The University of Sydney
School of Mathematics and Statistics

3. Hilbert Spaces and the Stone–Weierstrass Theorem

PMH3: Functional Analysis — Semester 1, 2017

Lecturer: Anne Thomas

1. Let $\mathscr{H}$ be a Hilbert space and let $T \in \mathscr{L}(\mathscr{H}, \mathscr{H})$. Use the Riesz Representation Theorem to show that the equation

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \qquad \text{for all } x, y \in \mathscr{H}$$

defines a unique linear operator $T^* : \mathscr{H} \to \mathscr{H}$. Moreover, show that T^* is continuous, and that $\|T^*\| = \|T\|$. (The operator T^* is called the *adjoint operator*). Describe the adjoint more precisely in the case that $\mathscr{H}$ is finite dimensional.

2. (a) A *unitary operator* on a Hilbert space $\mathscr{H}$ is an operator $U \in \mathscr{L}(\mathscr{H}, \mathscr{H})$ satisfying $U^*U = UU^* = I$, where $Ix = x$ is the identity operator. Show that if S is a Hilbert basis of $\mathscr{H}$ and if $U : \mathscr{H} \to \mathscr{H}$ is a unitary operator, then $\{Ue \mid e \in S\}$ is also a Hilbert basis of $\mathscr{H}$.

(b) Let $e_n(x) = \sqrt{2}\cos\left[\left(n + \frac{1}{2}\right)\pi x\right]$ for each $n \in \mathbb{N}$.

(i) Show that $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal system in $L^2_{\mathbb{C}}([0, 1])$.

(ii) Explain why the Stone–Weierstrass Theorem cannot be immediately applied to prove that $\{e_n \mid n \in \mathbb{N}\}$ is complete.

(iii) Verify that the continuous linear operator $U : L^2_{\mathbb{C}}([0, 1]) \to L^2_{\mathbb{C}}([0, 1])$ given by

$$(Uf)(x) = \frac{1}{\sqrt{2}}\left(f(x)e^{i\pi x/2} + f(1 - x)e^{-i\pi x/2}\right)$$

is unitary.

(iv) Deduce that $\{e_n(x) \mid n \in \mathbb{N}\}$ is complete in $L^2_{\mathbb{C}}([0, 1])$.

3. Let $T : L^2_{\mathbb{C}}([0, 1]) \to L^2_{\mathbb{C}}([0, 1])$ be the continuous linear operator $(Tf)(x) = \int_0^x f(y)\, dy$.

(a) Find a formula for the adjoint operator T^*, and hence show that

$$(T^*Tf)(x) = \int_0^1 f(y)dy - x\int_0^x f(y)dy - \int_x^1 yf(y)dy \quad \text{for almost all } x \in [0, 1].$$

(b) Let $e_n(x)$ be the function from Question 2(b). Show that e_n is an eigenfunction of T^*T for each $n \in \mathbb{N}$. Compute the corresponding eigenvalue.

(c) The operator T^*T is *compact* and *self-adjoint*. Later in the course we show that this implies that $\|T^*T\| = \sup\{|\lambda| : \lambda \text{ an eigenvalue of } T^*T\}$. Use this to compute $\|T\|$.

4. Let $S = \{e_n(x) \mid n \in \mathbb{N}\}$ be the orthonormal system from Question 2(b). Find the Fourier series of $f(x) = \sin \pi x$ relative to S. Hence find the exact value of the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n - 1)^2(2n + 3)^2}.$$

5. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

(a) Prove the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in \mathscr{H}.$$

(b) Prove the polarisation identity:

$$\langle x, y\rangle = \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & \text{if } \mathbb{K} = \mathbb{R} \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

6. Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$. Show that if the parallelogram identity holds for all $x, y \in X$ then X is a Hilbert space, with inner product given by the polarisation identity formula. Here is an outline in the case $\mathbb{K} = \mathbb{R}$.

(a) Write $x + z = [(x+y)/2 + z] + (x-y)/2$ and $y + z = [(x+y)/2 + z] - (x-y)/2$ to deduce that

$$\|x+z\|^2 + \|y+z\|^2 = 2\left(\|(x+y)/2 + z\|^2 + \|(x-y)/2\|^2\right) \quad \text{for all } x, y, z \in X.$$

(b) Use the previous part to show that $\langle x, z\rangle + \langle y, z\rangle = 2\langle (x+y)/2, z\rangle$, and deduce that $\langle x, z\rangle + \langle y, z\rangle = \langle x+y, z\rangle$ for all $x, y, z \in X$.

(c) Verify that $\langle -x, y\rangle = -\langle x, y\rangle$, and that $\langle nx, y\rangle = n\langle x, y\rangle$ for natural numbers n.

(d) Deduce that $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for $\lambda \in \mathbb{R}$.

7. Let $\mathscr{H}$ be a Hilbert space.

(a) Let $X \subseteq \mathscr{H}$ be a convex set. Suppose that $(x_n)_{n\geq 1}$ is a sequence in X such that

$$\lim_{n\to\infty} \|x_n\| = \inf_{x\in X} \|x\|.$$

Show that x_n converges in $\mathscr{H}$. Is the limit of x_n necessarily in X?

(b) Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in $\mathscr{H}$. Let $x \in \mathscr{H}$ be fixed. Show that the infimum

$$\inf\{\|x - y\| \,:\, y \in \operatorname{span}\{e_1, \ldots, e_n\}\}$$

is attained at exactly one point. Find this point.

(c) Show that if $T \in \mathscr{L}(\mathscr{H}, \mathscr{H})$ then $\|T^*\| = \|T\|$ and $\|T\|^2 = \|T^*T\|$.

8. Let $\mathscr{H}$ be a separable Hilbert space with orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$. Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a linear operator with

$$\langle Te_i, e_j\rangle = \alpha_{ij} \qquad \text{for all } i, j,$$

where α_{ij} are given complex numbers. Do these equations uniquely determine T? What about if we assume that T is continuous? What about if $\mathscr{H}$ is finite dimensional?

9. The Legendre polynomials can be defined recursively by $P_0(x) = 1$ and $P_1(x) = x$, and

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \qquad \text{for } n \geq 1.$$

The normalised Legendre polynomials are

$$e_n(x) = \sqrt{\frac{2n+1}{2}}\,P_n(x).$$

You are given the following facts (you might also like to prove them):

- The set $S = \{e_n(x) \mid n \in \mathbb{N}\}$ is orthonormal in $L^2_{\mathbb{R}}([-1,1])$. Indeed $\{e_n(x) \mid n \in \mathbb{N}\}$ is the outcome of applying the Gram-Schmidt orthonormalisation process to the sequence $1, x, x^2, x^3, \ldots$
- The Legendre polynomials can be expressed using Rodrigues' Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n.$$

(a) Show that $\mathcal{A} = \text{span}\{P_n(x) \mid n \geq 0\}$ is a unital subalgebra of $\mathcal{C}_{\mathbb{R}}([-1,1])$.

(b) Deduce that $S = \{e_n(x) \mid n \geq 0\}$ is a complete orthonormal system in $L^2_{\mathbb{R}}([-1,1])$.

(c) Compute the Fourier series of $f(x) = |x|$ relative to the orthonormal basis S.

10. Let $\mathscr{H}$ be a separable Hilbert space with complete orthonormal system $e_1, e_2, \ldots$

(a) Let $T : \mathscr{H} \to \mathscr{H}$ be a continuous linear operator. Suppose that there is a sequence $(v_i)_{i\geq 1}$ of strictly positive numbers, and numbers $\lambda, \mu \geq 0$, with the property that

$$\sum_{j=1}^{\infty} |\langle Te_j, e_i\rangle| v_j \leq \lambda v_i \qquad \text{for all } i \geq 1$$

$$\sum_{i=1}^{\infty} |\langle Te_j, e_i\rangle| v_i \leq \mu v_j \qquad \text{for all } j \geq 1.$$

Show that $\|T\| \leq \sqrt{\lambda\mu}$.
Hint: Apply Parseval's Identity to $\|Tx\|^2$. Expand x into its Fourier series to see that

$$|\langle Tx, e_i\rangle| \leq \sum_{j=1}^{\infty} |\langle x, e_j\rangle|\,|\langle Te_j, e_i\rangle|.$$

Now write $|\langle x, e_j\rangle|\,|\langle Te_j, e_i\rangle| = \left[|\langle x, e_j\rangle| v_j^{-1/2} |\langle Te_j, e_i\rangle|^{1/2}\right] \left[|\langle Te_j, e_i\rangle|^{1/2} v_j^{1/2}\right]$.

(b) Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a continuous linear operator with

$$\langle Te_j, e_i\rangle = (i+j-1)^{-1}.$$

Show that $\|T\| \leq \pi$. (In fact $\|T\| = \pi$, but you do not need to prove this).
Hint: Use (a) with $v_j = (j - \frac{1}{2})^{-1/2}$, and approximate sums by integrals.

11. Let X be an inner product space.

(a) Show that if $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(b) Show that if $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ then $x_n \to x$.

(c) Let $(e_j)_{j\geq 1}$ be an orthonormal sequence in X. Show that

$$\sum_{j=1}^{\infty} |\langle x, e_j\rangle\langle y, e_j\rangle| \leq \|x\|\|y\| \qquad \text{for all } x, y \in X.$$

12. One form of the Riemann-Lebesgue Lemma states that if $f \in \mathcal{C}([a,b])$ and $\lambda \in \mathbb{R}$ then

$$\int_a^b f(x)e^{i\lambda x}\,dx \to 0 \qquad \text{as } |\lambda| \to \infty.$$

(a) Integrate by parts to prove the Riemann-Lebesgue Lemma under the hypothesis that $f(x)$ has continuous derivative on $[a,b]$.

(b) Use the Stone–Weierstrass Theorem to complete the proof.

13. Let $q \in \mathbb{Z}_{\geq 1}$ and let Γ be the homogeneous tree with degree $q+1$ (that is, Γ is a graph with no loops in which every vertex has exactly $q+1$ neighbours). Let V be the vertex set of Γ. Define a linear operator P acting on the space $\mathscr{F}$ of all functions $f : V \to \mathbb{C}$ by

$$(Pf)(x) = \frac{1}{q+1} \sum_{y \sim x} f(y),$$

where the sum is over the neighbours of x. That is, $(Pf)(x)$ is the average value of f on the neighbours of x. Let $d(x, y)$ be the graph distance in the tree from x to y. Let $\ell^2(V)$ be the space of all functions $f : V \to \mathbb{C}$ such that $\sum_{x \in V} |f(x)|^2 < \infty$. Then $\ell^2(V)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle = \sum_{x \in V} f(x)\overline{g(x)}.$$

Fix a vertex $o \in V$ and define a function $\varphi : V \to \mathbb{C}$ by

$$\varphi(x) = q^{-d(o,x)/2} \left(1 + \frac{q-1}{q+1} d(o, x) \right) \qquad \text{for all } x \in V.$$

(a) Show that φ is not in $\ell^2(V)$.

(b) Show that φ is an eigenfunction for the linear operator $P : \mathscr{F} \to \mathscr{F}$.

(c) Show that if $f \in \ell^2(V)$ then $Pf \in \ell^2(V)$. Therefore we can consider P as an operator $P : \ell^2(V) \to \ell^2(V)$. Show that this operator is continuous with $\|P\| \leq 1$.

(d) Use Question 10 and part (b) to show that $\|P\| \leq \frac{2\sqrt{q}}{q+1}$.

(e) Show that $\|P\| = \frac{2\sqrt{q}}{q+1}$.

14. Recall that continuous functions on $[a, b]$ are uniformly continuous. We can in fact prove this using Stone–Weierstrass.

(a) Suppose that f has bounded derivative: $|f'(x)| \leq M$ for all $x \in [a, b]$. Show that f is uniformly continuous.

(b) Deduce that polynomials are uniformly continuous, and use Stone–Weierstrass to conclude that all continuous functions are uniformly continuous.

15. Let f be a strictly increasing continuous function $f : [0, 1] \to \mathbb{R}$. Show that the subalgebra of $\mathcal{C}([0, 1])$ generated by $\{1, f\}$ is dense in $\mathcal{C}([0, 1])$.