Representations of wreath products on cohomology of De Concini–Procesi compactifications

Anthony Henderson
University of Sydney
anthonyh@maths.usyd.edu.au

November 12, 2003

Hyperplane Complements and their Compactifications

Let V be a v. sp. over $\mathbb{C}$, $W < GL(V)$ a finite reflection group acting irreducibly on V with arrangement $\mathcal{A}$.
Let $\mathcal{M} = \mathbb{P}(V) \setminus \bigcup_{H \in \mathcal{A}} \mathbb{P}(H)$ be the **projective hyperplane complement**. Clearly W acts on this.

If $V = \mathbb{C}^n / \mathbb{C}(1, 1, \cdots, 1)$, $W = S_n$, $\mathcal{A} = \{\{z_i = z_j\}\}$:

$$\begin{aligned} \mathcal{M}(n) &= \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\} / (\mathbb{C}^\times \ltimes \mathbb{C}) \\ &\cong \{(z_1, \cdots, z_{n+1}) \in (\mathbb{P}^1)^{n+1} \mid z_i \neq z_j\} / PGL_2(\mathbb{C}) \\ &= \mathcal{M}_{0,n+1}. \end{aligned}$$

E.g. $\mathcal{M}(2) = \text{pt}$, $\mathcal{M}(3) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

If $V = \mathbb{C}^n$, $W = \mu_r \wr S_n$, $\mathcal{A} = \{\{z_i = 0\}, \{z_i = \zeta z_j\}\}$:

$$\mathcal{M}(r, n) = \{[z_1, \cdots, z_n] \in \mathbb{P}^{n-1} \mid z_i \neq 0, z_i^r \neq z_j^r\}.$$

E.g. $\mathcal{M}(r, 2) = \mathbb{P}^1 \setminus (\{0, \infty\} \cup \mu_r)$.

Let $\mathcal{F}$ be the set of hyperplane intersections $X \subset V$ which are irreducible, i.e. the induced arrangement on V/X is irreducible.

The **De Concini–Procesi wonderful compactification** $\overline{\mathcal{M}}$ is the closure of the image of the embedding

$$\mathcal{M} \hookrightarrow \prod_{X \in \mathcal{F}} \mathbb{P}(V/X).$$

This is smooth. (See Selecta Math. **1** (1995).)

In the S_n case, $\mathcal{F} = \{X_J \mid J \subseteq [n], |J| \geq 2\}$, where

$$X_J = \{(z_1, \cdots, z_n) \mid z_j \text{ equal}, \forall j \in J\}.$$

In fact $\overline{\mathcal{M}}(n) \cong \overline{\mathcal{M}}_{0,n+1}$, the moduli space of stable genus 0 curves with $n+1$ marked points.

E.g. $\overline{\mathcal{M}}(2) = \text{pt}$, $\overline{\mathcal{M}}(3) = \mathbb{P}^1$. Limit points in $\overline{\mathcal{M}}(4)$ are not just those in $\mathbb{P}(V)$: when z_1, z_2, z_3 coalesce, $\mathbb{P}(V/X_{123})$ remembers their "asymptotic configuration".

The equivariant Poincaré polynomials

$$\sum_{i=0}^{\dim\mathcal{M}} \mathrm{tr}(w, H^i(\mathcal{M}, \mathbb{C}))\, t^i,\ \ w \in W,$$

have been computed in many cases (incl. S_n, $\mu_r \wr S_n$) by Lehrer (e.g. see Invent. Math. **120** (1995)).

Problem Find formulas for the polynomials

$$\sum_{i=0}^{\dim\mathcal{M}} \mathrm{tr}(w, H^{2i}(\overline{\mathcal{M}}, \mathbb{C}))\, q^i,\ \ w \in W.$$

(The odd cohomology groups vanish.)

- Solved for S_n by Ginzburg and Kapranov (Duke Math. J. **76** (1994)).
- Non-equivariant ($w = 1$) case solved for $\mu_2 \wr S_n$ by Yuzvinsky (Invent. Math. **127** (1997)).
- Now solved for $\mu_r \wr S_n$, all w, all r (to appear in IMRN, see arXiv:math.RT/0307383).

The Symmetric Group Case

Let $\mathcal{T}(n)$ be the set of rooted trees with leaves $\leftrightarrow [n]$. We have a stratification $\overline{\mathcal{M}}(n) = \bigcup_{T\in\mathcal{T}(n)} \mathcal{M}_T$, where

$$\mathcal{M}_T \cong \prod_{v\in\mathsf{Int}(T)} \mathcal{M}(\mathsf{Fibre}(v)).$$

E.g. $\overline{\mathcal{M}}(4)$ has the open stratum $\mathcal{M}(4)$, ten strata isomorphic to $\mathcal{M}(3)$, and fifteen point strata.

Hence in Groth. group of mixed Hodge structures,

$$\sum_i H^{2i}(\overline{\mathcal{M}}(n)) = \sum_{T\in\mathcal{T}(n)} \bigotimes_{v\in\mathsf{Int}(T)} \left(\sum_i (-1)^i H_c^i(\mathcal{M}(\mathsf{Fibre}(v))) \right).$$

Want to obtain from this a recursion for LHS.

Let $\mathbf{B}$ be the category of finite sets and bijections.
A $\mathbf{B}$-**module** is a functor $U : \mathbf{B} \to \mathbb{C}$-mod.
Essentially $U = (U(n))_{n \geq 0}$, where $U(n)$ is a repn of S_n.

"Substitution" aka "partitional composition": if U, V are $\mathbf{B}$-modules, define a $\mathbf{B}$-module $U \circ V$ by

$$(U \circ V)(I) = \bigoplus_{\pi \in \mathrm{Par}(I)} \left(U(\pi) \otimes \bigotimes_{J \in \pi} V(J) \right).$$

If $V(I) = 0$ for $|I| \leq 1$, define $\mathbb{T}V$ by

$$\mathbb{T}V(I) = \bigoplus_{T \in \mathcal{T}(I)} \bigotimes_{v \in \mathrm{Int}(T)} V(\mathrm{Fibre}(v)).$$

Then $\mathbb{T}V \cong \mathbb{C}_{|I|=1} + V \circ \mathbb{T}V$.

If the B-module U is bounded ($U(I) = 0$ for $|I| \gg 0$),

$$F_U(M) = \bigoplus_{n \geq 0} (U(n) \otimes M^{\otimes n})^{S_n}$$

defines a **polynomial functor** $F_U : \mathbb{C}\text{-mod} \to \mathbb{C}\text{-mod}$. Define the characteristic $\mathrm{ch}(U) \in \Lambda$ by

$$\mathrm{ch}(U) = \sum_{n \geq 0} \frac{1}{n!} \sum_{w \in S_n} \mathrm{tr}(w, U(n))\, p_w.$$

Prop 1 *If* $\varphi \in \mathrm{End}(M)$,

$$\mathrm{ch}(U)[p_i \to \mathrm{tr}(\varphi^i, M)] = \mathrm{tr}(F_U(\phi), F_U(M)).$$

Prop 2 $F_{U \circ V} \cong F_U \circ F_V$.

Cor 1 $\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V)$ *(plethysm).*

Cor 2 $\mathrm{ch}(\mathbb{T}V) = p_1 + \mathrm{ch}(V) \circ \mathrm{ch}(\mathbb{T}V)$.

Ginzburg and Kapranov's Recursion

Define the following elements of $\mathbb{C}[q][[p_1, p_2, \cdots]]$:

$$\mathcal{P} = \sum_{n \geq 2} \frac{1}{n!} \sum_{w \in S_n} \sum_{s} (-1)^s \mathrm{tr}(w, H_c^s(\mathcal{M}(n))) q^{s-n+2} p_w,$$

$$\overline{\mathcal{P}} = \sum_{n \geq 1} \frac{1}{n!} \sum_{w \in S_n} \sum_{s} \mathrm{tr}(w, H^{2s}(\overline{\mathcal{M}}(n))) q^s p_w.$$

Thm 1 *(GK 1994, Getzler)*

$$\overline{\mathcal{P}} = p_1 + \mathcal{P} \circ \overline{\mathcal{P}}.$$

Thus the terms of $\overline{\mathcal{P}}$ can be computed recursively, using Lehrer's formula for $\mathcal{P}$.

The Wreath Product Case

Let $\mathcal{T}(r,n)$ be the set of trees $T \in \mathcal{T}(\mu_r \times [n])$ such that

1. T is μ_r-stable;
2. the μ_r-fixed edges form a path from the root;
3. μ_r acts freely on the other edges and vertices.

We have a stratification $\overline{\mathcal{M}}(r,n) = \bigcup_{T \in \mathcal{T}(r,n)} \mathcal{M}_T$, where

$$\mathcal{M}_T \cong \prod_{v \in \mathsf{Int}(T)^{\mu_r}} \mathcal{M}(r, \mathsf{Fibre}(v)^{\circ}) \times \prod_{\mathcal{O} \in \mathsf{Orb}(T)} \mathcal{M}(\mathsf{Fibre}(\mathcal{O})).$$

Let $\mathbf{B}_r$ be the category of finite sets with a free μ_r-action and μ_r-equivariant bijections. A $\mathbf{B}_r$-**module** is a functor $U : \mathbf{B}_r \to \mathbb{C}$-mod. Essentially $U = (U(r,n))_{n\geq 0}$, where $U(r,n)$ is a repn of $\mu_r \wr S_n$.

Substitution: if U is a $\mathbf{B}_r$-module and V a $\mathbf{B}$-module, define a $\mathbf{B}_r$-module $U \circ V$ by

$$(U \circ V)(I) = \bigoplus_{\pi \in \mathsf{Par}(I)_{\mathbf{B}_r}} \left(U(\pi) \otimes \bigotimes_{\mathcal{O} \in \mu_r \backslash \pi} V(\mathcal{O}) \right).$$

If $V(I) = 0$ for $|I| \leq 1$, define $\mathbb{T}_r(U,V)$ by

$$\mathbb{T}_r(U,V)(I) = \bigoplus_{T \in \mathcal{T}(r,I)} \left(\bigotimes_{v \in \mathsf{Int}(T)^{\mu_r}} U(\mathsf{Fibre}(v)^\circ) \otimes \bigotimes_{\mathcal{O} \in \mathsf{Orb}(T)} V(\mathsf{Fibre}(\mathcal{O})) \right).$$

Then $\mathbb{T}_r(U,V) \cong (\mathbb{C}_{|I|=0} + \mathbb{T}_r(U,V)) \cdot (U \circ \mathbb{T}V)$.

If the $\mathbf{B}_r$-module U is bounded,

$$F_U(M) = \bigoplus_{n\geq 0} (U(r,n) \otimes M^{\otimes n})^{\mu_r \wr S_n}$$

defines a polynomial functor $F_U : \mathbb{C}\mu_r\text{-mod} \to \mathbb{C}\text{-mod}$. Define the characteristic $\mathrm{ch}(U) \in \mathbb{C}[p_i(\zeta)]$ by

$$\mathrm{ch}(U) = \sum_{n\geq 0} \frac{1}{r^n n!} \sum_{w \in \mu_r \wr S_n} \mathrm{tr}(w, U(r,n))\, p_w.$$

Prop 3 *If* $\varphi \in \mathrm{End}_{\mu_r}(M)$,

$$\mathrm{ch}(U)[p_i(\zeta) \to \mathrm{tr}(\varphi^i \zeta, M)] = \mathrm{tr}(F_U(\phi), F_U(M)).$$

Prop 4 $F_{U \circ V} \cong F_U \circ F_V$, *where* F_V *means the induced functor* $\mathbb{C}\mu_r$-mod $\to$ $\mathbb{C}\mu_r$-mod.

Cor 3 $\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V)$, *where the plethysm is defined so that* $p_i(\zeta) \circ p_j = p_{ij}(\zeta^j)$.

Cor 4 $1 + \mathrm{ch}(\mathbb{T}_r(U,V)) = (1 - \mathrm{ch}(U) \circ \mathrm{ch}(\mathbb{T}V))^{-1}$.

The Main Theorem

Define the following elements of $\mathbb{C}[q][[p_i(\zeta)]]$:

$$\mathcal{P}(r) = \sum_{n\geq 1} \frac{1}{r^n n!} \sum_{w\in\mu_r\wr S_n} \sum_s (-1)^s \mathrm{tr}(w, H_c^s(\mathcal{M}(r,n)))q^{s-n+1}p_w,$$

$$\overline{\mathcal{P}}(r) = \sum_{n\geq 1} \frac{1}{r^n n!} \sum_{w\in\mu_r\wr S_n} \sum_s \mathrm{tr}(w, H^{2s}(\overline{\mathcal{M}}(r,n)))q^s p_w.$$

Thm 2 $1+\overline{\mathcal{P}}(r) = (1-\mathcal{P}(r)\circ\overline{\mathcal{P}})^{-1}$.

Thus $\overline{\mathcal{P}}(r)$ can be computed using Lehrer's formula for $\mathcal{P}(r)$ and Ginzburg–Kapranov's for $\overline{\mathcal{P}}$.