{O},\mathcal{O}',\mathcal{O}''}(x),$$

where $g_{\mathcal{O},\mathcal{O}',\mathcal{O}''}(x)$ has the following interpretation over F:

$$g_{\mathcal{O},\mathcal{O}',\mathcal{O}''}(x) = \#\text{ of filtrations} \quad \begin{tikzcd} V^{(1)} \arrow[r] & V^{(2)} \arrow[r] & \cdots & \underline{d}+\underline{d}'+\underline{d}'' \\ W^{(1)} \arrow[r] \arrow[u, phantom, "\cup"] & W^{(2)} \arrow[r] \arrow[u, phantom, "\cup"] & \cdots & \underline{d}+\underline{d}' \\ X^{(1)} \arrow[r] \arrow[u, phantom, "\cup"] & X^{(2)} \arrow[r] \arrow[u, phantom, "\cup"] & \cdots & \underline{d} \end{tikzcd}$$

where X is of type $\mathcal{O}$,
W/X is of type $\mathcal{O}'$,
V/W is of type $\mathcal{O}''$. □

Theorem (Ringel re-interpreted by Lusztig)

There is an isomorphism of $\mathbb{N}^{n-1}$-graded associative $\mathbb{C}(q)$-algebras

$$\Psi : U_q(\underline{n}^-) \xrightarrow{\sim} \mathcal{X}$$

uniquely determined by $\Psi(f_i) = 1_{\mathcal{O}_i}$ where $\mathcal{O}_i = E_{(0,\cdots,0,\underset{i}{1},0,\cdots,0)} = \{0\}$ for all $i \in \{1,\cdots,n-1\}$

Proof (sketch):

Step 1: existence of a graded algebra homomorphism $\Psi : U_q(\underline{n}^-) \longrightarrow \mathcal{X}$ such that $\Psi(f_i) = 1_{\mathcal{O}_i}$. (Uniqueness of Ψ is obvious since $f_1,\cdots,f_{n-1}$ generate $U_q(\underline{n}^-)$.)

For this we just have to check the relations

$$1_{\mathcal{O}_i} 1_{\mathcal{O}_j} = 1_{\mathcal{O}_j} 1_{\mathcal{O}_i} \quad \text{if } |i-j| \geq 2$$

$$1_{\mathcal{O}_i} 1_{\mathcal{O}_i} 1_{\mathcal{O}_j} + 1_{\mathcal{O}_j} 1_{\mathcal{O}_i} 1_{\mathcal{O}_i} = [2]_q \, 1_{\mathcal{O}_i} 1_{\mathcal{O}_j} 1_{\mathcal{O}_i} \quad \text{if } |i-j| = 1,$$

using the defn of multiplication in $\mathcal{X}$.

Example (really enough to prove the general case of $j = i+1$)

$n = 3$, functions in $\mathcal{X}_{(2,1)}$: $\quad x \in E_{(2,1)} = \mathrm{Hom}(\underset{\mathbb{C}^2}{\overset{\wr\wr}{V^{(1)}}}, \underset{\mathbb{C}^1}{\overset{\wr\wr}{V^{(2)}}})$

$$1_{\mathcal{O}_1} 1_{\mathcal{O}_1} 1_{\mathcal{O}_2}(x) = \begin{cases} q^{-1}(q^2+1) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$1_{\mathcal{O}_2} 1_{\mathcal{O}_1} 1_{\mathcal{O}_1}(x) = q^{-3}(q^2+1) \quad \text{for any } x$$

$$1_{\mathcal{O}_1} 1_{\mathcal{O}_2} 1_{\mathcal{O}_1}(x) = \begin{cases} q^{-2}(q^2+1) & \text{if } x = 0 \\ q^{-2} & \text{if } x \neq 0 \end{cases}$$

Step 2: to prove Ψ is an isomorphism, it suffices to prove that Ψ is surjective, since we already know that $\dim U_q(\underline{n}^-)_{\underline{d}} = \dim \mathcal{X}_{\underline{d}}$ for all $\underline{d}$.

Step 3: to prove that Ψ is surjective, it suffices to show that for any $G_{\underline{d}}$-orbit $\mathcal{O}$ in $E_{\underline{d}}$ there is some $(i_1, i_2, \cdots, i_d) \in I_{\underline{d}}$ such that $\Psi(f_{(i_1,i_2,\cdots,i_d)})$ is supported on the closure $\overline{\mathcal{O}}$ and not zero on $\mathcal{O}$. If $\mathcal{O}$ corresponds to a collection of segments with sum $\underline{d}$, order them lexicographically and read each segment backwards. $\square$

What is $\Psi(f_{(i_1,i_2,\cdots,i_d)}) = 1_{\mathcal{O}_{i_1}} 1_{\mathcal{O}_{i_2}} \cdots 1_{\mathcal{O}_{i_d}}$ as a function on $E_{\underline{d}}$?

$\uparrow$ some sequence in $I_{\underline{d}}$

Let $\mathcal{F}_{(i_1,i_2,\cdots,i_d)}$ be the variety of complete flags

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$$

where $V_s = V_{s-1} \oplus$ (some 1-dim. subspace of $V^{(i_s)}$), $\forall s \in \{1, \cdots, d\}$.

Let $\tilde{\mathcal{F}}_{(i_1,i_2,\cdots,i_d)}$ be the vector bundle over $\mathcal{F}_{(i_1,i_2,\cdots,i_d)}$

consisting of pairs $(x, (V_s)) \in E_{\underline{d}} \times \mathcal{F}_{(i_1,\cdots,i_d)}$ such that $x(V_s) \subseteq V_s \quad \forall s \in \{1, \cdots, d\}$.

Let $\overset{\pi_{\underline{i}}}{\pi_{(i_1,i_2,\cdots,i_d)}} : \tilde{\mathcal{F}}_{(i_1,i_2,\cdots,i_d)} \longrightarrow E_{\underline{d}}$ be the first projection.

∴ For $x \in E_{\underline{d}}$, the fibre $\pi_{\underline{i}}^{-1}(x)$ consists of all flags $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$ as above such that $x(V_s) \subseteq V_s \ \forall s$

This is a projective variety.

Up to isomorphism, $\pi_{\underline{i}}^{-1}(x)$ only depends on the $G_{\underline{d}}$-orbit of x, and $1_{\mathcal{O}_{i_1}} 1_{\mathcal{O}_{i_2}} \cdots 1_{\mathcal{O}_{i_d}}(x)$ involves its number of points over a finite field.

<u>Def</u> A projective variety Z has an <u>affine paving</u> if it has closed subvarieties

$$\phi = Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_t = Z$$

such that each $Z_i \setminus Z_{i-1} \cong \mathbb{A}^{s_i}$ for some $s_i \geq 0$.

If so, then $\# Z(\underset{\substack{\uparrow \\ \text{finite field with} \\ q^2 \text{ elements}}}{F}) = q^{2s_1} + q^{2s_2} + \cdots + q^{2s_t}$

$$= \sum_{p \geq 0} \dim H^p(Z)\, q^p$$

(the "Poincaré polynomial" of Z)

<u>Theorem</u> (R. Maksimau, 'Canonical basis, KLR-algebras and parity sheaves', J. Algebra 422 (2015))

For any $\underline{d} \in \mathbb{N}^{n-1}$ and $\underline{i} \in I_{\underline{d}}$, every fibre $\pi_{\underline{i}}^{-1}(x)$ has an affine paving.

<u>Corollary</u> (Lusztig by a different method)

$$\Psi(f_{\underline{i}})(x) = q^{-\dim \tilde{\mathcal{F}}_{\underline{i}}} \sum_{p \geq 0} \dim H^p(\pi_{\underline{i}}^{-1}(x))\, q^p.$$

Geometric Representation Theory Week 2 Lecture 4

Yesterday: $\Psi : U_q(\underline{n}^-)_{\underline{d}} \xrightarrow{\sim} \mathcal{X}_{\underline{d}} = \{ G_{\underline{d}}\text{-invariant functions } E_{\underline{d}} \to \mathbb{C}(q) \}$

For $\underline{i} \in I_{\underline{d}}$, $\Psi(f_{\underline{i}})(x) = q^{-\dim \widetilde{\mathcal{F}}_{\underline{i}}} \sum_{p \geq 0} \dim H^p(\pi_{\underline{i}}^{-1}(x))\, q^p$

where $\pi_{\underline{i}} : \widetilde{\mathcal{F}}_{\underline{i}} = \{ (x, (V_s)) \in E_{\underline{d}} \times \mathcal{F}_{\underline{i}} \mid x(V_s) \subseteq V_s \ \forall s \} \to E_{\underline{d}}$ is the first projection.

So far we have just replaced one vector space $U_q(\underline{n}^-)_{\underline{d}}$ with another $\mathcal{X}_{\underline{d}}$. But we are now in the context of a general philosophy (due to Grothendieck):

$$\begin{tikzcd} \text{vector space of functions} \arrow[r, squiggly, shift left, "\text{“categorify”}"] & \text{category of sheaves} \arrow[l, squiggly, shift left, "\text{“traces”/ “Poincaré polynomials”}"] \\ + & \oplus \\ \text{basis elements} & \text{indecomposables} \end{tikzcd}$$

Ref: Iversen, 'Cohomology of sheaves', P. Achar, 'Introduction to perverse sheaves', Wis web page K. Rietsch, 'An Intro to perverse sheaves', arXiv: 0307349

A sheaf $\mathcal{F}$ on a topological space X (for us, $E_{\underline{d}}$) consists of:

- for every open subset U of X, a vector space of sections $\Gamma(U, \mathcal{F})$
- whenever $V \subseteq U$, a linear restriction map $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$
  $s \mapsto s|^U_V$

satisfying:
① $s|^U_U = s$
② $(s|^U_V)|^V_W = s|^U_W$ if $W \subseteq V \subseteq U$
"gluing axiom" ③ if $U = \bigcup_i U_i$ is an open covering of U and $s_i \in \Gamma(U_i, \mathcal{F})$ satisfy $s_i|^{U_i}_{U_i \cap U_j} = s_j|^{U_j}_{U_i \cap U_j}$ for all i, j, there exists a unique $s \in \Gamma(U, \mathcal{F})$ such that $s|^U_{U_i} = s_i \ \forall i$.

Example Sheaf $\underline{\text{Cont}}_X$: $\Gamma(U, \underline{\text{Cont}}_X) = \{ \text{cts functions } U \to \mathbb{C} \}$ with usual restriction

Example More generally, for any vector bundle $Z \to X$, can take sections of that bundle.

Non-example Defining $\Gamma(U, \mathcal{F}) = \{ \underline{\text{constant}} \text{ functions } U \to \mathbb{C} \}$ doesn't satisfy ③ when U is disconnected.

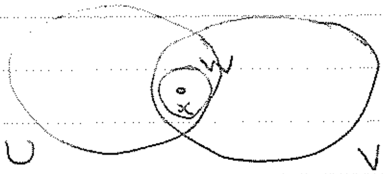
Example Constant sheaf $\underline{\mathbb{C}}_X$: $\Gamma(U, \underline{\mathbb{C}}_X) = \{ \text{locally constant functions } U \to \mathbb{C} \} \cong H^0(U)$.

The *stalk* $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ at a point $x \in X$ is the vector space of "germs of sections at x", i.e. equivalence classes of pairs (U, s),

U open $\subseteq X$
$x \in U$
$s \in \Gamma(U, \mathcal{F})$

where $(U,s) \sim (V,t)$ if $\exists\, W$ open $\subseteq U \cap V$, $x \in W$,

$$s|^U_W = t|^V_W .$$



Example $(\underline{\mathbb{C}}_X)_x = \mathbb{C}$ for all $x \in X$.

Given a cts map $\pi : Y \longrightarrow X$, there is a sheaf $\pi_* \underline{\mathbb{C}}_Y$ ("direct image" / "push-forward") defined by $\Gamma(U, \pi_* \underline{\mathbb{C}}_Y) = H^0(\pi^{-1}(U))$. If π is proper, its stalks are $(\pi_* \underline{\mathbb{C}}_Y)_x = H^0(\pi^{-1}(x))$.

More generally, for any $p \geq 0$ there is a sheaf $R^p \pi_* \underline{\mathbb{C}}_Y$ ("pth right-derived direct image") with stalks (still assuming π proper)

$$(R^p \pi_* \underline{\mathbb{C}}_Y)_x = H^p(\pi^{-1}(x)).$$

These can be bundled into one object $R\pi_* \underline{\mathbb{C}}_Y$, which is a *complex* of sheaves on X:

$$0 \longrightarrow \mathcal{F}^{(0)} \xrightarrow{d^{(0)}} \mathcal{F}^{(1)} \xrightarrow{d^{(1)}} \mathcal{F}^{(2)} \longrightarrow \cdots$$

where:
- each $d^{(p)}$ is a morphism of sheaves (obvious defn)
- $d^{(p+1)} d^{(p)} = 0 \quad \forall\, i$
- $R^p \pi_* \underline{\mathbb{C}}_Y \cong \mathcal{H}^p(R\pi_* \underline{\mathbb{C}}_Y) := \underset{\substack{\uparrow \\ \text{obvious defn}}}{\ker(d^{(p)})} \Big/ \underset{\substack{\uparrow \\ \text{not so obvious}}}{\mathrm{im}(d^{(p-1)})}$

We also need to consider *shifts* $R\pi_* \underline{\mathbb{C}}_Y[m]$, $m \in \mathbb{Z}$, which are the same complex but re-indexed so that

$$\begin{aligned} \mathcal{H}^p(R\pi_* \underline{\mathbb{C}}_Y[m]) &= \mathcal{H}^{p+m}(R\pi_* \underline{\mathbb{C}}_Y) \\ &= R^{p+m} \pi_* \underline{\mathbb{C}}_Y \quad \text{(zero unless } p \geq -m) \end{aligned}$$

Back to our situation: $\underline{d} \in \mathbb{N}^{n-1}$, $\underline{i} \in I_{\underline{d}}$, $\pi_{\underline{i}} : \tilde{\mathcal{F}}_{\underline{i}} \longrightarrow E_{\underline{d}}$.
Since $\pi_{\underline{i}}$ is a projective morphism, it is proper.
Let $\mathcal{L}_{\underline{i}} = R(\pi_{\underline{i}})_* \underline{\mathbb{C}}_{\tilde{\mathcal{F}}_{\underline{i}}} [\dim \tilde{\mathcal{F}}_{\underline{i}}]$, the Lusztig complex.

By defn, this is a complex of sheaves on $E_{\underline{d}}$ such that

$$\mathcal{H}^p(\mathcal{L}_{\underline{i}}) = R^{p+\dim \tilde{\mathcal{F}}_{\underline{i}}}(\pi_{\underline{i}})_* \underline{\mathbb{C}}_{\tilde{\mathcal{F}}_{\underline{i}}},$$

so $\mathcal{H}^p(\mathcal{L}_{\underline{i}})_x = H^{p+\dim \tilde{\mathcal{F}}_{\underline{i}}}(\pi_{\underline{i}}^{-1}(x))$ for all $x \in E_{\underline{d}}$.

So $\Psi(f_{\underline{i}})(x) = \sum_p \dim \mathcal{H}^p(\mathcal{L}_{\underline{i}})_x \, q^p$,

i.e. $\mathcal{L}_{\underline{i}}$ "categorifies" the function $\Psi(f_{\underline{i}})$.

Decomposition Theorem (Beilinson-Bernstein-Deligne, 'Faisceaux pervers', Astérisque 100 (1981))
(applied to our special situation)

Each $\mathcal{L}_{\underline{i}}$ is (quasi-)isomorphic to a direct sum of shifts of intersection cohomology complexes $IC(\overline{\mathcal{O}})$ where $\mathcal{O}$ runs over $G_{\underline{d}}$-orbits in $E_{\underline{d}}$. Moreover, for each $IC(\overline{\mathcal{O}})$ the shift $[m]$ occurs as often as $[-m]$.

otherwise known as simple perverse sheaves

Here, $IC(\overline{\mathcal{O}})$ is a certain complex of sheaves on $E_{\underline{d}}$ satisfying:

$$\mathcal{H}^p(IC(\overline{\mathcal{O}}))_x = \begin{cases} 0 & \text{if } x \notin \overline{\mathcal{O}} \\ 0 & \text{if } x \in \mathcal{O} \text{ and } p \neq -\dim \mathcal{O} \\ \mathbb{C} & \text{if } x \in \mathcal{O} \text{ and } p = -\dim \mathcal{O} \\ 0 & \text{if } x \in \overline{\mathcal{O}} \setminus \mathcal{O} \text{ and } p < -\dim \mathcal{O} \\ & \quad \text{or } p \geq -\dim \mathcal{O}_x \\ ? & \text{if } x \in \overline{\mathcal{O}} \setminus \mathcal{O} \text{ and} \\ & \quad -\dim \mathcal{O} \leq p < -\dim \mathcal{O}_x \end{cases}$$

orbit containing x

If $\overline{\mathcal{O}}$ is a smooth variety, $IC(\overline{\mathcal{O}}) \cong \underline{\mathbb{C}}_{\overline{\mathcal{O}}}[\dim \mathcal{O}]$;
otherwise, the ? has something to do with the singularities of $\overline{\mathcal{O}}$.

Define $f_{\overline{\mathcal{O}}} \in \mathcal{X}_{\underline{d}}$ by $f_{\overline{\mathcal{O}}}(x) = \sum_p \dim \mathcal{H}^p(IC(\overline{\mathcal{O}}))_x \, q^p$,
$b_{\overline{\mathcal{O}}} \in U_q(\underline{n}^-)_{\underline{d}}$ by $b_{\overline{\mathcal{O}}} = \Psi^{-1}(f_{\mathcal{O}})$.
Then $\{ b_{\overline{\mathcal{O}}} \mid \mathcal{O} \text{ a } G_{\underline{d}}\text{-orbit in } E_{\underline{d}} \}$ is Lusztig's canonical basis of $U_q(\underline{n}^-)$.

Example $n=3$, $\underline{d}=(2,2)$. Three orbits: $\{0\}$, $\mathcal{O}$ (rank 1), $E_{(2,2)} \setminus \overline{\mathcal{O}}$ (rank 2).

$$\overline{\mathcal{O}} = \mathcal{O} \cup \{0\} = \{ x \in \mathrm{Mat}_{2\times 2} \mid \mathrm{rank}(x) \leq 1 \}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad-bc = 0 \right\}, \text{ singular at } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\mathcal{H}^p(IC(\{0\}))_x = \begin{cases} 0 & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \text{ and } p \neq 0 \\ \mathbb{C} & \text{if } x=0 \text{ and } p=0 \end{cases}$$

$$\mathcal{H}^p(IC(\overline{\mathcal{O}}))_x = \begin{cases} 0 & \text{if } x \notin \overline{\mathcal{O}} \\ 0 & \text{if } x \in \mathcal{O} \text{ and } p \neq -3 \\ \mathbb{C} & \text{if } x \in \mathcal{O} \text{ and } p = -3 \\ 0 & \text{if } x = 0 \text{ and } p < -3 \text{ or } p \geq 0 \\ ? & \text{if } x = 0 \text{ and } -3 \leq p \leq -1 \end{cases}$$

We can determine the ? by considering

$$\mathcal{H}^p(\mathcal{L}_{(1,2,1,2)})_x = H^{p+3}(\pi^{-1}_{(1,2,1,2)}(x))$$

$$= \begin{cases} 0 & \text{if } x \in \overline{\mathcal{O}} & (\pi^{-1}_{(1,2,1,2)}(x) \text{ is empty} \\ 0 & \text{if } x \in \mathcal{O} \text{ and } p \neq -3 & (\pi^{-1}_{(1,2,1,2)}(x) \text{ is a} \\ \mathbb{C} & \text{if } x \in \mathcal{O} \text{ and } p = -3 & \text{single point}) \\ 0 & \text{if } x = 0 \text{ and } p \notin \{-3,-1,1\} & (\pi^{-1}_{(1,2,1,2)}(0) \cong \\ \mathbb{C} & \text{if } x = 0 \text{ and } p \in \{-3,1\} & \mathbb{P}^1 \times \mathbb{P}^1) \\ \mathbb{C}^2 & \text{if } x=0 \text{ and } p=-1 & \text{Betti numbers} \\ & & 1,0,2,0,1 \end{cases}$$

By the Decomposition Thm and these calculations,

$$(\dagger) \qquad \mathcal{L}_{(1,2,1,2)} \cong IC(\overline{\mathcal{O}}) \oplus \underset{\text{to get the } \mathbb{C} \text{ when } p=1}{IC(\{0\})[-1]} \oplus \underset{\text{must balance the shifts}}{IC(\{0\})[1]},$$

$$\text{so } \mathcal{H}^p(IC(\overline{\mathcal{O}}))_0 = \begin{cases} 0 & \text{if } p \notin \{-3,-1\} \\ \mathbb{C} & \text{if } p \in \{-3,-1\}. \end{cases}$$

Moreover, $(\dagger)$ implies after decategorification that

$$f_{(1,2,1,2)} = b_{\overline{\mathcal{O}}} + (q+q^{-1})\, b_{\{0\}}.$$

It is easy to see that $\mathcal{L}_{(1,1,2,2)} \cong IC(\{0\}) \oplus IC(\{0\}) \oplus IC(\{0\})[-2] \oplus IC(\{0\})[2]$,

so $f_{(1,1,2,2)} = (q^2+2+q^{-2})\, b_{\{0\}}$.

So $b_{\{0\}} = f_1^{(2)} f_2^{(2)}$, $b_{\overline{\mathcal{O}}} = f_1 f_2^{(2)} f_1$.

Similarly $b_{E_{(2,2)}} = f_2^{(2)} f_1^{(2)}$.