Non-Archimedean Geometry

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Plan

Introduction to non-Archimedean geometry.
Berkovich spaces.
Some recent progress.
The Archimedean Axiom

The Archimedean axiom states that sufficiently many copies of a given line segments together become longer than any other given line segment.

Attributed to Archimedes of Syracuse (287-212 BC). Archimedes himself attributed it to Eudoxus of Cnidus (408-355 BC).

Roughly speaking, non-Archimedean geometry is geometry without this axiom. In practice, this can mean different things. We shall view non-Archimedean geometry as an analogue of complex geometry.
Differential geometry

The main objects in differential geometry are manifolds. These locally look like Euclidean space $\mathbb{R}^n$.

Transition functions are smooth (differentiable functions). Manifolds can have interesting topology. Manifolds come with differential forms and other objects. Can add/study additional structures: line bundles, vector bundles, and connections and metrics on these.
Complex geometry

Complex manifolds locally look like balls in $\mathbb{C}^n$. Transition functions are analytic: locally given by convergent power series.

Complex manifolds are more rigid than differentiable manifolds. One reason is uniqueness of analytic continuation: two analytic functions that agree on an open set must be the same.

Example: the zero set $\bigcap_i \{ f_i = 0 \}$ of a set of polynomials $f_i$ with complex coefficients is often a complex manifold.
Valued fields

Now replace \( \mathbb{C} \) another valued field i.e. a field \( K \) with a norm \( | \cdot | : K \to [0, \infty) \) satisfying

\[
|a| = 0 \iff a = 0,
\]

\[
|a + b| \leq |a| + |b| \quad \text{and} \quad |ab| = |a||b|.
\]

Examples:

- \( \mathbb{C} \) or \( \mathbb{R} \) with the usual norm.
- \( \mathbb{Q} \) with the \( p \)-adic norm: \( |p| < 1 \).
- Formal Laurent series \( \mathbb{C}((t)) \): \( |t| < 1 \).
- Any field \( K \) with the \textit{trivial norm}: \( |a| = 1 \) if \( a \neq 0 \).

Can talk about convergent power series with coefficients in \( K \).

Can we do analytic geometry over general valued fields?
Non-Archimedean fields

The valued fields
- \( \mathbb{Q} \) with the \( p \)-adic norm
- Formal Laurent series \( \mathbb{C}((t)) \)
- Any field \( K \) with the trivial norm

are non-Archimedean: they satisfy the strong triangle inequality

\[
|a + b| \leq \max\{|a|, |b|\}.
\]

Relation to the Archimedean axiom: \( K \) is not non-Archimedean iff there exists an integer \( n > 1 \) such that

\[
|n| = |1 + 1 + \cdots + 1| > 1.
\]

**Fact:** most valued fields are non-Archimedean!
Can we do analytic geometry over general valued fields?
General curiosity!

Number theory: study complexity of numbers by looking at their size under different absolute values. Based on...

Ostrowski’s Theorem: any absolute value on \( \mathbb{Q} \) is equivalent to one of the following:

- The trivial norm.
- The usual Archimedean norm.
- The \( p \)-adic norm for some prime \( p \).

Degenerations. Can use \( \mathbb{C}((t)) \) to describe degenerations of complex manifolds.

Can also use \( \mathbb{C} \) with the trivial norm to describe degenerations.
Naive approach

Can try to develop analytic geometry as with complex manifolds: glueing together balls using analytic functions as “glue”.

**Problem 1**: balls in $K^n$ are totally disconnected and look like:

A discrete set if $K$ is trivially valued.
A Cantor set otherwise.

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Glueing together such sets won’t yield interesting topology!

**Problem 2**: uniqueness of analytic continuation fails.

Too many analytic functions with the naive definition. (This is related to Problem 1).
There have been several approaches to developing analytic geometry over non-Archimedean fields:

- M. Krasner (1940’s): function theory.
- J. Tate (1960’s): rigid spaces.

I will focus on Berkovich spaces.

Berkovich spaces are obtained by gluing together basic building blocks, called **affinoids**.

Definition of affinoids and (especially) gluing is a bit tricky…

… but it’s easy to define what the Berkovich affine space is!
The Berkovich affine space

The Berkovich affine $n$-space $\mathbb{A}_K^{n,\text{an}}$ over $K$ is the set of all multiplicative seminorms

$$|\cdot| : K[z_1, \ldots, z_n] \to [0, \infty)$$

that extend the given norm on $K$.

Every point $a \in K^n$ defines a seminorm in $\mathbb{A}_K^{n,\text{an}}$:

$$f \mapsto |f(a)|$$

so we get an injective map $K^n \hookrightarrow \mathbb{A}_K^{n,\text{an}}$.

If $K = \mathbb{C}$ with the usual norm, Gelfand-Mazur implies that this map is surjective, so $\mathbb{A}_\mathbb{C}^{n,\text{an}} = \mathbb{C}^n$.

However, if $K$ is non-Archimedean, $\mathbb{A}_K^{n,\text{an}} \supseteq K^n$.

**Thm.** $\mathbb{A}_K^{n,\text{an}}$ is locally compact and path connected.

Can similarly define the analytification $X^{\text{an}}$ of a general algebraic variety (or scheme) $X$ over $K$. 
The Berkovich affine line

Can describe the Berkovich affine line $\mathbb{A}^{1,\text{an}}_K$ quite concretely. Exemplify for a *trivially valued and algebraically closed* field $K$. $\mathbb{A}^{1,\text{an}}_K$ is the set of seminorms $|\cdot|$ on $K[z]$ that are trivial on $K$.

Case 1: $|\cdot|$ is trivial on $K$.

Case 2: $|z| > 1$. The seminorm is then uniquely determined by $r := |z| \in (1, \infty)$:

$$|\sum_j c_j z^j| = \max_j |c_j z^j| = \max_{c_j \neq 0} r^j$$

Case 3: $\exists! a \in K$ such that $|z - a| < 1$. The seminorm is then uniquely determined by $a$ and by $r := |z - a| \in [0, 1)$:

$$|\sum_j c_j (z - a)^j| = \max_j |c_j (z - a)^j| = \max_{c_j \neq 0} r^j$$
The Berkovich affine line

\( \mathbb{A}^{1, \text{an}}_K \) looks like a tree with endpoints given by \( K \cup \{\infty\} \) and a simple branch point.

The branch point is the trivial norm (called the Gauss point).
When $K$ is non-Archimedean and non-trivially valued, the Berkovich affine line looks like a tree with lots of branching.

Every point in $K \cup \{\infty\}$ determines an endpoint in $\mathbb{A}^{1,\text{an}}_K$ but there may be other endpoints, too (depends on $K$).
The Berkovich projective line and other curves

Get the Berkovich projective line $\mathbb{P}_{\mathcal{K}}^{1,\text{an}} = \mathbb{A}_{\mathcal{K}}^{1,\text{an}} \cup \{\infty\}$ by adding one point. This is also a tree.

Berkovich curves of higher genus look like trees except that they may have finitely many loops. (Matt Baker calls them *arboreta*).
Higher dimensional Berkovich spaces are harder to visualize, but:

- The Berkovich affine plane over a trivially valued field “locally” looks like a cone over an $\mathbb{R}$-tree.
- Many Berkovich spaces can be viewed as an inverse limit of simplicial complexes (Kontsevich-Soibelman, Payne, Foster-Gross-Payne, Boucksom-Favre-J).

In any case, it is not always necessary to visualize a Berkovich space in order to work with it!
Recent work/progress on Berkovich spaces

- Topological structure of Berkovich spaces (Berkovich, Hrushkovski-Loeser, Thuillier).
- Differential forms (and currents) on Berkovich spaces. (Chambert-Loir and Ducros).
- Dynamics (Rivera, Favre, J, Kiwi, Baker, Rumely, Yuan, Zhang, Ruggiero, Gignac, DeMarco, Faber, . . .)
- Connections to tropical geometry [Payne, Baker-Payne-Rabinowitz, Foster-Gross-Payne, . . .]
- Differential Equations [Kedlaya, Poineau, Pulita . . .]
- Potential Theory [Zhang, Kontsevich-Tschinkel, Thuillier, Baker-Rumely, Boucksom-Favre-J . . .]
- Metrics on line bundles on Berkovich spaces (Zhang, Gubler, Chambert-Loir, Ducros, Yuan, Liu, Boucksom-Favre-J).
- etc. . .
Now describe two types of results in non-Archimedean geometry and dynamics.

(1) A non-Archimedean Calabi-Yau theorem.

(2) Degree growth of plane polynomial maps.

The result in (1) is *formulated* in terms of Berkovich spaces (but may be useful in the study of Kähler metrics).

In (2) the formulation of the problem does not involve Berkovich spaces, but the *method* does.
Complex Calabi-Yau Theorem

\[ X = \text{smooth, complex projective variety} \]
\[ L \rightarrow X \text{ ample line bundle.} \]
\[ c_1(L, \| \cdot \|) \text{ curvature form of smooth metric } \| \cdot \| \text{ on } L. \]

**Calabi-Yau Theorem:** if \( \mu \) is a smooth positive measure on \( X \) of mass \( c_1(L)^n \), then there exists a unique (up to scaling) positive metric \( \| \cdot \| \) on \( L \) such that

\[ c_1(L, \| \cdot \|)^n = \mu. \quad (1) \]

Can view (1) as nonlinear PDE, a *Monge-Ampère equation.*  
Kołodziej, and later Guedj and Zeriahi, generalized this to the case when \( \mu \) is a positive, not too singular measure. Metric is then no longer smooth.
A non-Archimedean Calabi-Yau theorem

$K = \text{discretely valued field of residue characteristic 0, or trivially valued field of characteristic 0.}$

$X = \text{smooth projective variety over } K.$

$L \rightarrow X \text{ ample line bundle.}$

$\mu = \text{not too singular measure on Berkovich space } X^{\text{an}}.$

**Thm** [Boucksom-Favre-J, 2012,2013] There exists a semipositive continuous metric $\| \cdot \|$ on $L$ such that $c_1(L, \| \cdot \|)^n = \mu.$ The metric is unique up to a multiplicative constant.

All concepts in theorem were previously introduced by Berkovich, Zhang, Gubler, Chambert-Loir . . .

Previous results:

- Kontsevich-Tschinkel (2001): sketch for $\mu = \text{Dirac mass.}$
- Thuillier (2005): $X = \text{curve.}$
- Liu (2010): existence when $X = \text{abelian variety.}$
Degree growth of plane polynomial maps

\( f : \mathbb{C}^2 \to \mathbb{C}^2 \) polynomial mapping.

\( f(x, y) = (p(x, y), q(x, y)) \).

\( \deg f := \max\{\deg p, \deg q\} \)

\( f^n := f \circ f \circ \cdots \circ f \) (\( n \) times).

Easy to see that \( \deg f^{n+m} \leq \deg f^n \cdot f^m \).

Define (first) dynamical degree as

\[
\lambda_1 := \lim_{n \to \infty} \left( \frac{\deg f^n}{n} \right)^{1/n} = \inf_{n \to \infty} \left( \frac{\deg f^n}{n} \right)^{1/n}
\]

Called algebraic entropy by Bellon-Viallet.

Example: \( f(x, y) = (y, xy) \)

\( f^2(x, y) = (xy, xy^2), f^3(x, y) = (xy^2, x^2y^3), \ldots \)

\( \deg f^n = \text{Fibonacci}_{n+2}, \)

\( \lambda_1 = \frac{\sqrt{5}+1}{2} \).
Degree growth of plane polynomial maps

**Thm** (Favre-J, 2011) For any polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^2$ the following holds:

- The dynamical degree $\lambda_1$ is a quadratic integer.
- The sequence $(\deg f^n)_{n=1}^{\infty}$ satisfies integral linear recursion: there exists $k \geq 1$ and $c_j \in \mathbb{Z}$, $1 \leq j \leq k$, such that

$$\deg f^n = \sum_{j=1}^{k} c_j \deg f^{n-j}$$

Related work by Gignac and Ruggiero (2013).

Main idea is to study the induced action on the Berkovich affine plane $\mathbb{A}_{\mathbb{C}}^{2,\text{an}}$ over the field $\mathbb{C}$ equipped with the trivial norm.

Exploit the fact that this affine plane has an invariant subset with the structure of a cone over a tree.

Unclear if the theorem is true in dimension $\geq 3$. 
Degree growth: outline of proof

Look at induced map on the Berkovich space $A^2_{\mathbb{C}}$. $A^2_{\mathbb{C}}$ contains invariant subset which is a cone over an $\mathbb{R}$-tree. Perron-Frobenius-type argument gives point $x \in A^2_{\mathbb{C}}$ such that

$$f(x) = \lambda_1 \cdot x$$

Several cases. In one case, the value group of $x$ has rank two, generated by $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$, with $\alpha_1/\alpha_2 \notin \mathbb{Q}$. The equation $f(x) = \lambda_1 \cdot x$ leads to

$$\lambda_1 \alpha_i \in \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \quad i = 1, 2.$$ 

Thus $\lambda_1$ is a quadratic integer!

The proof that the sequence $(\deg f^n)_{n=1}^{\infty}$ satisfies linear recursion requires more work but uses that the ray $\mathbb{R}_{>0}$ is attracting.