A numerical analysis framework for linear and non-linear elasticity equations

J. Droniou

School of Mathematical Sciences, Monash University

AustMS 2013, 30/09/2013.

Joint work with B. P. Lamichhane (U. Newcastle)
Plan

1. Numerical methods for elasticity equations

2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. Conclusion
Plan

1. Numerical methods for elasticity equations

2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. Conclusion
Tasks of the numerical analyst

1. Design numerical methods.
2. Test them in simple and real-world applications (benchmarking).
3. Analyse their convergence and other properties.
Tasks of the numerical analyst

1. Design numerical methods.
2. Test them in simple and real-world applications (benchmarking).
3. Analyse their convergence and other properties.
Tasks of the numerical analyst

1. Design numerical methods.
2. Test them in simple and real-world applications (benchmarking).
3. Analyse their convergence and other properties under assumptions compatible with real-world applications.
Linear and non-linear elasticity

\begin{align*}
-\text{div}(\sigma(x, \varepsilon(\bar{u}))) &= F, \quad \text{in } \Omega, \\
\varepsilon(\bar{u}) &= \frac{\nabla \bar{u} + (\nabla \bar{u})^T}{2}, \quad \text{in } \Omega, \\
\bar{u} &= 0, \quad \text{on } \Gamma_D, \\
\sigma(x, \varepsilon(\bar{u})) n &= g, \quad \text{on } \Gamma_N,
\end{align*}

▶ Example: linear elasticity $\sigma(x, \varepsilon(\bar{u})) = C\varepsilon(\bar{u})$. 

\begin{align*}
\text{Weak formulation:}
\begin{align*}
&\text{Find } u \in H_1(\Gamma_D(\Omega)) \text{ such that, for any } v \in H_1(\Gamma_D(\Omega)), \\
&\int_{\Omega} \sigma(x, \varepsilon(u))(x):\varepsilon(v)(x) dx = \\
&\int_{\Omega} F(x) \cdot v(x) dx + \\
&\int_{\Gamma_N} g(x) \cdot \gamma(v)(x) dS(x).
\end{align*}
\end{align*}

where $H_1(\Gamma_D(\Omega)) = \{ v \in H_1(\Omega) : \gamma(v) = 0 \text{ on } \Gamma_D \}$ and $\gamma$ is the trace operator.
Linear and non-linear elasticity

\[
\begin{aligned}
    -\text{div}(\sigma(x, \varepsilon(\vec{u}))) &= F, \quad \text{in } \Omega, \\
    \varepsilon(\vec{u}) &= \frac{\nabla \vec{u} + (\nabla \vec{u})^T}{2}, \quad \text{in } \Omega, \\
    \vec{u} &= 0, \quad \text{on } \Gamma_D, \\
    \sigma(x, \varepsilon(\vec{u}))n &= g, \quad \text{on } \Gamma_N,
\end{aligned}
\]

Example: linear elasticity \( \sigma(x, \varepsilon(\vec{u})) = C \varepsilon(\vec{u}) \).

Weak formulation:

\[
\begin{aligned}
    \text{Find } \vec{u} \in H^1_{\Gamma_D}(\Omega) \text{ such that, for any } \vec{v} \in H^1_{\Gamma_D}(\Omega), \\
    \int_{\Omega} \sigma(x, \varepsilon(\vec{u})(x)) : \varepsilon(\vec{v})(x)dx &= \int_{\Omega} F(x) \cdot \vec{v}(x)dx \\
    &\quad + \int_{\Gamma_N} g(x) \cdot \gamma(\vec{v})(x) dS(x).
\end{aligned}
\]

where \( H^1_{\Gamma_D}(\Omega) = \{ \vec{v} \in H^1(\Omega)^d : \gamma(\vec{v}) = 0 \text{ on } \Gamma_D \} \) and \( \gamma \) is the trace operator.
Numerical methods and their convergence analysis

**Methods:**
- Finite Element (or Mixed FE) based.
- Sometimes with projections or modifications to stabilise in the nearly-incompressible limit (mostly for linear elasticity).
Numerical methods and their convergence analysis

Methods:

- Finite Element (or Mixed FE) based.

- Sometimes with projections or modifications to stabilise in the nearly-incompressible limit (mostly for linear elasticity).

Convergence analysis

- Based on error estimates, establish optimal orders of convergence.

- Mostly/only done for:
  - Linear elasticity: conforming methods, or non-conforming methods when $\overline{u} \in H^2$.
  - Non-linear elasticity: conforming methods, under sometimes very strong assumptions on $\overline{u}$ (e.g. $C^2(\overline{\Omega})$).
References

General theory


Linear elasticity


Non-linear elasticity

- Braess & Ming, 2005.
1 Numerical methods for elasticity equations

2 **Gradient Schemes for elasticity equations**
   - 4 discrete elements
   - 3 properties
   - Convergence results

3 Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4 Conclusion
Gradient schemes for diffusion equations

► Developed for diffusion equations (Eymard, Guichard, Herbin, Gallouët, D.: 2012+): linear, non-linear, stationary, transient, non-local...

► Unified convergence analysis of numerous numerical schemes for anisotropic diffusion equations for numerous models.

<table>
<thead>
<tr>
<th>METHODS</th>
<th>MODELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE</td>
<td>Linear diffusion</td>
</tr>
<tr>
<td>MFE</td>
<td>Non-linear diffusion</td>
</tr>
<tr>
<td>MPFA</td>
<td>Multi-phase flow</td>
</tr>
<tr>
<td>MFV</td>
<td>Stefan problem</td>
</tr>
<tr>
<td>DDFV</td>
<td>Image processing</td>
</tr>
<tr>
<td>MFD</td>
<td>Non-conservative eq.</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Gradient schemes for diffusion equations

- Developed for diffusion equations (Eymard, Guichard, Herbin, Gallouët, D.: 2012+): linear, non-linear, stationary, transient, non-local...

- Unified convergence analysis of numerous numerical schemes for anisotropic diffusion equations for numerous models.

**METHODS**
- FE
- MFE
- MPFA
- MFV
- DDFV
- MFD

**MODELS**
- Linear diffusion
- Non-linear diffusion
- Multi-phase flow
- Stefan problem
- Image processing
- Non-conservative eq.
1 Numerical methods for elasticity equations

2 Gradient Schemes for elasticity equations
   • 4 discrete elements
   • 3 properties
   • Convergence results

3 Examples of Gradient Schemes for elasticity equations
   • Displacement-based formulation
   • Stabilised nodal strain formulation
   • Hu-Washizu-based formulation

4 Conclusion
Weak formulation of the elasticity equations:

\begin{align*}
\text{Find } \overline{u} \in H^1_{\Gamma_D}(\Omega) \text{ such that, for any } v \in H^1_{\Gamma_D}(\Omega),
\int_{\Omega} \sigma(x, \varepsilon(\overline{u})(x)) : \varepsilon(v)(x) dx &= \int_{\Omega} F(x) \cdot v(x) dx \\
&\quad + \int_{\Gamma_N} g(x) \cdot \gamma(v)(x) dS(x).
\end{align*}
A Gradient Discretisation is $\mathcal{D} = (X_D, \Pi_D, T_D, \nabla_D)$ with

- $X_{D,\Gamma_D} = \mathbb{R}^{d.o.f.}$ discrete space (with Dirichlet boundary conditions on $\Gamma_D$),

- $\Pi_D : X_{D,\Gamma_D} \rightarrow L^2(\Omega)$ a reconstruction of functions,

- $T_D : X_{D,\Gamma_D} \rightarrow L^2(\partial\Omega)$ a discrete trace operator,

- $\nabla_D : X_{D,\Gamma_D} \rightarrow L^2(\Omega)^d$ a discrete gradient such that $\| \cdot \|_D = \| \nabla_D \cdot \|_{L^2(\Omega)^d}$ is a norm on $X_{D,\Gamma_D}$. 
Continuous equation

\[
\begin{cases}
\text{Find } \bar{u} \in H^1_{\Gamma_D}(\Omega) \text{ such that, for any } v \in H^1_{\Gamma_D}(\Omega), \\
\int_\Omega \sigma(x, \varepsilon(\bar{u})(x)) : \varepsilon(v)(x) \, dx = \int_\Omega F(x) \cdot v(x) \, dx \\
\hspace{5em} + \int_{\Gamma_N} g(x) \cdot \gamma(v)(x) \, dS(x).
\end{cases}
\]
Gradient Scheme for elasticity equations

Continuous equation

\[
\begin{cases}
\text{Find } \bar{u} \in H^1_{\Gamma_D}(\Omega) \text{ such that, for any } v \in H^1_{\Gamma_D}(\Omega), \\
\int_{\Omega} \sigma(x, \varepsilon(\bar{u})(x)) : \varepsilon(v)(x) \, dx &= \int_{\Omega} F(x) \cdot v(x) \, dx \\
&+ \int_{\Gamma_N} g(x) \cdot \gamma(v)(x) \, dS(x).
\end{cases}
\]

Discretisation

\[
\begin{cases}
\text{Find } u \in X_{D,\Gamma_D} \text{ such that, for any } v \in X_{D,\Gamma_D}, \\
\int_{\Omega} \sigma(x, \varepsilon_D(u)(x)) : \varepsilon_D(v)(x) \, dx &= \int_{\Omega} F(x) \cdot \Pi_D(v)(x) \, dx \\
&+ \int_{\Gamma_N} g(x) \cdot T_D(v)(x) \, dS(x).
\end{cases}
\]

where \( \varepsilon_D(u) = \frac{\nabla_D u + (\nabla_D u)^T}{2} \).
Plan

1. Numerical methods for elasticity equations

2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. Conclusion
Gradient Scheme framework for elasticity: 3 properties to ensure convergence

A sequence \((D_m)\) of Gradient discretisations is:

**Coercive** if there exists \(C\) such that, for all \(m\) and \(v \in X_{D_m, \Gamma_D}\),

\[
\|\Pi_{D_m} v\|_{L^2} \leq C \|\nabla_{D_m} v\|_{L^2},
\]

\[
\|T_{D_m} v\|_{L^2} \leq C \|\nabla_{D_m} v\|_{L^2},
\]

\[
\|\nabla_{D_m} v\|_{L^2} \leq C \|\varepsilon_{D_m} v\|_{L^2}
\]

(*Poincaré’s, trace and Körm’s inequalities*).
A sequence \((D_m)\) of Gradient discretisations is:

**Consistent** if, for all \(\varphi \in H^1_{\Gamma_D}(\Omega)\)

\[
S_{D_m}(\varphi) := \min_{v \in X_{D_m, \Gamma_D}} (\|\Pi_{D_m} v - \varphi\|_{L^2} + \|\nabla_{D_m} v - \nabla \varphi\|_{L^2})
\]

tends to 0 as \(m \to \infty\).

*(Ultimate density of the range of the discrete reconstruction and trace).*
Gradient Scheme framework for elasticity: 3 properties to ensure convergence

A sequence \((\mathcal{D}_m)\) of Gradient discretisations is:

**Limit-conforming** if, for all \(\boldsymbol{\tau} \in (L^2)^{d \times d}\) such that \(\text{div}(\boldsymbol{\tau}) \in (L^2)^d\) and \(\gamma_n(\boldsymbol{\tau}) \in L^2(\Gamma_N)\),

\[
W_{\mathcal{D}_m}(\varphi) := \max_{\begin{subarray}{c} \mathbf{v} \in X_{\mathcal{D}_m,\Gamma_D} \\ \mathbf{v} \neq 0 \end{subarray}} \left| \int_{\Omega} (\nabla_{\mathcal{D}_m} \mathbf{v}) : \boldsymbol{\tau} + (\Pi_{\mathcal{D}_m} \mathbf{v}) \text{div}(\boldsymbol{\tau}) - \int_{\Gamma_N} \gamma_n(\boldsymbol{\tau}) \cdot \mathbf{T}_{\mathcal{D}_m}(\mathbf{v}) \right| \right|_{L^p} \rightarrow 0 \text{ as } m \rightarrow \infty.
\]

\((\lim_m (\nabla_{\mathcal{D}_m})^* \approx -\text{div} \text{ and } \lim_{m \rightarrow \infty} \mathbf{T}_{\mathcal{D}_m} \approx \gamma \text{ in weak topology})\).
Plan

1. Numerical methods for elasticity equations

2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. Conclusion
Linear elasticity: \(-\text{div}(\mathbb{C}\varepsilon(\overline{u})) = F\text{ in } \Omega\).

**Theorem (Error estimates for linear elasticity)**

Assume that \(\mathbb{C}\) is bounded and coercive, that \(F \in (L^2)^d\) and that \(g \in (L^2)^d\). If \((\mathcal{D}_m)\) is a **coercive** family of Gradient Discretization then

\[
\|\overline{u} - \Pi_{\mathcal{D}_m} u_m\|_{L^2(\Omega)} + \|\nabla\overline{u} - \nabla\mathcal{D}_m u_m\|_{L^2(\Omega)^d} \leq CW_{\mathcal{D}_m}(\mathbb{C}\varepsilon(\overline{u})) + CS_{\mathcal{D}_m}(\overline{u}).
\]

In particular, if \((\mathcal{D}_m)\) is **consistent** and **limit-conforming**, then
\(\Pi_{\mathcal{D}_m} u_m \to \overline{u}\) and \(\nabla\mathcal{D}_m u_m \to \nabla\overline{u}\) in \(L^2\).
Linear elasticity: \(-\text{div}(C \varepsilon(\mathbf{u})) = \mathbf{F} \text{ in } \Omega\). 

**Theorem (Error estimates for linear elasticity)**

Assume that \(C\) is bounded and coercive, that \(\mathbf{F} \in (L^2)^d\) and that \(g \in (L^2)^d\). If \(\mathcal{D}_m\) is a **coercive** family of Gradient Discretization then

\[
\|\mathbf{u} - \Pi_{\mathcal{D}_m} \mathbf{u}_m\|_{L^2(\Omega)} + \|\nabla \mathbf{u} - \nabla_{\mathcal{D}_m} \mathbf{u}_m\|_{L^2(\Omega)^d} \leq CW_{\mathcal{D}_m}(C \varepsilon(\mathbf{u})) + CS_{\mathcal{D}_m}(\mathbf{u}).
\]

In particular, if \(\mathcal{D}_m\) is **consistent** and **limit-conforming**, then

\(\Pi_{\mathcal{D}_m} \mathbf{u}_m \to \mathbf{u}\) and \(\nabla_{\mathcal{D}_m} \mathbf{u}_m \to \nabla \mathbf{u}\) in \(L^2\).

\(\mathbf{C}(\mathbf{x})\) may be discontinuous, no regularity assumption on \(\mathbf{u}\).
Linear elasticity:  \(-\text{div}(\mathbb{C} \varepsilon(\bar{u})) = F \text{ in } \Omega\).

**Theorem (Error estimates for linear elasticity)**

Assume that \(\mathbb{C}\) is bounded and coercive, that \(F \in (L^2)^d\) and that \(g \in (L^2)^d\). If \((D_m)\) is a **coercive** family of Gradient Discretization then

\[
\|\bar{u} - \Pi_{D_m} u_m\|_{L^2(\Omega)} + \|\nabla \bar{u} - \nabla D_m u_m\|_{L^2(\Omega)^d} \leq CW_{D_m}(\mathbb{C} \varepsilon(\bar{u})) + CS_{D_m}(\bar{u}).
\]

*In particular, if \((D_m)\) is **consistent** and **limit-conforming**, then \(\Pi_{D_m} u_m \to \bar{u}\) and \(\nabla D_m u_m \to \nabla \bar{u}\) in \(L^2\).*

▶ \(\mathbb{C}(x)\) may be discontinuous, no regularity assumption on \(\bar{u}\).

▶ Error estimates if \(\mathbb{C}\) is Lipschitz and \(\bar{u} \in H^2\):

\[
W_{D_m}(\mathbb{C} \varepsilon(\bar{u})) + S_{D_m}(\bar{u}) = \mathcal{O}(h_m) \quad (h_m = \text{mesh size}).
\]
Non-linear elasticity: \(-\text{div}(\sigma(\cdot, \varepsilon(\bar{u}))) = F \text{ in } \Omega.\)

**Theorem (Convergence for non-linear elasticity)**

Assume that \(\sigma\) has a linear growth, is coercive and strictly monotone, that \(F \in (L^2)^d\) and that \(g \in (L^2)^d\).

If \((D_m)\) is a coercive, consistent and limit-conforming family of Gradient Discretization then, up to a subsequence, \(\Pi_{D_m} u_m \to \bar{u}\) and \(\nabla D_m u_m \to \nabla \bar{u}\) in \(L^2\).
Non-linear elasticity: \(-\text{div}(\sigma(\cdot, \varepsilon(\overline{u}))) = F \text{ in } \Omega.\)

Theorem (Convergence for non-linear elasticity)

Assume that \(\sigma\) has a linear growth, is coercive and strictly monotone, that \(F \in (L^2)^d\) and that \(g \in (L^2)^d\).

If \((D_m)\) is a coercive, consistent and limit-conforming family of Gradient Discretization then, up to a subsequence, \(\Pi_{D_m} u_m \rightarrow \overline{u}\) and \(\nabla D_m u_m \rightarrow \nabla \overline{u}\) in \(L^2\).

Covered models:

- Damage models \(\sigma(x, \varepsilon) = (1 - D(\varepsilon))C(x)\varepsilon\) (Cervera, Chiumenti, Codina 2010).

- non-linear Hencky-von Mises elasticity
\[ \sigma = \lambda(\text{dev}(\varepsilon)) \text{tr}(\varepsilon)I + 2\mu(\text{dev}(\varepsilon))\varepsilon.\]
Plan

1. Numerical methods for elasticity equations
2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results
3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation
4. Conclusion
Plan

1. Numerical methods for elasticity equations

2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results

3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation

4. Conclusion
Conforming methods (Galerkin approximation)

Replace $\mathbf{H}^1_{\Gamma_D}$ with $\mathbf{X}_{D,\Gamma_D}$ in the weak continuous formulation!

- $\mathbf{X}_{D,\Gamma_D} =$ finite-dimensional subspace of $\mathbf{H}^1_{\Gamma_D}(\Omega)$,
- $\Pi_D = \text{Id}$, $\mathcal{T}_D = \gamma$ and $\nabla_D = \nabla$.

**Example**: any low- or high-degree conforming Finite Element method (e.g. $P1$ on triangles or simplices, bilinear functions on quadrilaterals, etc.)
Given $\mathcal{T}$ a triangulation of $\Omega$,

- $X_{\mathcal{T},\Gamma_D} =$ space of piecewise linear functions on $\mathcal{T}$, which are continuous at the edge mid-points,
- $\Pi_D = \text{Id}$, $T_D =$ restriction to $\partial \Omega$ and $\nabla_D =$ broken gradient.
Given $T$ a triangulation of $\Omega$,

- $X_{D,\Gamma_D} =$ space of piecewise linear functions on $T$, which are continuous at the edge mid-points,
- $\Pi_D = \text{Id}$, $\mathcal{T}_D =$ restriction to $\partial \Omega$ and $\nabla_D =$ broken gradient.

▶ May not be coercive (no Körn inequality) if $\Gamma_D \neq \partial \Omega$.
Higher order methods (still Gradient Schemes) then required.
Plan

1. Numerical methods for elasticity equations
2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results
3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation
4. Conclusion
\( \mathbf{V}_h \) standard Finite Element space on a partition \( \mathcal{T}_h \) of \( \Omega \).
\( \mathcal{T}_h^* \) = dual mesh.
In the weak formulation of the FE scheme, replace

\[ \int_{\Omega} C \varepsilon(u_h) : \varepsilon(v_h) \]

with

\[ \int_{\Omega} \Pi^* \varepsilon(u_h) : C \varepsilon(v_h) + \int_{\Omega} D(\varepsilon(u_h) - \Pi^* \varepsilon(u_h)) : \varepsilon(v_h) \, dx \]

where \( D \) is symmetric definite positive and \( \Pi^* \) = orthogonal projection on piecewise constant functions on \( T_h^* \).

Gradient Discretisation

- $X_D, \Gamma_D = V_h$, $\Pi_D = \text{Id}$, $T_D = \gamma$.
- For $v \in X_D, \Gamma_D$,

\[
\nabla_D v = \Pi_h \nabla v + C^{-1/2} D^{1/2}(\nabla v - \Pi_h \nabla v)
\]

(for $C$ and $D$ piecewise constant on $T_h^*$).

- Orthogonality properties of $\Pi_h^*$ and $I - \Pi_h^*$ eliminate the cross products in $\int_\Omega C \varepsilon_D(u) : \varepsilon_D(v)$. 
Gradient Discretisation

- $X_D, \Gamma_D = \mathbf{V}_h$, $\Pi_D = \text{Id}$, $\mathcal{T}_D = \gamma$.

- For $v \in X_D, \Gamma_D$,

$$\nabla_D v = \Pi^*_h \nabla v + C^{-1/2} D^{1/2}(\nabla v - \Pi^*_h \nabla v)$$

(for $C$ and $D$ piecewise constant on $\mathcal{T}_h^*$).

▶ Orthogonality properties of $\Pi^*_h$ and $\text{I} - \Pi^*_h$ eliminate the cross products in $\int_\Omega C \varepsilon_D(u) : \varepsilon_D(v)$.

▶ Consistency and limit-conformity follow because

$$\nabla_D v = \nabla v + L_h \nabla v$$

where $L_h : (L^2)^d \rightarrow (L^2)^d$ is self-adjoint, bounded and converges pointwise to 0.
Plan

1 Numerical methods for elasticity equations

2 Gradient Schemes for elasticity equations
   • 4 discrete elements
   • 3 properties
   • Convergence results

3 Examples of Gradient Schemes for elasticity equations
   • Displacement-based formulation
   • Stabilised nodal strain formulation
   • Hu-Washizu-based formulation

4 Conclusion
Based on a 3-field formulation ($u$, $\varepsilon$ and $\sigma$ approximated in three different spaces).

Gives stable numerical scheme in the nearly-incompressible limit.

Can be reduced to a displacement formulation by static condensation.

(Lamichhane, Reddy & Wohlmuth, 2006).
Reduced displacement formulation of the Hu-Washizu method

$V_h$ space of bilinear conforming Finite Element on quadrilaterals. In the weak formulation of the FE scheme, replace

$$\int_\Omega C \varepsilon(u_h) : \varepsilon(v_h)$$

with

$$\int_\Omega C_h P_{S_h} \varepsilon(u_h) : P_{S_h} \varepsilon(v_h) \, dx.$$
Reduced displacement formulation of the Hu-Washizu method

$V_h$ space of bilinear conforming Finite Element on quadrilaterals. In the weak formulation of the FE scheme, replace

$$\int_\Omega C\varepsilon(u_h) : \varepsilon(v_h)$$

with

$$\int_\Omega C_h P_{S_h}\varepsilon(u_h) : P_{S_h}\varepsilon(v_h) \, dx.$$

- $S_h = \text{a suitable sub-space of } V_h \text{ (several possible examples)},$
- $P_{S_h} = \text{orthogonal projection on } S_h,$
- $C_h = \text{approximation of } C \text{ defined by}$

$$\forall \tau \in V_h, : \; C_h \tau = C P_{S_h^c} \tau + \theta P_{S_h^t} \tau$$

where

$$S_h^c = \{ \tau \in S_h : C \tau \in S_h \}, \quad S_h = S_h^c \oplus S_h^t.$$
Gradient Discretisation

- \( X_D, \Gamma_D = V_h, \Pi_D = \text{Id}, \ T_D = \gamma, \)
- For \( v \in X_D, \Gamma_D, \)

\[
\nabla_D v = P_{S_h} \nabla v + \sqrt{\theta} C^{-1/2} P_{S_h^t} \nabla v.
\]

▶ The particular choices of \( S_h \) (and orthogonality properties) eliminate the cross products.
Gradient Discretisation

- \( \mathbf{X}_D, \Gamma_D = \mathbf{V}_h, \Pi_D = \text{Id}, \ T_D = \gamma, \)
- For \( \mathbf{v} \in \mathbf{X}_D, \Gamma_D, \)

\[
\nabla_D \mathbf{v} = P_{S_h} \nabla \mathbf{v} + \sqrt{\theta} \mathcal{C}^{-1/2} P_{S_h} \nabla \mathbf{v}.
\]

- The particular choices of \( S_h \) (and orthogonality properties) eliminate the cross products.

- Consistency and limit-conformity follow because

\[
\nabla_D \mathbf{v} = \nabla \mathbf{v} + \mathcal{L}_h \nabla \mathbf{v}
\]

where \( \mathcal{L}_h : (L^2)^d \to (L^2)^d \) is self-adjoint, bounded and converges pointwise to 0.
Plan

1. Numerical methods for elasticity equations
2. Gradient Schemes for elasticity equations
   - 4 discrete elements
   - 3 properties
   - Convergence results
3. Examples of Gradient Schemes for elasticity equations
   - Displacement-based formulation
   - Stabilised nodal strain formulation
   - Hu-Washizu-based formulation
4. Conclusion
Add two branches...

**Methods**
- FE
- MFE
- MPFA
- MFV
- DDFV
- MFD
- Stabilised/projected FE

**Models**
- Linear diffusion
- Non-linear diffusion
- Multi-phase flow
- Stefan problem
- Image processing
- Non-conservative eq.
- Linear & nonlinear elasticity

**Gradient Scheme framework**
Thanks.