THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

2. Linear operators and the operator norm

PMH3: Functional Analysis — Semester 1, 2017

Lecturer: Anne Thomas

At a later stage a selection of these questions will be chosen for an assignment.

1. Compute the operator norms of the following linear operators. Here, ℓ^p has the norm $\|\cdot\|_p$, for $1 \leq p \leq \infty$, and $L^2(\mathbb{R})$ has the norm $\|\cdot\|_2$.

(a) $T : \ell^1 \to \ell^1$, with $Tx = (x_1, x_2/2, x_3/3, x_4/4, \dots)$.

(b) $T : \ell^2 \to \ell^1$, with $Tx = (x_1, x_2/2, x_3/3, x_4/4, \dots)$.

(c) $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with $Tf(x) = \displaystyle\int_{\mathbb{R}} f(x-y)e^{-y^2}\,dy$.

2. Calculate the operator norm of the following linear operators. Here, ℓ^p has the norm $\|\cdot\|_p$, for $1 \leq p \leq \infty$, and $\mathcal{C}([-1,1])$ has the norm $\|\cdot\|_\infty$.

(a) $T : \ell^1 \to \ell^\infty$ with $(x_i)_{i\geq 1} \mapsto (2^{-i}x_i)_{i\geq 1}$.

(b) $T : \ell^2 \to \ell^1$ with $(x_i)_{i\geq 1} \mapsto (2^{-i}x_i)_{i\geq 1}$.

(c) $T : \mathcal{C}([-1,1]) \to \mathbb{K}$ with $f \mapsto \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} f(1/n)$.

3. Let $(\mathscr{P}([0,1]), \|\cdot\|_\infty)$ be the normed vector space of all polynomials $p : [0,1] \to \mathbb{K}$.

(a) Is $D : \mathscr{P}([0,1]) \to \mathscr{P}([0,1])$, $Dp = p'$ bounded? If so, compute $\|D\|$.

(b) Is $T : \mathscr{P}([0,1]) \to \mathscr{P}([0,1])$, $Tf(x) = \int_0^x f(t)\,dt$ bounded? If so, compute $\|T\|$.

4. Let X and Y be normed vector spaces and let $T \in \mathscr{L}(X,Y)$. Show that if X is a Banach space then $\ker(T)$ is a Banach space.

5. Let X be a normed vector space, let $S \subseteq X$ be a dense subspace of X, and let Y be a Banach space. Show that every continuous linear operator $T : S \to Y$ extends uniquely to a continuous linear operator $\tilde{T} : X \to Y$. Moreover, show that $\|\tilde{T}\| = \|T\|$.

6. Show that the integral operator K on $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ defined by

$$Kf(x) = \int_0^1 k(x,y)f(y)\,dy,$$

where $k \in \mathcal{C}([0,1]\times[0,1])$, has norm $\|K\| = \max_{0\leq x\leq 1} \displaystyle\int_0^1 |k(x,y)|\,dy$. Calculate this explicitly in the case where $k(x,y) = (1+x^2+y^2)^{-1}$.

7. Calculate the operator norm of the bounded linear operator $T : \ell^2 \to \ell^2$ given by

$$T(x_1, x_2, \dots) = (x_1 + x_2, x_2 + x_3, x_3 + x_4, \dots).$$

8. Let X and Y be normed vector spaces and let $T \in \mathrm{Hom}(X,Y)$ be bijective. Define the inverse operator $T^{-1} : Y \to X$ by $T^{-1}y = x$ if and only if $Tx = y$.

(a) Show that $T^{-1} \in \mathrm{Hom}(Y,X)$.

(b) Show, by way of example, that T^{-1} need not be continuous, even if T is.

(c) Suppose there is a constant $a > 0$ such that $\|Tx\| \geq a\|x\|$ for all $x \in X$. Show that T^{-1} is continuous.

9. Let $\ell^2(\mathbb{Z}) = \{(x_i)_{i\in\mathbb{Z}} \mid \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty\}$. This is a Banach space with the obvious norm. Let $p, q > 0$. Compute the operator norm of the bounded linear operator

$$P : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad P(x_i)_{i\in\mathbb{Z}} = (px_{i-1} + qx_{i+1})_{i\in\mathbb{Z}}.$$

10. Let $S, T : \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ be the linear operators

$$Sf(x) = x\int_a^b f(t)\,dt \qquad \text{and} \qquad Tf(x) = xf(x) \quad \text{for all } f \in \mathcal{C}([a,b]).$$

Do these operators commute? Compute $\|S\|$, $\|T\|$, $\|ST\|$ and $\|TS\|$.

11. Let $A : \mathbb{K}^n \to \mathbb{K}^m$ be a linear operator. We can consider $A = (a_{ij})$ as a matrix relative to the standard bases of $\mathbb{K}^n$ and $\mathbb{K}^m$. If we put the p-norm on $\mathbb{K}^n$ and the q-norm on $\mathbb{K}^m$ then the operator norm of A is $\|A\|_{p,q} = \sup\{\|Ax\|_q : \|x\|_p \leq 1\}$. We write $\|A\|_p = \|A\|_{p,p}$. Show that

(a) $\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$.

(b) $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}$.

(c) $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$.

(d) $\|A\|_2 \leq \sqrt{\|A\|_1\|A\|_\infty}$.

Here A^* is conjugate transpose. What can you say about general $\|A\|_{p,q}$?

12. Let X and Y be normed vector spaces over $\mathbb{K}$.

(a) Show that if $\dim(X) < \infty$ then each $T \in \mathrm{Hom}(X,Y)$ is continuous.

(b) Suppose that $T \in \mathrm{Hom}(X,X)$ is surjective. Show that if $\dim(X) < \infty$ then T is injective. Is T necessarily injective if $\dim(X) = \infty$?

(c) Show that if $\dim(X) = \infty$ then $X^*\backslash X'$ has a linearly independent uncountable subset.

(d) Write down a discontinuous linear functional in the case where $X = \mathcal{C}([0,1])$ with norm $\|\cdot\|_1$. Include a proof that your functional is discontinuous, and make your example constructive (independent of Zorn's Lemma).