THE UNIVERSITY OF SYDNEY

MATH2969/2069

Graph Theory **Tutorial 1 (Week 8)** **2008**

1. Draw a picture of each of the following graphs, and state whether or not it is simple.

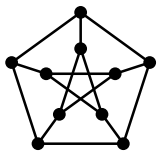
(a) $G_1 = (V_1, E_1)$, where $V_1 = \{a, b, c, d, e\}$ and
$E_1 = \{ab,\ bc,\ ac,\ ad,\ de\}$.

(b) $G_2 = (V_2, E_2)$, where $V_2 = \{P, Q, R, S, T\}$ and
$E_2 = \{PQ,\ PR,\ PS,\ PT,\ TR,\ PR\}$.

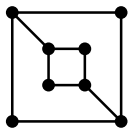
(c) $G_3 = (V_3, E_3)$, where $V_3 = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E_3 = \{v_1v_1,\ v_1v_2,\ v_2v_3,\ v_3v_4,\ v_5v_4,\ v_4v_5\}$.

2. For each of the following graphs write down the number of vertices, the number of edges and the degree sequence. Verify the hand-shaking lemma in each case.

(*i*)



(*ii*)

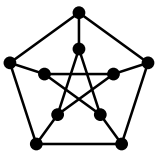


3. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph with degree sequence d. Determine whether or not the following sequences are graphic. If the sequence is graphic, draw a corresponding graph.

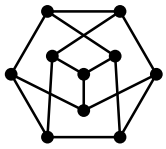
(a) $(2, 3, 3, 4, 4, 5)$ (b) $(2, 3, 4, 4, 5)$ (c) $(1, 1, 1, 1, 4)$
(d) $(1, 3, 3, 3)$ (e) $(1, 2, 2, 3, 4, 4)$ (f) $(1, 3, 3, 4, 5, 6, 6)$
(g) $(2, 2, 2)$

4. (a) By suitably lettering the vertices, prove that the following two graphs are isomorphic:

G :

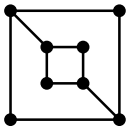


G' :

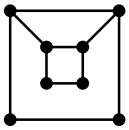


(b) Explain why (i.e., prove that) the following two graphs are not isomorphic:

H :



H' :



5. List all non-identical simple labelled graphs with 4 vertices and 3 edges. How many of these are not isomorphic as unlabelled graphs?

6. (*i*) What is the maximum number of edges in a simple graph on n vertices?

(*ii*) How many simple labelled graphs with n vertices are there?

7. Explain why, in the solution using Graph Theory of the 'Instant Insanity' puzzle, the chosen subgraphs should satisfy the conditions that they

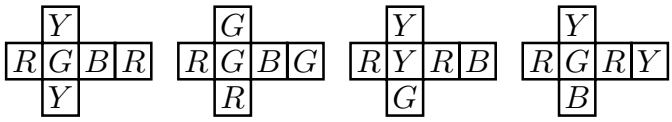
(a) have no edges in common;

(b) contain exactly one edge from each cube;

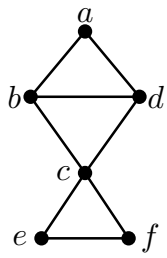
(c) contain only vertices of degree two.

Illustrate your answer by finding a solution for the following set of cubes.

(There are several solutions.)



8. For each of the following sequences of vertices, state whether or not it represents a walk, trail, path, closed walk, closed trail, or cycle in the graph illustrated.



(i) $abcefcbd$ (ii) $abcefcd$

(iii) $abcefcdba$ (iv) $bcefcdb$

(v) $bcdb$ (vi) $abefcd$

9. Suppose that a graph G is regular of degree r, where r is odd.

(i) Prove that G has an even number of vertices.

(ii) Prove that the number of edges in G is a multiple of r.

10. Let u and v be distinct vertices of a graph. Prove that there is a walk from u to v if and only if there is a path from u to v.

11. A simple graph has 20 vertices. Any two distinct vertices u and v are such that $\deg(u) + \deg(v) \geq 19$. Prove that the graph is connected.

12. (i) Draw the graphs formed by the vertices and edges of a tetrahedron, a cube, and an octahedron.

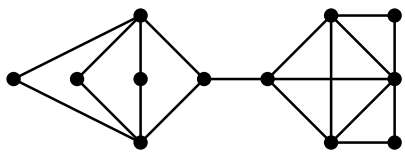
(ii) Are any of these graphs Eulerian?

(iii) Find a Hamiltonian cycle in each graph.

13. (a) Is it possible to draw a sketch of K_5 without lifting your pen from the paper, and without retracing any edges?

(b) For which values of n (≥ 2) is K_n (i) Eulerian? (ii) semi-Eulerian?

14. Find, if possible, an Euler trail or a semi-Euler trail in this graph:



15. (a) A domino is a rectangular tile containing a number of dots (i say) near one end, and a number of dots (j say) near the other end. Call this an $[i, j]$ type domino. Is it possible to arrange 10 dominoes, types $[1, 2]$, $[1, 3]$, $[1, 4]$, $[1, 5]$, $[2, 3]$, $[2, 4]$, $[2, 5]$, $[3, 4]$, $[3, 5]$, $[4, 5]$, end to end so that the numbers of dots on touching ends of adjacent dominoes are always equal?

(b) Repeat the above question in the more general case, when there are $\binom{n}{2}$ dominoes ($n \geq 2$), types all combinations $[i, j]$ with $1 \leq i < j \leq n$.