1. (i) The order of \( G = \text{no. of 2-subsets} \)
\[
= \binom{5}{2} = 10.
\]
The vertex \( \{a, b\} \) is adjacent to a vertex \( \{c, d\} \) in 3 ways, as \( \{c, d\} \) is a 2-subset of \( S \setminus \{a, b\} \).
Thus, the degree of each vertex of \( G \) is 3.
By the Handshaking Lemma,
\[
3 \times 10 = 2 \ (\text{sum of degrees of } G)
\]
do sum of \( G = 15 \).

(ii) The vertices \( \{1, 2\}, \{3, 4\}, \{5, 1\}, \{2, 3\}, \{4, 5\} \)
are the vertices of a 5-cycle: \( G_1 \).
Similarly,
\( \{1, 3\}, \{5, 2\}, \{4, 1\}, \{3, 5\}, \{2, 4\} \)
are the vertices of another 5-cycle: \( G_2 \).

N.B. An edge of \( G_1 \) is not an edge of \( G_2 \) and vice versa.
1. (iii) Non-adjacent vertices are 2-subsets (of \( S \)) sharing one element, their union \( U \) has size 3. (i.e. \( |U| = 3 \)). A vertex adjacent to both \( u \) and \( v \) is a 2-subset disjoint from both. Since the 2-subsets are chosen from \( \{1, 2, 3, 4, 5\} \), there is exactly one 2-subset disjoint from \( U \).

(iv) \( G \) is simple, so \( G \) has no 1-cycle or 2-cycle. A 3-cycle would need three pairwise-disjoint 2-subsets, which cannot occur among 5 elements.

A 4-cycle, in the absence of 3-cycles, requires non-adjacent vertices with two common neighbours (which cannot happen by part (iii)). Finally, we exhibit a 5-cycle, \( \{1, 2\}, \{3, 4\}, \{5, 1\}, \{2, 3\}, \{4, 5\} \). Thus \( G \) has girth 5.
2. (i) If \( f \) is an isomorphism from \( G \) to \( H \), then \( f \) is a bijection of the vertex sets of \( G \) and \( H \) preserving adjacency and non-adjacency, and hence \( f \) preserves non-adjacency and adjacency in \( \overline{G} \) and is an isomorphism from \( \overline{G} \) to \( \overline{H} \).

The same argument applies for the converse, since the complement of \( \overline{G} \) is \( G \).

(ii) By part (i), two simple graphs \( G \) and \( H \) are isomorphic iff \( \overline{G} \) and \( \overline{H} \) are isomorphic.

Hence it is sufficient to count the number of 2-regular simple graphs of order 7. (The complement of a 4-regular simple graph of order 7 is a 2-regular simple graph of order 7).

Every component of a 2-regular graph is a cycle.

In a simple graph, each cycle has at least 3 vertices.

Thus, each different 2-regular simple graph is determined by
2. (ii) (cont'd)
partitioning 7 into integers of size at least 3 to be the sizes of the cycles.

The only two graphs are exist are $C_7$:

and $C_3 \times C_4$ (one component $C_3$, one component $C_4$).
3. (i) 

\[ A(G) = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0
\end{bmatrix} \]

\[ D(G) = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix} \]

(ii) \[ M = D - A = \begin{bmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 3 & -1 & -1 & 0 \\
0 & -1 & 3 & -1 & -1 \\
0 & -1 & -1 & 3 & -1 \\
-1 & 0 & -1 & -1 & 3
\end{bmatrix} \]
3. (ii) (cont'd)

\[
\{\text{no. of spanning trees of } G\} = \left\{\frac{\text{any cofactor of } M}{M}\right\}
\]

Taking the (1,1) cofactor of \( M \) we get

\[
(-1)^{1+1} \begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{vmatrix} = 24.
\]

The spanning trees appear below

(assume in each case edges labelled as in part (i)).
4. A planar graph of order \( n \) has at most \( 3n - 6 \) edges.

Hence each planar graph \( G \) of order 11 has at most 27 edges.

It is also a spanning subgraph of \( K_{11} \). \( K_{11} \) has 55 edges. Thus the complement of \( G \) has at least 28 edges and is thus non-planar.

Now consider a planar simple graph \( G \) of order \( > 11 \). Any subgraph \( H \) of \( G \) of order 11 is simple and planar.

From above, \( H \) is non-planar. Hence \( G \) is non-planar.

There are many examples of a self-complementary simple planar graph of order 8.

Here is one:

\[ G \]

\[ \overline{G} \]
5. \[ P_G(t) = P_{G+e}(t) + P_{G\setminus e}(t) \] (1.)

\[ P_G(t) = P_{G-e}(t) - P_{G\setminus e}(t) \] (2.)

Let \( G_1 \):

[Diagram of \( G_1 \)]

Let \( G_2 \):

[Diagram of \( G_2 \)]

Then \( P_G(t) = P_{G_1}(t) \cdot P_{G_2}(t) \).

\( G_1, G_2 \) : components of \( G \).

We find \( P_{G_1}(t) \) first.
5. (cont'd) 

By (1.):

\[ G = K_6 + K_5 + K_5 \]

\[ = t(t-1)(t-2)(t-3)(t-4)(t-5) + 2 \cdot t(t-1)(t-2)(t-3)(t-4) \]

\[ p_{G_1}(t) = t(t-1)(t-2)(t-3)^2(t-4). \]
5. (cont'd)

We now find $P_{G_2}(t)$.
And denote $T_n^2$: tree of order $n$.
We use (2):

\[
\begin{align*}
&= (u \Rightarrow v) - (v \Rightarrow u) - (u = v) \\
&= T_6 - (u \Rightarrow v) - T_5 + (v \Rightarrow u) \\
&= T_6 - T_5 + (u \Rightarrow v) - T_5 + T_4 - T_3 \\
&= T_6 - 2T_5 + 2T_4 - T_3 - K_3 \\
&= t(t-1)^5 - 2t(t-1)^4 + 2t(t-1)^3 - t(t-1)^2 - (t-1)(t-2)^2 \\
\end{align*}
\]

\[
\begin{align*}
P_{G_1}(t) &= t(t-1)(t-2)(t^3 - 4t^2 + 6t - 4) = t(t-1)(t-2)(t^2 + t + 2) \\
P_{G_1}(t) &= t^2(t-1)(t-2)^2(t-3)(t-4)(t^3 - 4t^2 + 6t - 4) \\
&= t^{12} - 20t^{11} + 176t^{10} - 900t^9 + 2978t^8 - 6660t^7 + 10298t^6 \\
&\quad - 10900t^5 + 7576t^4 - 3120t^3 + 876t^2.
\end{align*}
\]
Observe $P_G(0) = P_G(1) = \ldots = P_G(4) = 0$,
[not into factors of $P_G(t)$]
but $P_G(5) > 0$, so $\chi(G) = 5$.

This is no surprising,

$$\chi(G) = \max \{ \chi(G_1), \chi(G_2) \}$$

and

$G_1$ has $K_5$ as subgraph so
$\chi(G_1) \geq 5$, and by Brooks' Theorem
$\chi(G_1) \leq 5$.

Thus, $\chi(G_1) = 5$;
giving $\chi(G) = 5$. 