Chevalley groups and finite geometry

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Theis problem: Is there an avoid in the finite Hermitian variety \( H(5, q^2) \)?

No for \( q=2 \) [de Beule - Metsch 2006]

Unknown for \( q > 2 \).

Idea: Craft a hammer

\[
\text{Big words: using Schur faith, Chevalley groups, Steinberg presentations, sheaf cohomology, intersection cohomology}
\]
Incidence geometries

An incidence geometry \( G \) is a triple \( (P, L, I) \) where \( P \) and \( L \) are sets and \( I \subseteq P \times L \).

Example

**Quadrangle:**

\[
\begin{array}{ccc}
|   & l_1 & P_2 \\
\hline
l_4 & b_1 & b_2 \\
P_4 & b_3 & b_4 \\
\end{array}
\]

\[P = \{p_1, p_2, p_3, p_4\}\]
\[L = \{l_1, l_2, l_3, l_4\}\]
\[I = \{(p_1, l_1), (p_2, l_4), (p_3, l_1), \ldots\}\]

An **ovoid** in \( G \) is a set of points \( O \subseteq P \) such that every line \( L \in L \) is incident with \( O \) exactly once.

Example For \( G \) = quadrangle, \( O = \{p_1, p_3\} \) and \( O = \{p_2, p_4\} \) are ovoids.

The finite Hermitian variety \( H(3, q^2) \)

\[H_{q^2} = \text{a finite field with } q^2 \text{ elements, } q \text{ a prime power.}\]
\[V = H_{q^2}^4, \text{ a vector space over } H_{q^2}.\]
The Frobenius automorphism is
\[ \Phi : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^2, \quad c \mapsto c^q \]

The Hermitian form is
\[ \beta : V \times V \rightarrow \mathbb{F}_q \]
\[ \beta(v, w) = v_1 \overline{w}_1 + v_2 \overline{w}_2 + v_3 \overline{w}_3 + v_4 \overline{w}_4. \]

A subspace \( W \subseteq V \) is \textit{totally isotropic} if
\[ \beta(w, w) = 0 \]
for all \( w, w \in W \).

The finite Hermitian variety \( H(3, q^2) \) is the incidence geometry \( g = (P, L, X) \) with
\[ P = \{ 1 \text{-dimensional totally isotropic subspace of } V \} \]
\[ L = \{ 2 \text{-dimensional totally isotropic subspace of } V \} \]
\[ X = \{ (p, l) \in P \times L | p \subseteq l \}. \]

Example of an avoid
\[ \Theta = \{ <v_1, v_2, v_3, 0> | v_1, v_2, v_3 \in \mathbb{F}_q \} \land \mathcal{P} \]

is an avoid in \( H(3, q) \).
Fact: Overods exist in $H(3,q^2)$ for $q \geq 2$.

Recall: (Tham's problem) Is there an overod in $H(5,q^2)$ for $q \geq 2$?

Conversion to Chevalley group world

Studying $H(3,q^2) = $ Studying the Chevalley group $U_4(J_{q^2})$.

Overods in $U_4(J_{q^2})$

Let $G = U_4(J_{q^2})$ and let $P_1$ and $P_2$ be certain parabolic subgroups of $G$. A set of $P_1$ cosets $O = \{g_1P_1, g_2P_1, \ldots, g_kP_1\}$ is an overod if

$$U \left< e_1, e_2, \ldots, e_3 g_iP_1P_2 = G \right.$$

and the union is disjoint.

Analogously, let:

Studying words in $H(5,q^2) = $ Studying words in $U_6(J_{q^2})$.

Let $G = U_6(J_{q^2})$ and let $P_1$ and $P_2$ be certain parabolic subgroups of $G$. A set of $P_1$ cosets $O = \{g_1P_1, g_2P_1, \ldots, g_kP_1\}$ is an overod if

$$U \left< e_i, e_2, \ldots, e_5 g_iP_1P_2P_3 = G \right.$$

and the union is disjoint.
Current task: Find 'good' generators and relations for $U_4(\mathbb{F}_2)$.

[Steinberg, 'Notes on Chevalley groups', 1967... off by a minus sign!]

Future tasks: Find 'good' generators and relations for $U_6(\mathbb{F}_2)$, $Sp_4(\mathbb{F}_2)$.

related to other to be determined