On Klyachko’s model for the representations of finite general linear groups

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Abstract

Let $G = \text{GL}(n, q)$, the group of $n \times n$ invertible matrices over $F_q$, the field of $q$ elements. A theorem of A. A. Klyachko [5] gives a collection of subgroups $\{ G_d \mid 0 \leq 2d \leq n \}$ of $G$, and for each $d$ a degree 1 complex character $\lambda_d$ of $G_d$, such that the induced characters $\lambda_d^G$ are all multiplicity free, pairwise disjoint, and between them contain as constituents all irreducible complex characters of $G$.

In this paper we derive, for each $g \in G$, a formula relating numbers of $g$-invariant bilinear forms of certain kinds with values of the Gel’fand-Graev character, and show that Klyachko’s theorem follows as a corollary of this.†

§1 Introduction

Let $g \in G$ and let $U$ be a $g$-invariant subspace of $V = F_q^n$, the space of $n$-component column vectors over $F_q$. We shall say that a bilinear form $f: U \times U \to F_q$ is symmetric modulo $g$ if $f(x, y) = f(gy, x)$ for all $x, y \in U$, and we let $\text{Sym}(U, g)$ be the set of all such forms. We denote by $s_g(U)$ the number $f \in \text{Sym}(U, g)$ that are non-degenerate. We also let $\text{Alt}(U, g)$ be the set of all $g$-invariant alternating bilinear forms $U \times U \to F_q$, and write $S_g(U)$ for the number of nondegenerate elements of $\text{Alt}(U, g)$.

Let $\psi$ be a fixed nontrivial homomorphism from the additive group of $F_q$ to $\mathbb{C}^\times$, the multiplicative group of $\mathbb{C}$. The Gel’fand-Graev character of $G$, to be discussed in more detail below, is the character $\Gamma$ of $G$ induced from the degree 1 character $\lambda$ of $X$, the group of all upper unitriangular matrices, given by the formula

$$\lambda(x) = \psi\left(\sum_{i=1}^{n-1} x_{i,i+1}\right)$$

for all $x \in X$ (where we use the notation $x_{i,j}$ for the $(i, j)$-entry of a matrix $x$). For each $g$-invariant subspace $U$ of $V$ we denote by $\Gamma(g, V/U)$ the value of the Gel’fand-Graev character of $\text{GL}(V/U)$ on the transformation of $V/U$ induced by $g$. Our main result is as follows.

(1.1) Theorem. If $g$ is any element of $G$ then $s_g(V) = \sum_U \Gamma(g, V/U)S_g(U)$, where the sum is over all $g$-invariant subspaces $U$ of $V$.

For any matrix $g$, let $g^t$ denote the transpose of $g$. For each positive integer $d$ with $0 \leq 2d \leq n$ choose a nonsingular skew-symmetric $2d \times 2d$ matrix $j_d$ over $F_q$, and define

$$S_d = \{ g \in \text{GL}(2d, q) \mid g^t j_d g = j_d \},$$

† This is a slightly streamlined account of the second author’s PhD thesis (University of Sydney, 1993).
a realization of the symplectic group \( \text{Sp}(2d, q) \). Let \( X_d \) be the group of all upper unitriangular \((n - 2d) \times (n - 2d)\) matrices. Define

\[
G_d = \{ \begin{pmatrix} g & h \\ 0 & x \end{pmatrix} \mid g \in S_d, x \in X_d \},
\]

which is clearly a subgroup of \( G \), and define a character \( \lambda_d \) of \( G_d \) by

\[
\lambda_d \begin{pmatrix} g & h \\ 0 & x \end{pmatrix} = \psi \left( \sum_{i=1}^{n-2d-1} x_{i,i+1} \right).
\]

Observe that \( \lambda_0^G \) is the Gel'fand-Graev character.

Klyachko’s Theorem can be stated as follows.

(1.2) Theorem. With the notation as above, \[
\sum_{d=0}^{[n/2]} \lambda_d^G = \sum_{\chi \in \text{Irr}(G)} \chi.
\]

(Here \( \text{Irr}(G) \) denotes the set of all irreducible complex characters of \( G \).)

Klyachko’s proof of this proceeded by analysing endomorphism algebras of the relevant induced modules, and homomorphisms between them. Another proof was given by Inglis and Saxl [3], who used the classification of the irreducible characters of \( \text{GL}(n, q) \) and identified the constituents of each \( \lambda_d^G \). Our proof uses properties of the twisted indicator function \( \varepsilon \) of Kawanaka and Matsuyama [4] (a generalization of the indicator function of Frobenius and Schur [1]) to show that \( \sum_{\chi \in \text{Irr}(G)} \varepsilon(\chi) \chi(g) \) equals \( s_g(V) \). Combined with Theorem (1.1) and the straightforward fact (also proved below) that

\[
\sum_{d=0}^{[n/2]} \lambda_d^G(g) = \sum_U \Gamma(g, V/U) S_g(U),
\]

this shows that \( \varepsilon(\chi) \) is the multiplicity of \( \chi \) in \( \sum_{d=0}^{[n/2]} \lambda_d^G \). Hence \( \varepsilon(\chi) \geq 0 \) for all \( \chi \). However, the only possible values for \( \varepsilon(\chi) \) (in any case) are 0, 1 and \(-1\), and it is easy to show that in this case 0 does not occur. Hence Klyachko’s Theorem follows.

§2 The twisted indicator function

In order to make this work self-contained we include an account of the twisted indicator function. It is assumed that \( G \) is a finite group and \( \sigma : G \rightarrow G \) an anti-automorphism of \( G \) of order 2. In the case considered by Frobenius and Schur, \( \sigma \) is taken to be the anti-automorphism given by \( g \mapsto g^{-1} \) for \( g \in G \). We shall apply the theory in the case \( G = \text{GL}(n, q) \), with \( \sigma \) defined by \( g^\sigma = g^t \).

Let \( R : G \rightarrow \text{GL}(d, \mathbb{C}) \) be an irreducible matrix representation of \( G \). Then \( R^* : g \mapsto R(g^\sigma)^t \) is obviously also an irreducible representation of \( G \). We are interested in whether or not \( R^* \) is equivalent to \( R \). Suppose that \( R^* \) is, in fact, equivalent to \( R \); that is, there is some \( X \in \text{GL}(d, \mathbb{C}) \) such that \( X^{-1} R(g) X = R(g^\sigma)^t \) for all \( g \in G \). Replacing \( g \) by \( g^\sigma \), taking transposes of both sides, and using the fact that \( \sigma \) has order 2, now yields \( X^t R(g^\sigma)^t (X^t)^{-1} = R(g) \), whence

\[
(X^t)^{-1} R(g) X^t = R(g^\sigma)^t = X^{-1} R(g) X \quad \text{for all } g \in G.
\]
Hence $X^tX^{-1}$ commutes with $R(g)$ for all $g \in G$. Schur’s Lemma now yields that $X^tX^{-1} = \lambda I$ for some $\lambda \in \mathbb{C}$, and we conclude that $X$ is either a symmetric or a skew-symmetric matrix.

Suppose now that $G$ has $s$ conjugacy classes, and for each irreducible character $\chi_k$ of $G$ (for $1 \leq k \leq s$) choose a fixed matrix representation $R^{(k)}$ that is unitary (so that $R^{(k)}(g)^t = \overline{R^{(k)}(g^{-1})}$ for each $g \in G$, where here the overline denotes complex conjugation). For each $g \in G$, let $R^{(k)}(g)$ have $(i, j)$-entry $R^{(k)}_{i,j}(g)$, and let the degree of $R^{(k)}$ be $d_k$. There are $\sum_{k=1}^{s} d_k^2 = |G|$ coordinate functions $g \mapsto R^{(k)}_{i,j}(g)$, parametrized by the set $\mathcal{I}$ consisting of all triples $(k, i, j)$ with $k \in \{1, 2, \ldots, s\}$ and $i, j \in \{1, 2, \ldots, d_k\}$. We place the numbers $R^{(k)}_{i,j}(g)$ in a $|G| \times |G|$ matrix $T$ whose rows are indexed by $\mathcal{I}$ and whose columns are indexed by the elements of $G$.

Orthogonality of coordinate functions and the assumption that each $R^{(k)}$ is unitary gives

$$
\sum_{g \in G} R^{(m)}_{i,j}(g) \overline{R^{(l)}_{r,s}(g)} = \frac{|G| \delta_{lm} \delta_{ij} \delta_{rs}}{d_l}.
$$

Since this shows that $T(T^t)$ is diagonal, with nonzero diagonal entries, we conclude that $T$ is nonsingular.

Let $k \mapsto k^*$ be the permutation of $\{1, 2, \ldots, s\}$ such that $R^{(k^*)}$ is equivalent to $R^{(k)^*}$ for each $k$, and for each $k$ choose a matrix $X^{(k)}$ such that

$$
R^{(k^*)}(g) = X^{(k)^{-1}}R^{(k^*)}(g)X^{(k)}
$$

for all $g \in G$. We define a function $\varepsilon: \{1, 2, \ldots, s\} \to \{-1, 0, 1\}$ as follows:

$$
\varepsilon(k) = \begin{cases} 
+1 & \text{if } k^* = k \text{ and } X^{(k)} \text{ is symmetric}, \\
-1 & \text{if } k^* = k \text{ and } X^{(k)} \text{ is skew-symmetric}, \\
0 & \text{if } k^* \neq k.
\end{cases}
$$

Now fix $g \in G$, and let $P$ denote the permutation matrix corresponding to the permutation of $G$ given by $x \mapsto x^g$ for $x \in G$. Thus the rows and columns of $P$ are indexed by elements of $G$, the $(x, y)$-entry $P_{x,y}$ of $P$ being given by

$$
P_{x,y} = \begin{cases} 
1 & \text{if } x = y^g, \\
0 & \text{otherwise}.
\end{cases}
$$

Observe that the general entry of $TP$, in the $((k, i, j), y)$-position, is given by

$$
[TP]_{(k, i, j), y} = \sum_{x \in G} T_{(k, i, j), x} P_{x,y} = \sum_{x \in G} R^{(k)}_{i,j}(x) P_{x,y} = R^{(k)}_{i,j}(y^g) = \sum_{t} R^{(k)}_{i,t}(y^g) R^{(k)}_{t,j}(g).
$$

But now $R^{(k^*)}(g) = R^{(k)}(y^g)^t$; hence $R^{(k)}_{i,t}(y^g) = R^{(k^*)}_{i,t}(y)$. Thus

$$
R^{(k)}_{i,t}(y^g) = [X^{(k)^{-1}}R^{(k^*)}(y)X^{(k)}]_{t,i} = \sum_{m,n} [X^{(k)^{-1}}]_{t,m} R^{(k^*)}_{m,n}(y) [X^{(k)}]_{n,i},
$$
and so the \((k, i, j), y\)-entry of \(TP\) is

\[
[TP]_{(k, i, j), y} = \sum_{m, n} \left( \sum_{l} [X^{(k)}]_{l, m} [X^{(k)}]_{n, i} R^{(k)}_{l, j} (g) \right) R^{(k^*)}_{m, n} (y).
\]

However, the right hand side of this formula is also the \((k, i, j), y\)-entry of \(QT\), where \(Q\) is the matrix whose rows and columns are indexed by \(I\), and whose general entry, in the \((k, i, j), (r, m, n)\)-position, is given by

\[
Q_{(k, i, j), (r, m, n)} = \delta_{rk^*} \left( \sum_{l} [X^{(k)}]_{l, i} [X^{(k)}]_{j, i} R^{(k)}_{l, j} (g) \right).
\]

It follows that \(Q = TP^{-1}\), and, in particular, the trace of \(Q\) equals the trace of \(P\).

Since \(P\) is simply a permutation matrix, its trace is the number of fixed points of the permutation, which is the number of elements \(x \in G\) with \(x^\sigma g = x\). Alternatively put, it is the number of \(x\) such that \(g = (x^\sigma)^{-1} x\). As for the trace of \(Q\), we find that

\[
\text{Trace } Q = \sum_{k, i, j} \delta_{kk^*} \left( \sum_{l} [X^{(k)}]_{l, i} [X^{(k)}]_{j, i} R^{(k)}_{l, j} (g) \right)
\]

\[
= \sum_{k, i, j} \sum_{l} \varepsilon(k) [X^{(k)}]_{l, i} [X^{(k)}]_{j, i} R^{(k)}_{l, j} (g)
\]

since \(\varepsilon(k)[X^{(k)}]_{i, j}\) is zero if \(k \neq k^*\), and equals \([X^{(k)}]_{j, i}\) if \(k = k^*\). Thus

\[
\text{Trace } Q = \sum_{k, i, j} \sum_{l} \varepsilon(k) \delta_{ij} R^{(k)}_{l, j} (g) = \sum_{k, j} \varepsilon(k) R^{(k)}_{i, j} (g) = \sum_{k} \varepsilon(k) \chi_k (g).
\]

Clearly \(\varepsilon(k)\) depends only on the character \(\chi_k\), and not on the choice of representation \(R^{(k)}\). So for each irreducible character \(\chi_k\) we define \(\varepsilon_\sigma(\chi_k) = \varepsilon(k)\); we call \(\varepsilon_\sigma\) the indicator function corresponding to the antiautomorphism \(\sigma\). Our calculations above have established the following result.

(2.1) **Theorem.** Let \(g\) be an arbitrary element of \(G\). Then \(\sum_{\chi \in \text{Irr}(G)} \varepsilon_\sigma(\chi) \chi(g)\) is equal to the number of \(x \in G\) such that \(g = (x^\sigma)^{-1} x\).

Inverting this relationship using orthogonality of characters gives a formula for \(\varepsilon_\sigma(\chi)\), for each irreducible character \(\chi\).

(2.2) **Theorem.** For each \(\chi \in \text{Irr}(G)\) we have

\[
\varepsilon_\sigma(\chi) = \frac{1}{|G|} \sum_{x \in G} \chi((x^\sigma)^{-1} x).
\]

Furthermore, this quantity is 0, 1 or \(-1\), as described above.
§3  The Gel’fand Graev character

Continuing our policy of making this paper self-contained, in this section we derive the formula for the value of the Gel’fand-Graev character of $G = \text{GL}(n, q)$ at an arbitrary element of $G$. Although the formula is well-known, we were unable to find an elementary derivation of it in the literature.

We define a based flag in a vector space $W$ to be a chain of subspaces

$$\{0\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = W,$$

such that $\dim W_i = i$ for all $i$, together with a choice of basis vector in each of the one-dimensional quotient spaces $W_i/W_{i-1}$. An ordered basis $w_1, w_2, \ldots, w_k$ of $W$ determines a based flag, which we denote by $\mathcal{B}(w_1, w_2, \ldots, w_k)$, and clearly $\text{GL}(W)$ permutes the based flags so that $g(\mathcal{B}(w_1, w_2, \ldots, w_k)) = \mathcal{B}(gw_1, gw_2, \ldots, gw_k)$ for all $g \in \text{GL}(W)$ and all bases $w_1, w_2, \ldots, w_k$.

Before restricting our attention to the case $d = 0$, we consider the character $\lambda^G_d$ for an arbitrary integer $d$ satisfying $0 \leq 2d \leq n$. Let $e_1, e_2, \ldots, e_m$ be the standard basis of $V = \mathbb{F}_q^n$ and $V_0 \subset V_1 \subset \cdots \subset V_n$ the corresponding flag of subspaces. Let $F_d$ be the bilinear form on $V_{2d}$ defined by

$$F_d(x, y) = x^t \begin{pmatrix} jd & 0 \\ 0 & 0 \end{pmatrix} y$$

for all $x, y \in V_{2d}$, and let $\mathcal{E}$ be the based flag in $V/V_{2d}$ given by

$$\mathcal{E} = \mathcal{B}(w_1, w_2, \ldots, w_{n-2d}),$$

where $w_i = e_{2d+i} + V_{2d}$. Then the group $G_d$ consists of all $g \in G$ that preserve the subspace $V_{2d}$, the form $F_d$ and the based flag $\mathcal{E}$. Note that $G$ acts transitively on the set of triples $(U, F, \mathcal{B})$ consisting of a $2d$-dimensional subspace $U$ of $V$, a nondegenerate alternating bilinear form $F$ on $U$, and a based flag $\mathcal{B}$ in $V/U$; hence the left cosets of $G_d$ in $G$ are parametrized by these triples. Let $T$ be a set of representatives of these cosets.

For each $h \in G$, define $h\lambda_d; hG_d h^{-1} \to \mathbb{C}^\times$ by $(h\lambda_d)(t) = \lambda_d(h^{-1}th)$ for all $t \in hG_d h^{-1}$. Then for each $g \in G$ we have $\lambda^G_d(g) = \sum (h\lambda_d)(g)$, summed over $h \in T$ such that $g \in hG_d h^{-1}$. This amounts to summing over triples $(U, F, \mathcal{B})$ fixed by $g$.

Now let $h \in G$ and $g \in hG_d h^{-1}$. Thus $h^{-1}gh = \begin{pmatrix} s & t \\ 0 & 2 \end{pmatrix} \in G_d$, where $x \in X_d$ and $s \in S_d$, and for all $j \in \{1, 2, \ldots, n - 2d\}$ we have

$$(h^{-1}gh)w_j = w_j + \sum_{i=1}^{j-1} x_{i,j}w_i$$

since $x$ is upper unitriangular. Writing $W_j = V_{2d+j}/V_{2d}$, it follows that if $j < n-2d$ then $g - 1$ induces a map $hW_j/hW_{j-1} \to hW_j/hW_{j-1}$ such that

$$(g - 1)(hw_{j+1} + hW_j) = x_{j,j+1}hw_j + hW_{j-1}.$$ 

In particular, it follows that the coefficients $x_{j,j+1}$ depend only on $g$ and the based flag $h\mathcal{E} = B(hw_1, hw_2, \ldots, hw_{n-2d})$ in $V/hV_{2d}$. We define

$$\psi_{h\mathcal{E}}(g) = \psi \left( \sum_{i=1}^{n-2d-1} x_{i,i+1} \right)$$

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(where $\psi$ is our fixed nontrivial homomorphism $F_q^+ \to \mathbb{C}^\times$), and note that, by our definitions,

$$(h\lambda_d)(g) = \lambda_d(h^{-1}gh) = \psi\left(\sum_{i=1}^{n-2d-1} x_{i,i+1}\right) = \psi_h \varepsilon(g).$$

Hence we have the following result.

(3.1) **Proposition.** For all $d$ with $0 \leq 2d \leq n$ and all $g \in G$,

$$\lambda_d^G(g) = \sum_{U,F,B} \psi_B(g),$$

where the sum is over all $g$-invariant subspaces $U$ of $V$ of dimension $2d$, all nondegenerate $F \in \text{Alt}(U,g)$, and all based flags $B$ in $V/U$ fixed by $g$.

In the case $d = 0$ this gives $\Gamma(g) = \sum_B \psi_B(g)$, summed over based flags in $V$ fixed by $g$, where here $\Gamma$ is the Gel'fand-Graev character. Applying this with $V/U$ in place of $U$ (where $U$ is any $g$-invariant subspace) gives $\Gamma(g,V/U) = \sum_B \psi_B(g)$ where $B$ runs over $g$-fixed based flags in $V/U$. Combining this with Proposition (3.1) we obtain the formula

$$\lambda_d^G(g) = \sum_U \Gamma(g,V/U) S_g(U)$$

where $U$ runs through all $2d$-dimensional $g$-invariant subspaces, and since $S_g(U)$ is zero for odd dimensional subspaces $U$,

$$\sum_{d=0}^{[n/2]} \lambda_d^G(g) = \sum_U \Gamma(g,V/U) S_g(U) \tag{1}$$

where $U$ runs through all $g$-invariant subspaces.

We turn now to the investigation of the Gel'fand-Graev character. Let $F$ be a flag in $V$ of the form $\{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = V$ and let $g$ be an element of $G$ that centralizes $F$, in the sense that $g$ acts trivially on all the 1-dimensional quotient spaces. There are $(q-1)^n$ based flags $B$ associated with $F$, all having the form $B = B(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n)$, where $v_1, v_2, \ldots, v_n$ is a fixed basis of $V$ adapted to the flag $F$ and the $\lambda_i$ are nonzero scalars. We find that

$$\psi_B(g) = \psi\left(\sum_{i=1}^{n-1} \mu_i \frac{\lambda_{i+1}}{\lambda_i}\right) = \psi(\mu_1 \frac{\lambda_2}{\lambda_1}) \psi(\mu_2 \frac{\lambda_3}{\lambda_2}) \cdots \psi(\mu_{n-1} \frac{\lambda_n}{\lambda_{n-1}})$$

where the scalars $\mu_i$ are such that $(g-1)v_{i+1} \equiv \mu_i v_i$ modulo $U_{i-1}$. Summing over all values of $\lambda_n$, then $\lambda_{n-1}$, then $\lambda_{n-2}$, and so on, and using the fact that $\sum_{i=1}^{\lambda_{i+1}} \psi(\mu_i \lambda_{i+1}/\lambda_i)$ is $q-1$ if $\mu_i = 0$ and $-1$ if $\mu_i \neq 0$, gives

$$\sum_B \psi_B(g) = (q-1)^{n-c(g,F)} (-1)^{c(g,F)}$$

where $B$ runs through the based flags associated with the fixed flag $F$, and $c(g,F)$ is the number of $\mu_i$ that are nonzero. The value of $\Gamma(g)$ is obtained by summing over all possibilities for $F$. 

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(3.2) Proposition. For all $g \in G$ we have
\[ \Gamma(g) = \sum_{\mathcal{F}} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \]

where $\mathcal{F}$ runs through all flags centralized by $g$.

It is of course the case that if $g$ is not unipotent then the sum in Proposition (3.2) is empty, and hence $\Gamma(g) = 0$. We assume henceforth in this section that $g$ is unipotent.

We shall show that in fact the sum in Proposition (3.2) depends only on the dimension of the kernel of $1 - g$. For each $1$-dimensional subspace $U$ of this kernel we define $F(U; V)$ to be the set of flags $\{0\} = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = V$ centralized by $g$ such that $U_1 = U$. We define also
\[ \Delta(g, U; V) = \sum_{\mathcal{F} \in F(U; V)} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \]

so that $\Gamma(g) = \sum_U \Delta(g, U; V)$.

(3.3) Lemma. Let $k$ be the dimension of $\ker(1-g)$, and let $U$ be any $1$-dimensional subspace of $\ker(1-g)$. Then
\[ \Delta(g, U; V) = (-1)^{(n-k)}(q^{k-1} - 1)(q^{k-2} - 1) \cdots (q - 1)(q - 1). \]

Proof. We use induction on $n = \dim V$. If $n = 1$ we have $V = U = \ker(1-g)$, and $c(g, \mathcal{F}) = 0$ for the unique flag $\mathcal{F}$. Hence $\Delta(g, U; V) = (q - 1)$ as required.

Whenever $W$ is a two-dimensional $g$-invariant subspace of $V$ such that $U \subset W$, let $F(U; W; V)$ be the set of flags $\mathcal{F}$ of the form
\[ \{0\} = V_0 \subset U \subset W \subset V_3 \subset \cdots \subset V_n = V \]
centralized by $g$. Note first of all that
\[ \Delta(g, U; V) = \sum_{\mathcal{F} \in F(U; V)} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \]
\[ = \sum_W \sum_{\mathcal{F} \in F(U; W; V)} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}, \]

where $W$ runs over all two-dimensional $g$-invariant subspaces of $V$ which contain $U$. So
\[ \Delta(g, U; V) = \sum_{W \in S_1} \sum_{\mathcal{F} \in F(U; W; V)} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})} \]
\[ + \sum_{W \in S_2} \sum_{\mathcal{F} \in F(U; W; V)} (q - 1)^{n-c(g,\mathcal{F})}(-1)^{c(g,\mathcal{F})}, \]

where $S_1$ consists of those $W$ such that $(1 - g)W = 0$ and $S_2$ consists of those $W$ such that $(1 - g)W = U$.

The natural map $V \to V/U$ induces a one-to-one correspondence between $F(U; W; V)$ and $F(W/U; V/U)$; denote this by $\mathcal{F} \mapsto \mathcal{F}'$. Note that, by definition,
\[ \Delta(g, W/U; V/U) = \sum_{\mathcal{F}' \in F(W/U; V/U)} (q - 1)^{n-1-c(g,\mathcal{F}')}(-1)^{c(g,\mathcal{F}')} \]

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since $V/U$ has dimension $n - 1$, and note also that

$$c(g, \mathcal{F}) = \begin{cases} c(g, \mathcal{F}') & \text{if } (1 - g)W = 0 \\ c(g, \mathcal{F}) + 1 & \text{if } (1 - g)W = U. \end{cases}$$

We now treat separately the cases $U \not\subseteq (1 - g)V$ and $U \subseteq (1 - g)V$. If $U \not\subseteq (1 - g)V$ then $(1 - g)W \neq U$ for any $W \subseteq V$, and so $S_2$ is empty. Hence

$$\Delta(g, U; V) = \sum_{W \in S_1} \sum_{\mathcal{F} \in \mathcal{F}(U, W; V)} (q - 1)^{n - c(g, \mathcal{F})}(-1)^{c(g, \mathcal{F})}$$

$$= \sum_{W \in S_1} \sum_{\mathcal{F}' \in \mathcal{F}(W/U; V/U)} (q - 1)^{n - c(g, \mathcal{F}')}(-1)^{c(g, \mathcal{F}')}$$

$$= \sum_{W \in S_1} (q - 1)\Delta(g, W/U; V/U)$$

since $W \in S_1$ implies $(1 - g)W = 0$, and so $c(g, \mathcal{F}) = c(g, \mathcal{F}')$. Furthermore, since $(1 - g)v \in U$ implies $(1 - g)v = 0$, the kernel of $1 - g$ in its action on $V/U$ is $\ker(1 - g)/U$, which has dimension $k - 1$. So the inductive hypothesis yields

$$\Delta(g, W/U; V/U) = ((-1)^{(n - 1) - (k - 1)}(q^{k - 2} - 1) \cdots (q - 1))(q - 1),$$

and thus

$$\Delta(g, U; V) = \sum_{W} ((-1)^{n - k(q^{k - 2} - 1) \cdots (q - 1)})(q - 1)$$

where the sum is over those $W$ such that $W/U$ is a one-dimensional subspace of the $(k - 1)$-dimensional space $\ker(1 - g)/U$. Since the number of such $W$ is $\frac{k^2 - 1}{q - 1}$ we conclude that

$$\Delta(g, U; V) = ((-1)^{n - k(q^{k - 2} - 1) \cdots (q - 1)})(q - 1),$$

as required.

On the other hand, suppose that $U \subseteq (1 - g)V$. As before we observe that if $W \in S_1$ then $(1 - g)W = 0$; hence $c(g, \mathcal{F}) = c(g, \mathcal{F}')$ for all $\mathcal{F} \in \mathcal{F}(U, W; V)$. If $W \in S_2$ then $(1 - g)W = U$; in this case $c(g, \mathcal{F}) = c(g, \mathcal{F}') + 1$ for all $\mathcal{F} \in \mathcal{F}(U, W; V)$. Hence

$$\Delta(g, U; V) = \sum_{W \in S_1} \sum_{\mathcal{F} \in \mathcal{F}(U, W; V)} (q - 1)^{n - c(g, \mathcal{F})}(-1)^{c(g, \mathcal{F})}$$

$$+ \sum_{W \in S_2} \sum_{\mathcal{F} \in \mathcal{F}(U, W; V)} (q - 1)^{n - c(g, \mathcal{F})}(-1)^{c(g, \mathcal{F})}$$

$$= \sum_{W \in S_1} \sum_{\mathcal{F}' \in \mathcal{F}(W/U; V/U)} (q - 1)^{n - c(g, \mathcal{F}')}(-1)^{c(g, \mathcal{F}')}$$

$$+ \sum_{W \in S_2} \sum_{\mathcal{F}' \in \mathcal{F}(W/U; V/U)} (q - 1)^{n - c(g, \mathcal{F}') + 1}(-1)^{c(g, \mathcal{F}') + 1}$$

$$= \sum_{W \in S_1} (q - 1)\Delta(g, W/U; V/U) + \sum_{W \in S_2} (-1)\Delta(g, W/U; V/U).$$

Since $U \subseteq (1 - g)V$ it follows that $(1 - g)(V/U) = (1 - g)V/U$, and so the dimension of the kernel of $1 - g$ on $V/U$ equals $\dim V - \dim(1 - g)V = \dim(\ker(1 - g)) = k$. So our inductive hypothesis now yields that

$$\Delta(g, U; V) = ((-1)^{n - 1 - k(q^{k - 2} - 1) \cdots (q - 1)})(q - 1)((q - 1)|S_1| + (-1)|S_2|).$$
Now \( W \in S_1 \) if and only if \( W/U \) is a one-dimensional subspace of the \((k - 1)\)-dimensional space \( \ker(1 - g)/U \); hence \(|S_1| = \frac{q^{k-1} - 1}{q - 1} \). Similarly \( W \in S_2 \) if and only if \( W \notin S_1 \) and \( W/U \) is a one-dimensional subspace of the \(k\)-dimensional space which is the kernel of \( 1 - g \) on \( V/U \); hence \(|S_2| = \frac{q^k - 1}{q - 1} - \frac{q^{k-1} - 1}{q - 1} \). Thus
\[
(q - 1)|S_1| + (-1)|S_2| = (q - 1) \left( \frac{q^{k-1} - 1}{q - 1} \right) + (-1) \left( \frac{q^k - 1}{q - 1} - \frac{q^{k-1} - 1}{q - 1} \right)
\]
\[
= \frac{q^k - q^{k-1} - (q - 1) - q^k + q^{k-1}}{q - 1}
\]
\[
= \frac{-(q - 1)}{q - 1} = -1;
\]
and so in this case we end up with
\[
\Delta(g, U; V) = ((-1)^{n-k}(q^{k-1} - 1) \cdots (q - 1))(q - 1),
\]
which is what we were required to prove.

As an immediate corollary of Proposition (3.2) we obtain the following formula for the values of the Gel’fand-Graev character.

(3.4) Theorem. Let \( g \in G \) and let \( k = \dim(\ker(1 - g)) \). Then
\[
\Gamma(g) = \begin{cases} 
(-1)^{n-k}(q^k - 1)(q^{k-1} - 1) \cdots (q - 1) & \text{if } g \text{ is unipotent,} \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. We may assume that \( g \) is unipotent, since we have already noted that \( \Gamma(g) = 0 \) otherwise. Now \( \Gamma(g) = \sum_U \Delta(g, U; V) \), where \( U \) runs through all 1-dimensional subspaces of \( \ker(1 - g) \), and by Lemma (3.3) we have
\[
\Delta(g, U; V) = ((-1)^{n-k}(q^{k-1} - 1)(q^{k-2} - 1) \cdots (q - 1))(q - 1)
\]
for each of the \((q^k - 1)/(q - 1)\) such subspaces \( U \). Hence the result follows.

§4 Klyachko’s Theorem

Let \( \varepsilon = \varepsilon_t \) be the indicator function corresponding to the transpose antiautomorphism of \( G = \text{GL}(n, q) \). Let \( \chi \) be any irreducible complex character of \( G \), choose a matrix representation \( R \) with character \( \chi \), and let \( \chi^* \) be the character of the representation \( R^* : g \mapsto R(g^t) \). Then for all \( g \in G \) we have
\[
\chi^*(g) = \text{trace} R(g^t) = \text{trace} R(g^t) = \chi(g^t).
\]
But it is an elementary fact that (over any field) each square matrix is similar to its transpose; so \( g \) and \( g^t \) are conjugate elements of \( G \), and therefore \( \chi^* = \chi \). Thus the representations \( R^* \) and \( R \) are equivalent, and, consequently, \( \varepsilon(\chi) = \pm 1 \).

By Theorem (2.1), for each \( g \in G \) the sum \( \sum_{\chi} \varepsilon(\chi) \chi(g) \) equals the number of nonsingular matrices \( x \) such that \( x^t g x = x \). Given such a matrix \( x \), let \( f \) be the bilinear form \( V \times V \to \mathbb{F}_q \) defined by
\[
f(u, v) = u^t x v \quad \text{for all } u, v \in V,
\]
noting that \( f \) is nondegenerate since \( x \) is nonsingular. For all \( u, v \in V \),

\[
f(u, v) = u^t x v = u^t x^t(gv) = (gv)^t x u = f(gv, u)
\]

and so \( f \in \text{Sym}(V, g) \). Conversely, a nondegenerate element of \( \text{Sym}(V, g) \) gives a nonsingular \( x \) satisfying \( x^t g = x \). Thus it follows that \( \sum \chi(\epsilon(\chi)(g)) = s_g(V) \). Now once we have proved Theorem (1.1) it will follow, in view of Eq. (1) above, that

\[
\sum_{d=0}^{[n/2]} \lambda_d^f(g) = \sum \chi(\epsilon(\chi)(g)),
\]

showing that each \( \epsilon(\chi) \) is positive, and hence establishing Klyachko’s Theorem.

(4.1) Lemma. Let \( f \) be a \( g \)-invariant bilinear form on \( V \), and \( j \) a nonnegative integer. Let \( K_j \) and \( I_j \) be the subspaces of \( V \) defined by \( K_j = \ker(1 - g)^j \) and \( I_j = (1 - g)^j V \). Then \( f(u, v) = 0 = f(v, u) \) for all \( u \in K_j \) and \( v \in I_j \). Furthermore, if \( f \) is nondegenerate then

\[
I_j = \{ v \in V \mid f(v, u) = 0 \text{ for all } u \in K_j \} = \{ v \in V \mid f(u, v) = 0 \text{ for all } u \in K_j \},
\]

and likewise

\[
K_j = \{ u \in V \mid f(u, v) = 0 \text{ for all } v \in I_j \} = \{ u \in V \mid f(u, v) = 0 \text{ for all } v \in I_j \}.
\]

Proof. Since \( I_0 = V \) and \( K_0 = \{0\} \), in the case \( j = 0 \) it is trivial that \( f(u, v) = 0 \) for all \( u \in K_j \) and \( v \in I_j \). Proceeding inductively, let \( j > 0 \) and \( u \in K_j \), and note that each element of \( I_j \) can be expressed in the form \((1 - g)v\) with \( v \in I_{j-1} \). Now

\[
f(u, (1 - g)v) = f(u, v) - f(u, gv) = f(gu, gv) - f(u, gv) = -f((1 - g)u, v) = 0
\]

by the inductive hypothesis, since \((1 - g)u \in K_{j-1}\); this completes the induction. The proof that \( f(v, u) = 0 \) for all \( v \in I_j \) and \( u \in K_j \) is totally analogous.

The remaining assertions follow immediately by dimension arguments, since the dimension of \( I_j \) is the codimension of \( K_j \). \hfill \Box

Note that if \( f \in \text{Sym}(V, g) \) then \( f(u, v) = f(gv, u) = f(gu, gv) \) for all \( u, v \in V \), and so \( f \) is necessarily \( g \)-invariant. In particular, Lemma (4.1) applies. Note also that if \( f \in \text{Sym}(V, g) \) then

\[
\{ u \in V \mid f(u, v) = 0 \text{ for all } v \in V \} = \{ u \in V \mid f(gv, u) = 0 \text{ for all } v \in V \} = \{ u \in V \mid f(v, u) = 0 \text{ for all } v \in V \}
\]

showing that \( f \) is a reflexive form: one whose left and right radicals coincide. (Of course, alternating forms are also reflexive.) Factoring out the radical yields a nondegenerate form on the quotient space.

Given a nondegenerate alternating bilinear form \( F \) on \( V \), there is a natural way to associate with each \( g \in \text{GL}(V) \) that stabilizes \( F \) a bilinear form \( f \) on the space \((1 - g)V \). (The form \( f \) associated with \( g \) plays an important role the classification of conjugacy classes in symplectic groups: see Wall [6].)
(4.2) Proposition. Let $F$ be a nondegenerate form in $\text{Alt}(V, g)$. Then there is a nondegenerate $f \in \text{Sym}((1-g)V, g)$ satisfying $f((1-g)v, u) = F(v, u)$ for all $v \in V$ and $u \in (1-g)V$.

Proof. Restriction of $F$ yields a bilinear map $V \times (1-g)V \to \mathbb{F}_q$, which induces a bilinear map $(V/\ker(1-g)) \times (1-g)V \to \mathbb{F}_q$, since by Lemma (4.1) we have $F(u, v) = 0$ for all $u \in \ker(1-g)$ and $v \in (1-g)V$. Identifying $V/\ker(1-g)$ with $(1-g)V$ in the natural way yields $f$. Since $F$ is alternating and $g$-invariant we find that $F(v, (1-g)u) = F(gv-v, gu) = F(gu, (1-g)v)$ for all $u, v \in V$, from which it follows that $f((1-g)v, (1-g)u) = f((1-g)gu, (1-g)v)$, and $f \in \text{Sym}((1-g)V, g)$. If $u \in \text{rad} f$ then for all $v \in V$ we have $F(v, u) = f((1-g)v, u) = 0$, and this gives $u = 0$ since $F$ is nondegenerate. Hence $f$ is nondegenerate. □

By a parallel argument, reversing the roles of alternating forms and forms that are symmetric modulo $g$, we obtain the following result.

(4.3) Proposition. Let $f$ be a nondegenerate form in $\text{Sym}(V, g)$. Then there is a nondegenerate $F \in \text{Alt}((1-g)V, g)$ satisfying $F(u, (1-g)v) = f(u, v)$ for all $v \in V$ and $u \in (1-g)V$.

Observe that combining Propositions (4.2) and (4.3) gives a map from the $g$-invariant nondegenerate alternating bilinear forms on $V$ to those on $(1-g)^2V$. This map can be described as follows: restrict the given form on $V$ to the subspace $(1-g)V$, and then factor out the radical, which is $(1-g)V \cap \ker(1-g)$; the resulting space is naturally isomorphic to $(1-g)^2V$. Note also that in the case that $\ker(1-g) = \{0\}$, Propositions (4.2) and (4.3) both yield bijections between the sets of nondegenerate elements of $\text{Alt}(V, g)$ and $\text{Sym}(V, g)$. Thus we have the following fact.

(4.4) Proposition. Let $g \in G$ be such that $1-g$ is an invertible map $V \to V$. Then $S_g(V) = s_g(V)$.

Given any $g \in G$ there is an integer $p$ (which is $\dim V$ at most) such that $(1-g)^pV = (1-g)^pV$ for all $r \geq p$. We have $V = V_1 \oplus V_2$, where

$$V_1 = \{u \in V \mid (1-g)^ku = 0 \text{ for some integer } k\}$$

(the generalized 1-eigenspace) and $V_2 = (1-g)^pV$. By Lemma (4.1) we know that every $g$-invariant bilinear form $f$ on $V$ satisfies $f(u, v) = 0 = f(v, u)$ for all $u \in V_1$ and $v \in V_2$; hence each such form $f$ is determined by its restrictions to $V_1$ and $V_2$, and is nondegenerate precisely when both these restrictions are nondegenerate. Furthermore, a form $f$ can be found with any prescribed restrictions to $V_1$ and $V_2$. As an easy consequence of these considerations we obtain the following result.

(4.5) Proposition. Let $g, V_1$ and $V_2$ be as above. Then $S_g(V) = S_g(V_1)S_g(V_2)$ and $s_g(V) = s_g(V_1)s_g(V_2)$.

Propositions (4.5) and (4.4) enable us to reduce the proof of Theorem (1.1) to the case of unipotent elements $g$ (those for which $V_2 = 0$). For suppose that Theorem (1.1) holds for such elements $g$. Since an arbitrary element $g$ is unipotent on its generalized 1-eigenspace, we have

$$s_g(V_1) = \sum_{U_1 \subseteq V_1} \Gamma(g, V_1/U_1)S_g(U_1).$$
If \( U \) is a \( g \)-invariant subspace of \( V \) with \( V_2 \subseteq U \) then \( U = (U \cap V_1) \oplus V_2 \), and Proposition (4.5) (applied with \( U \) in place of \( V \)) together with Proposition (4.4) yields
\[
S_g(U) = S_g(U \cap V_1)S_g(V_2) = S_g(U \cap V_1)s_g(V_2).
\]
Now \( U_1 \mapsto U_1 + V_2 \) and \( U \mapsto U \cap V_1 \) are mutually inverse bijections between the sets of \( g \)-invariant subspaces of \( V_1 \) and \( g \)-invariant subspaces of \( V \) containing \( V_2 \). Furthermore, since
\[
V_1/U_1 = V_1/(U \cap V_1) \cong (V_1 + U)/U = (V_1 + V_2)/U = V/U
\]
as \( g \)-modules, it follows that \( \Gamma(g,V_1/U_1) = \Gamma(g,V/U) \). Thus
\[
s_g(V) = s_g(V_1)s_g(V_2) = \sum U \Gamma(g,V/U)S_g(U \cap V_1)s_g(V_2) = \sum U \Gamma(g,V/U)S_g(U)
\]
where \( U \) runs through all \( g \)-invariant subspaces of \( V \) containing \( V_2 \). However, \( \Gamma(g,V/U) = 0 \) for \( g \)-invariant subspaces \( U \) that do not contain \( V_2 \), since the Gel'fand-Graev character vanishes on elements that are not unipotent. Hence
\[
s_g(V) = \sum U \Gamma(g,V/U)S_g(U)
\]
with \( U \) running through all \( g \)-invariant subspaces of \( V \), as required.

Our remaining task is to prove Theorem (1.1) for unipotent \( g \).

Let \( g \in G \) be unipotent, and write \( I = (1-g)V \) and \( K = \ker(1-g) \). For each \( f \in \text{Sym}(V,g) \) we define
\[
 f K^\perp = \{ x \in V \mid f(x,v) = 0 \text{ for all } v \in K \} ,
\]
and note by Proposition (4.1) that \( I \subseteq f K^\perp \), equality holding if \( f \) is nondegenerate. The converse of this is also true.

(4.6) Proposition. Let \( f \in \text{Sym}(V,g) \) be such that \( I = f K^\perp \). Then \( f \) is nondegenerate.

Proof. Let \( R \) be the radical of \( f \) and \( \overline{V} = V/R \), and let \( \overline{f} \in \text{Sym}(\overline{V},g) \) be the form on \( \overline{V} \) induced by \( f \). Noting that \( R \subseteq f K^\perp = I \), write \( \overline{I} = I/R \) and \( \overline{K} = (K + R)/R \). Then
\[
\overline{f K^\perp} = \{ \overline{x} \in \overline{V} \mid \overline{f}(\overline{x},\overline{v}) = 0 \text{ for all } \overline{v} \in \overline{K} \}
\]
\[
= \{ x + R \mid f(x,v) = 0 \text{ for all } v \in K \}
\]
\[
= f K^\perp/R = I/R = \overline{I},
\]
and since \( \overline{f} \) is nondegenerate it follows that the dimension of \( \overline{K} \) equals the codimension of \( \overline{I} \). But the codimension of \( \overline{I} \) is the same as the codimension of \( I \), which equals \( \dim K \). So \( \dim K = \dim(K + R)/R \), whence the sum \( K + R \) is direct. But, since \( 1-g \) is nilpotent, all nonzero \((1-g)\)-invariant subspaces intersect \( K \), the kernel of \( 1-g \), nontrivially. Hence \( R \) is zero, as required. \( \square \)
Consider subspaces \( Y \) of \( V \) such that \( I \subseteq Y \). (Note that all such subspaces are \( g \)-invariant). For each such \( Y \) let \( \mathcal{R}_Y \) be the set of all ordered pairs \((F, f)\) such that \( F \in \text{Alt}(Y, g) \) and \( f \in \text{Sym}(V, g) \), and

\[
\int y, v \int F(y, (1-g)v) \quad \text{for all} \ y \in Y \text{ and} \ v \in V.
\]

Thus \( F \) is required to extend the form on \( I \) that is derived from \( f \) in the manner described in Proposition (4.3). Note, however, that we do not here require the forms to be nondegenerate.

\begin{proposition}
Let \( r \) be the codimension of \( Y \) in \( V \). For each \( F \in \text{Alt}(Y, g) \) there are precisely \( q^{(r+1)/2} \) forms \( f \in \text{Sym}(V, g) \) such that \((F, f) \in \mathcal{R}_Y\).
\end{proposition}

\textbf{Proof.} Choose a subspace \( W \) such that \( V = Y \oplus W \), and observe that since \((1-g)W \subseteq Y\),

\[
(w, w') \mapsto F((1-g)w, (1-g)w')
\]
defines an alternating bilinear form on \( W \). The number of bilinear forms \( f_0 \) on \( W \) such that

\[
f_0(w, w') - f_0(w', w) = F((1-g)w, (1-g)w')
\]
is the same as the number of symmetric bilinear forms on \( W \), namely \( q^{(r+1)/2} \). It is readily checked that for each such \( f_0 \),

\[
f(y + w, y' + w') = F(y, (1-g)(y' + w')) + F(yy', (1-g)w) + f_0(w, w')
\]
defines an \( f \) such that \((F, f) \in \mathcal{R}_Y\), and, conversely, every suitable \( f \) has this form for some such \( f_0 \). We leave the details to the reader. \( \square \)

\begin{proposition}
Let \( m = \dim Y/I \), and let \( f \in \text{Sym}(V, g) \). The number of forms \( F \in \text{Alt}(Y, g) \) such that \((F, f) \in \mathcal{R}_Y \) is \( q^m \) if \( Y \subseteq fK^\perp \), and zero otherwise.
\end{proposition}

\textbf{Proof.} Let \( v \in K \) (so that \((1-g)v = 0\)), and suppose there exists a form \( F \) on \( Y \) such that \((F, f) \in \mathcal{R}_Y \). Then for all \( y \in Y \),

\[
0 = F(y, (1-g)v) = f(y, v),
\]

and so \( Y \subseteq fK^\perp \). This proves the second assertion.

For the other, suppose that the condition \( Y \subseteq fK^\perp \) is satisfied, and choose any space \( X \) such that \( Y = I \oplus X \). If \((F, f) \in \mathcal{R}_Y \) and \( F_0 \) is the restriction of \( F \) to \( X \), then \( F_0 \) is an alternating bilinear form on \( X \), and for all \( v, v' \in V \) and \( x, x' \in X \),

\[
F((1-g)v + x, (1-g)v' + x') = f((1-g)v + x, v') - f(v', v) + F_0(x, x').
\]

Observe that this equals \( f(x, v') - f(x', v) + f(v, v') - f(v', v) + F_0(x, x') \). We leave it to the reader to check that, conversely, for any alternating bilinear form \( F_0 \) on \( X \) these formulas yield a well defined \( F \in \text{Alt}(Y, g) \) satisfying \((F, f) \in \mathcal{R}_Y \). Thus the total number of such forms \( F \) is the number of alternating bilinear forms on \( X \), which is \( q^m \). \( \square \)
We shall require the following two elementary facts, the proofs of which we leave to the reader.

(4.9) Lemma. Let $V$ be a finite dimensional vector space over $\mathbb{F}_q$. Then
\[
\sum_{Y} (-1)^{\dim Y} q^{\left(\frac{1+\dim Y}{2}\right)} = (1 - q)(1 - q^2) \cdots (1 - q^{\dim V})
\]
and
\[
\sum_{Y} (-1)^{\dim Y} q^{\left(\frac{\dim Y}{2}\right)} = \begin{cases} 1 & \text{if } V = 0, \\ 0 & \text{otherwise}, \end{cases}
\]
where in each case $Y$ runs through all subspaces of $V$.

We require one further preliminary result before we can complete the proof of Theorem (1.1).

(4.10) Lemma. Let $g$ be a unipotent element of $G$, and let $\overline{S}_g(V)$ be the total number of $g$-invariant alternating bilinear forms on $V$. Then $\overline{S}_g(V) = \sum_U S_g(U)$, where $U$ runs through all $g$-invariant subspaces of $V$.

Proof. Let $V^*$ be the dual of $V$, made into a $g$-module via the contragredient action. Since $g$ is unipotent it is clear that $g$ and $(g^{-1})^t$ have the same Jordan canonical form; so $V^*$ and $V$ are isomorphic $g$-modules. Hence $S_g(V) = S_g(V^*)$, and also $\overline{S}_g(V) = \overline{S}_g(V^*)$.

If $U$ is a subspace of $V$, let $\text{Ann}(U)$ be the subspace of $V^*$ consisting of those linear functionals that vanish on $U$. Then $U \leftrightarrow \text{Ann}(U)$ gives a bijective correspondence between the $g$-invariant subspaces of $V$ and those of $V^*$, and since $V^*/\text{Ann}(U) \cong U^*$, it follows that
\[
\sum_{U} S_g(U) = \sum_{U} S_g(U^*) = \sum_{W} S_g(V^*/W)
\]
where $U$ runs through the $g$-invariant subspaces of $V$ and $W$ runs through the $g$-invariant subspaces of $V^*$. But since each $F \in \text{Alt}(V^*, g)$ gives rise to a non-degenerate element of $\text{Alt}(V^*/W, g)$, where $W$ is the radical of $F$, and conversely each $g$-invariant nondegenerate alternating bilinear form on a quotient space $V^*/W$ yields an $F \in \text{Alt}(V^*, g)$ with radical $W$, it follows that $S_g(V^*/W)$ is the number of such forms with radical $W$, and $\sum_W S_g(V^*/W) = \overline{S}_g(V^*) = \overline{S}_g(V)$. $\square$

We are now able to complete the proof of the main theorem. Let $g \in G$ be unipotent, and let $U$ be an arbitrary $g$-invariant subspace of $V$. Observe that $S_g(U) = (-1)^{\dim U} S_g(U)$, since nondegenerate alternating forms exist only on even dimensional subspaces. Let $r$ be the codimension of $U + I$ in $V$, where $I = (1 - g)V$, and note that $r$ is the dimension of the kernel of the action of $g$ on $V/U$. Hence by Theorem (3.4),
\[
\Gamma(g, V/U) S_g(U) = \Gamma(g, V/U) (-1)^{\dim U} S_g(U)
\]
\[
= (-1)^{n-r} (q^r - 1)(q^{r-1} - 1) \cdots (q - 1) S_g(U)
\]
\[
= (-1)^n S_g(U) (1 - q)(1 - q^2) \cdots (1 - q^r)
\]
\[
= (-1)^n S_g(U) \sum_{Y} (-1)^{\text{codim } Y} q^{\left(\frac{1+\text{codim } Y}{2}\right)}
\]

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where $Y$ runs through all subspaces of $V$ containing $U + I$, this last step following from Lemma (4.9). Hence

$$\sum_U \Gamma(g, V/U) S_g(U) = \sum_{Y \supseteq I} (-1)^{\dim Y} q^{(\dim Y / 2)} \left( \sum_{U \subseteq Y} S_g(U) \right),$$

since $Y$ contains $U + I$ if and only if it contains both $U$ and $I$. By Lemma (4.10) and Proposition (4.7),

$$\sum_U \Gamma(g, V/U) S_g(U) = \sum_{Y \supseteq I} (-1)^{\dim Y} \sum_{F \in \text{Alt}(Y, g)} q^{(\dim Y / 2)}$$

$$= \sum_{Y \supseteq I} (-1)^{\dim Y} \sum_{F \in \text{Alt}(Y, g)} \sum_f 1$$

where $f$ runs through the forms in $\text{Sym}(V, g)$ such that $(F, f) \in R_Y$.

By Proposition (4.8), for each $f \in \text{Sym}(V, g)$ the number of $F \in \text{Alt}(Y, g)$ such that $(F, f) \in R_Y$ is 0 unless $Y \subseteq fK^\perp$, in which case it is $q^{(\dim Y / 2)}$. Thus

$$\sum_U \Gamma(g, V/U) S_g(U) = \sum_{f \in \text{Sym}(V, g)} \sum_{Y \supseteq I} (-1)^{\dim Y} \sum_{\{ F | (F, f) \in R_Y \}} q^{(\dim Y / 2)}$$

$$= \sum_{f \in \text{Sym}(V, g)} \sum_{Y \supseteq I} (-1)^{\dim Y} \sum_f 1$$

by Proposition (4.9). But by Proposition (4.3) and the fact that there are no nondegenerate alternating bilinear forms on $I$ if $\dim I$ is odd, we conclude that

$$\sum_U \Gamma(g, V/U) S_g(U) = \left| \{ f \in \text{Sym}(V, g) | I = fK^\perp \} \right| = s_g(V)$$

by Proposition (4.6).

References