THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

4. Uniform boundedness and the Open Mapping Theorem

PM3: Functional Analysis — Semester 1, 2017

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1. (a) Show that a Hamel basis of a Banach space X is either finite or uncountable.

(b) Let X be a normed vector space. Show that a subset $E \subseteq X$ is bounded if and only if $\sup_{x \in E} |\varphi(x)| < \infty$ for each $\varphi \in X'$.

(c) Let $c_0 = \{(x_i)_{i \geq 1} \mid x_i \in \mathbb{K} \text{ and } \lim_{i \to \infty} x_i = 0\}$. Suppose that $y_1, y_2, \ldots \in \mathbb{K}$ are such that

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for all $(x_i)_{i \geq 1} \in c_0$. Use Uniform Boundedness to show that $(y_i)_{i \geq 1} \in \ell^1$.

2. Let $(T_n)_{n \geq 1}$ be a sequence of operators $T_n \in \mathscr{L}(X, Y)$ where X and Y are Banach spaces. Suppose that for each $x \in X$ the sequence $T_n x$ converges in Y to a limit denoted by Tx. Show that if $x_n \to x$ then $T_n x_n \to Tx$.

3. Let $\mathscr{H}$ be a Hilbert space, and suppose that $T \in \text{Hom}(\mathscr{H}, \mathscr{H})$. Suppose that there exists an operator $\tilde{T} : \mathscr{H} \to \mathscr{H}$ such that

$$\langle Tx, y \rangle = \langle x, \tilde{T} y \rangle \qquad \text{for all } x, y \in \mathscr{H}.$$

Show that T is continuous.

4. Let X, Y, Z be Banach, and $T : X \times Y \to Z$ bilinear. Suppose that T is continuous in both variables. Show that there is a constant $M > 0$ such that $\|T(x, y)\| \leq M\|x\|\|y\|$.

5. Let X and Y be Banach spaces. By Corollary 20.3 of the course notes, if $T_n \in \mathscr{L}(X, Y)$ converges pointwise to $T : X \to Y$ then $T \in \mathscr{L}(X, Y)$ with

$$\|T\| \leq \liminf_{n \to \infty} \|T_n\|$$

(a) Give an example of Banach spaces X and Y and a sequence $T_n \in \mathscr{L}(X, Y)$ such that the inequality is strict.

(b) Give an example of a normed vector space X, a Banach space Y, and a sequence $T_n \in \mathscr{L}(X, Y)$ such that $\sup_{n \geq 1} \|T_n x\| < \infty$ for all $x \in X$, yet $\sup_{n \geq 1} \|T_n\| = \infty$. (Hence completeness of X is needed for the Principle of Uniform Boundedness).

6. Let X, Y be Banach spaces and let $T \in \text{Hom}(X, Y)$. Give another proof of the Closed Graph Theorem along the following lines. Suppose that $\Gamma(T)$ is closed in $X \times Y$.

(a) Show that $\Gamma(T)$ is a Banach space with norm $\|(x, Tx)\| = \|x\| + \|Tx\|$.

(b) Let $\pi_1 : \Gamma(T) \to X$ and $\pi_2 : \Gamma(T) \to Y$ be the natural projection maps. Show that they are linear and continuous, and that π_1 is bijective.

(c) Write T in terms of π_1 and π_2 and deduce that T is continuous.

7. The *nth Fourier coefficient* of $f \in L^1([-\pi, \pi])$ is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$

The Riemann–Lebesgue Lemma (see Question 8) says that if $f \in L^1([-\pi, \pi])$ then

$$\lim_{|n| \to \infty} \hat{f}(n) = 0.$$

Thus we have a linear map $T : L^1([-\pi, \pi]) \to c_0(\mathbb{Z})$ given by

$$Tf = (\hat{f}(n))_{n \in \mathbb{Z}},$$

where $c_0(\mathbb{Z})$ is the Banach space of all (two sided) sequences $(a_n)_{n \in \mathbb{Z}}$ in $\mathbb{K}$ with $a_n \to 0$ as $|n| \to \infty$ equipped with the sup norm $\|(a_n)\|_\infty = \sup_{n \in \mathbb{Z}} |a_n|$. As we see below, T is continuous and injective. It is natural to ask if T is surjective. The answer (perhaps surprisingly) is no.

(a) Show that T is continuous and injective.

(b) Show that T is not surjective.

8. One form of the Riemann–Lebesgue Lemma states that if $f \in L^1([a, b])$ and $\lambda \in \mathbb{R}$ then

$$\int_a^b f(x) e^{i\lambda x} \, dx \to 0 \qquad \text{as } |\lambda| \to \infty.$$

Integrate by parts to prove the Riemann–Lebesgue Lemma under the hypothesis that $f(x)$ has continuous derivative on $[a, b]$, and complete the proof using the Stone–Weierstrass Theorem.