Polynomial functors and representations of wreath products

Anthony Henderson
University of Sydney
anthonyh@maths.usyd.edu.au

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A Theorem of Lehrer's

Let $n \geq 0$. The symmetric group S_n has a reflection action on $\mathbb{C}^n$ by permuting coordinates. Let

$$M[n] := \{(z_1, \cdots, z_n) \in \mathbb{C}^n \,|\, z_i \neq z_j, \ \forall i \neq j\}$$

be the **hyperplane complement**. Since S_n preserves $M[n]$, it acts on its cohomology. For $w \in S_n$, define

$$P_{M[n]}(w,q) := \sum_{i=0}^{n} (-1)^i \operatorname{tr}(w, H^i(M[n], \mathbb{C}))\, q^{n-i}.$$

Thm 1 *(Lehrer 1987) If w has cycle-type (i^{a_i}),*

$$P_{M[n]}(w,q) = \prod_{i \geq 1} R_i(R_i - i) \cdots (R_i - (a_i - 1)i),$$

where

$$R_i := \sum_{d|i} \mu(d) q^{i/d}.$$

E.g. $R_1 = q,\ R_2 = q^2 - q,\ R_3 = q^3 - q,\ R_4 = q^4 - q^2.$

$$P_{M[n]}(1,q) = q(q-1)\cdots(q-(n-1)). \quad \text{(Arnold)}$$

Stratify $\mathbb{C}^n$ according to intersections of the hyperplanes. Strata are indexed by Par$[n]$ = {set partitions of $[n]$}.

E.g. Elements of Par[3], with corresponding strata:

$\{\{1\},\{2\},\{3\}\}$	$M[3]$
$\{\{1,2\},\{3\}\}$	$\{z_1 = z_2 \neq z_3\}$
$\{\{1,3\},\{2\}\}$	$\{z_1 = z_3 \neq z_2\}$
$\{\{2,3\},\{1\}\}$	$\{z_2 = z_3 \neq z_1\}$
$\{\{1,2,3\}\}$	$\{z_1 = z_2 = z_3\}$

Stratum indexed by $\pi \in$ Par$[n]$ is isomorphic to $M(\pi)$. Moreover, S_n permutes and acts on the strata as it does on Par$[n]$.

Mixed Hodge theory says $P_X(1,q)$ is additive for our stratification. Hence we get a recurrence relation:

$$q^n = \sum_{\pi \in \mathsf{Par}[n]} P_{M(\pi)}(1,q).$$

Rephrase using $A(x) = \sum_{n \geq 0} P_{M[n]}(1,q)\frac{x^n}{n!}$:

$$\exp(qx) = A(\exp(x) - 1).$$

Hence

$$A(x) = \exp(q \log(1+x)) = (1+x)^q,$$

reproving Arnold's result.

For general w, we again have a recurrence relation:

$$q^n = \sum_{\pi \in \mathsf{Par}[n]^w} P_{M(\pi)}(w_\pi, q).$$

How can we transform this into Lehrer's formula?

Polynomial Functors

A functor $F : \mathbb{C}\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}$ is **polynomial** if $F :$ Hom$(M, N) \to$ Hom$(F(M), F(N))$ is polynomial.

Thm 2 *(Schur) Every polynomial functor is isomorphic to one of the form*

$$F_U : M \mapsto \bigoplus_{n \geq 0} (U[n] \otimes M^{\otimes n})^{S_n},$$

where $U = (U[n])_{n \geq 0}$, $U[n]$ a repn of S_n.

E.g. If $\lambda \vdash n$, $F_{V_\lambda} = F_\lambda$ (Schur functor). If $d \geq \ell(\lambda)$, $F_\lambda(\mathbb{C}^d)$ is the irrep of $GL_d(\mathbb{C})$ with highest weight λ.

$$F_{(2)}(M) = (\mathbf{1} \otimes M^{\otimes 2})^{S_2} \cong \mathrm{Sym}^2 M,$$
$$F_{(1^2)}(M) = (\varepsilon \otimes M^{\otimes 2})^{S_2} \cong \Lambda^2 M.$$

Let $\Lambda = \mathbb{C}[p_i \,|\, i \in \mathbb{Z}^+]$. For U as above, set

$$\mathrm{ch}(U) := \sum_{n\geq 0} \frac{1}{n!} \sum_{w \in S_n} \mathrm{tr}(w, U[n])\, p_w \ \in \Lambda,$$

where $p_w = \prod_i p_i^{a_i}$ if w has cycle-type (i^{a_i}).

E.g. $\mathrm{ch}(V_\lambda) = \frac{1}{n!}\sum_{w\in S_n} \chi^\lambda(w) p_w = s_\lambda$ (Schur function).

Prop 1 *If* $M \in \mathbb{C}\text{-}\mathbf{mod}$, $\varphi \in \mathrm{End}(M)$, *then*

$$\mathrm{tr}(F_U(\varphi), F_U(M)) = \mathrm{ch}(U)|_{p_i \to \mathrm{tr}(\varphi^i, M)}.$$

So the characters of the polynomial irreps of $GL_d(\mathbb{C})$ are the Schur fns with $p_i \to i$th power sum of eigenvalues.

Composition of two poly functors is a poly functor.

Prop 2 *If* $V[0] = 0$, $F_U \circ F_V \cong F_{U \circ V}$, *where*

$$(U \circ V)[n] := \bigoplus_{\pi \in \mathsf{Par}[n]} U(\pi) \otimes \bigotimes_{J \in \pi} V(J).$$

Define an operation of **plethysm** $\circ : \Lambda \times \Lambda \to \Lambda$ by

1. $\forall g$, $\Lambda \to \Lambda : f \mapsto f \circ g$ is a $\mathbb{C}$-alg hom;

2. $\forall i$, $\Lambda \to \Lambda : g \mapsto p_i \circ g$ is a $\mathbb{C}$-alg hom;

3. $p_i \circ p_j = p_{ij}$.

Propositions 1 and 2 imply:

Thm 3 *If* $V[0] = 0$, $\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V)$.

E.g. To find the character of the S_4-repn

$$\bigoplus_{\pi \in \mathsf{Par}^{2,2}[4]} \varepsilon(\pi) = \mathsf{Ind}_{(S_2 \times S_2) \rtimes S_2}^{S_4} (1 \rtimes \varepsilon),$$

Theorem 3 tells us to calculate

$$\begin{aligned} s_{(1^2)} \circ s_{(2)} &= \frac{p_1^2 - p_2}{2} \circ \frac{p_1^2 + p_2}{2} \\ &= \frac{1}{2}\left((p_1 \circ \frac{p_1^2 + p_2}{2})^2 - p_2 \circ \frac{p_1^2 + p_2}{2} \right) \\ &= \frac{1}{2}\left(\frac{(p_1^2 + p_2)^2}{4} - \frac{p_2^2 + p_4}{2} \right) \\ &= \frac{3p_1^4}{24} + \frac{p_2 p_1^2}{4} + \frac{-p_2^2}{8} + \frac{0 p_3 p_1}{3} + \frac{-p_4}{4}. \end{aligned}$$

So character values are $3, 1, -1, 0, -1$. Since this is $s_{(31)}$, this repn is the irrep $V_{(31)}$. Also, by Proposition 2,

$$F_{(31)} \cong \Lambda^2 \circ \mathsf{Sym}^2.$$

Proof of Theorem 1

(Adapted from a preprint of Getzler.) Define

$$\mathrm{ch}_q(H^\bullet M) := \sum_{n\geq 0} \frac{1}{n!} \sum_{w\in S_n} P_{M[n]}(w,q)\, p_w.$$

By Theorem 3, the previous recursion implies

$$\exp(\sum_{i\geq 1} \frac{q^i p_i}{i}) = \mathrm{ch}_q(H^\bullet M) \circ (\exp(\sum_{i\geq 1} \frac{p_i}{i}) - 1).$$

The plethystic inverse of $\exp(\sum_{i\geq 1} \frac{p_i}{i}) - 1$ is

$$\sum_{d\geq 1} \frac{\mu(d)}{d} \log(1+p_d).$$

Hence

$$\mathrm{ch}_q(H^\bullet M) = \exp(\sum_{i\geq 1} \frac{q^i p_i}{i}) \circ (\sum_{d\geq 1} \frac{\mu(d)}{d} \log(1+p_d)),$$

which after rearranging gives Lehrer's result.

The Wreath Product Case

Let $r \geq 2$. Define the wreath product

$$\begin{aligned} W(r,n) &= \mu_r \wr S_n \\ &= \{\text{monomial } n \times n \text{ matrices, entries in } \mu_r\}. \end{aligned}$$

Its hyperplane complement is

$$M[n]_r := \{(z_1, \cdots, z_n) \in (\mathbb{C}^\times)^n \mid z_i \neq \zeta z_j, \zeta \in \mu_r\}.$$

Thm 4 *(Lehrer 1995) If w has cycle-type $((i,\zeta)^{a_i(\zeta)})$,*

$$P_{M[n]_r}(w,q) = \prod_{\substack{i \geq 1 \\ \zeta \in \mu_r}} R_{r,i,\zeta}(R_{r,i,\zeta} - ri) \cdots (R_{r,i,\zeta} - (a_i(\zeta)-1)ri),$$

where

$$R_{r,i,\zeta} := \sum_{d|i} \#\{d\textit{th roots of } \zeta\}\mu(d)(q^{i/d} - 1).$$

$(\mathbb{C}^\times)^n$ has a stratification with strata indexed by

$\mathsf{Par}[n]_r = \{$partitions of $\mu_r \times [n]$ on which μ_r acts freely$\}$.

E.g. Elements of $\mathsf{Par}[2]_2$ with corresponding strata:

$$\begin{array}{cc} \{\{+1\},\{-1\},\{+2\},\{-2\}\} & M[2]_2 \\ \{\{+1,+2\},\{-1,-2\}\} & \{z_1 = z_2\} \\ \{\{+1,-2\},\{-1,+2\}\} & \{z_1 = -z_2\} \end{array}$$

Stratum indexed by $\pi \in \mathsf{Par}[n]_r$ is isomorphic to $M(\pi)_r$. Moreover, $W(r,n)$ permutes and acts on the strata as it does on $\mathsf{Par}[n]_r$.

Need to consider polynomial functors $F : \mathbb{C}\mu_r$-**mod** $\to$ $\mathbb{C}$-$\mathbf{mod}$.

Thm 5 *(Macdonald) Every such functor is isomorphic to one of the form*

$$F_U : M \mapsto \bigoplus_{n\geq 0} (U[n]_r \otimes M^{\otimes n})^{W(r,n)},$$

where $U = (U[n]_r)_{n\geq 0}$, $U[n]_r$ *a repn of* $W(r,n)$.

Let $\Lambda(r) = \mathbb{C}[p_i(\zeta) \mid i \in \mathbb{Z}^+, \zeta \in \mu_r]$. Define

$$\mathrm{ch}(U) := \sum_{n\geq 0} \frac{1}{r^n n!} \sum_{w\in W(r,n)} \mathrm{tr}(w, U[n]_r)\, p_w \ \in \Lambda(r),$$

where $p_w = \prod_i p_i(\zeta)^{a_i(\zeta)}$ if w has cycle-type $((i,\zeta)^{a_i(\zeta)})$.

Prop 3 *If* $M \in \mathbb{C}\mu_r$-**mod**, $\varphi \in \mathrm{End}_{\mu_r}(M)$, *then*

$$\mathrm{tr}(F_U(\varphi), F_U(M)) = \mathrm{ch}(U)|_{p_i(\zeta)\to \mathrm{tr}(\varphi^i\zeta, M)}.$$

If $F : \mathbb{C}\mu_r\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}$ and $G : \mathbb{C}\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}$ are poly functors, so is $F \circ G^{(r)} : \mathbb{C}\mu_r\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}$.

Prop 4 *If* $V[0] = 0$, $F_U \circ F_V^{(r)} \cong F_{U \circ V}$, *where*

$$(U \circ V)[n]_r := \bigoplus_{\pi \in \mathrm{Par}[n]_r} U(\pi)_r \otimes \bigotimes_{\mathcal{O} \in \mu_r \backslash \pi} V(\mathcal{O}).$$

Define a generalized plethysm $\circ : \Lambda(r) \times \Lambda \to \Lambda(r)$ by

1. $\forall g$, $\Lambda(r) \to \Lambda(r) : f \mapsto f \circ g$ is a $\mathbb{C}$-alg hom;

2. $\forall i, \zeta$, $\Lambda \to \Lambda(r) : g \mapsto p_i(\zeta) \circ g$ is a $\mathbb{C}$-alg hom;

3. $p_i(\zeta) \circ p_j = p_{ij}(\zeta^j)$.

Propositions 3 and 4 imply:

Thm 6 *If* $V[0] = 0$, $\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V)$.

Proof of Theorem 4

Define

$$\mathrm{ch}_q(H^\bullet M_r) := \sum_{n\geq 0} \frac{1}{r^n n!} \sum_{w\in W(r,n)} P_{M[n]_r}(w,q)\, p_w.$$

Applying Theorem 6 to the stratification of $(\mathbb{C}^\times)^n$ gives

$$\exp\Big(\sum_{\substack{i\geq 1\\ \zeta\in\mu_r}} \frac{(q^i-1)p_i(\zeta)}{ri}\Big) = \mathrm{ch}_q(H^\bullet M_r) \circ \Big(\exp\Big(\sum_{i\geq 1}\frac{p_i}{i}\Big)-1\Big).$$

Hence

$$\mathrm{ch}_q(H^\bullet M_r) = \exp\Big(\sum_{\substack{i\geq 1\\ \zeta\in\mu_r}} \frac{(q^i-1)p_i(\zeta)}{ri}\Big)\circ\Big(\sum_{d\geq 1}\frac{\mu(d)}{d}\log(1+p_d)\Big),$$

which after rearranging gives the result.