Spaces of holomorphic maps from Stein manifolds to Oka manifolds

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Deforming continuous maps to holomorphic maps

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All complex Lie groups and their homogeneous spaces are Oka.
\( \mathbb{C}^* \) is Oka but \( \mathbb{D}^* \) is not.
The problem

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Can it be done for all $f$ at once, in a way that depends continuously on $f$ and leaves $f$ fixed if it is holomorphic to begin with?

Parametrised versions of Gromov’s theorem are known for ‘small’ parameter spaces: Euclidean compacts, e.g. finite polyhedra. Hence also for CW complexes (FL 2004). But $C$ and $O$ are CW complexes only in trivial cases. They are metrisable, but a metrisable CW complex is locally compact. Using homotopy theory and infinite-dimensional topology, we can solve the problem for reasonable $S$ and arbitrary $X$. 
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Reformulate the problem

By basic algebraic topology, the following are equivalent.

(i) $\mathcal{O}(S, X)$ is a deformation retract of $\mathcal{C}(S, X)$.

(ii) The inclusion $\iota : \mathcal{O}(S, X) \hookrightarrow \mathcal{C}(S, X)$ is a homotopy equivalence and has the homotopy extension property.
By basic algebraic topology, the following are equivalent.

(i) \( O(S, X) \) is a deformation retract of \( C(S, X) \).

(ii) The inclusion \( \iota : O(S, X) \hookrightarrow C(S, X) \) is a homotopy equivalence and has the homotopy extension property. That is, \( \iota \) is an acyclic cofibration in the h-structure on \( \text{Top} \).
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The \( h \)-structure (\( h \) for Hurewicz) is one of the two classical frameworks for standard homotopy theory. The other is the \( q \)-structure (\( q \) for Quillen).
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A parametrised version of Gromov’s theorem for finite polyhedra implies that \( \iota \) is a weak homotopy equivalence. How can we bridge the gap?
ANRs and the mixed structure on Top

Two main topological ingredients:
The brand new m-structure \((m\ \text{for} \ \text{mixed})\), due to Cole (2006): a third framework for standard homotopy theory.
The theory of ANRs (absolute neighbourhood retracts for metric spaces).
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To cut a long story short:

**Theorem** (FL). Suppose $C(S, X)$ is ANR. Then $O(S, X)$ is a deformation retract of $C(S, X)$ if and only if $O(S, X)$ is ANR.

**Theorem** (Milnor 1959, Smrekar-Yamashita 2009). $C(S, X)$ is ANR if $S$ is finitely dominated.

We need a good sufficient condition for $O(S, X)$ to be ANR.
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We need a good sufficient condition for \( O(S,X) \) to be ANR.
**Absolute neighbourhood retracts**

A metrisable space $A$ is ANR if whenever $A$ is embedded as a closed subspace of a metric space $B$, some neighbourhood of $A$ in $B$ retracts onto $A$. 

There are several other characterisations, including a 'combinatorial' one (Dugundji-Lefschetz) that we use.

ANRs have many nice properties. Being ANR is a local property. Every ANR is locally contractible (and conversely for finite-dimensional spaces).

A CW complex is ANR if and only if it is locally finite.

ANRs and CW complexes have the same homotopy types.

A metrisable space $A$ is ANR if and only if every open subset has the homotopy type of a CW complex (Cauty 1994).
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The main result

**Theorem.** Let $X$ be an Oka manifold and let $S$ be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold. Then $\mathcal{O}(S, X)$ is a deformation retract of $\mathcal{C}(S, X)$. 


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The hypothesis on $S$ is not necessary, but I do not know whether it can be omitted. For example:

- $\mathcal{O}(S, \mathbb{C}^n)$ is a deformation retract of $\mathcal{C}(S, \mathbb{C}^n)$.
- $\mathcal{O}(C \setminus N, \mathbb{C}^*)$ is a deformation retract of $\mathcal{C}(C \setminus N, \mathbb{C}^*)$.

Still, $\mathcal{C}$ and $\mathcal{O}$ are not ANR: they are not semilocally contractible, so they do not even have the homotopy type of an ANR (or of a CW complex).
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