Computer Tutorial 4

This tutorial is about the vector space of polynomials over $\mathbb{R}$. There is one new \texttt{MAGMA} command you will need to use (\texttt{PolynomialAlgebra}) and one new control structure (\texttt{for ... end for}). And of course you can still use the commands you learnt in previous tutorials (e.g. \texttt{RealField VectorSpace print func Sqrt InnerProduct Arccos NullSpace sub KMatrixSpace Transpose Solution}).

Remember that all \texttt{MAGMA} commands end with a semicolon (;) (which can go on the next line, if necessary). You should frequently get \texttt{MAGMA} to print the values of the variables you are working with, to see the results of your actions. (e.g. if you define a subspace by a command like

\texttt{SS := sub< V | u, v >;}

then you should type \texttt{SS}; (or \texttt{print SS;}) to find out what \texttt{MAGMA} thinks \texttt{SS} is. To see what variables the \texttt{MAGMA} system has defined, type \texttt{ShowIdentifiers();}.

1. Use \texttt{MAGMA} to add and multiply the polynomials $f(x) = x^7 - 5x^4 + 2x - 1$ and $g(x) = 3x^3 - 2x^2 + x - 1$. To get started type

\texttt{R := RealField();}
\texttt{P<x> := PolynomialAlgebra(R);}  
\texttt{(Now \texttt{MAGMA} knows that \texttt{P} is the set of all polynomials in the variable \texttt{x}.)}

\begin{itemize}
  \item [(i)] The next step is to enter the polynomials themselves. That is, $f := x^7 - 5x^4 + 2x - 1$; $g := 3x^3 - 2x^2 + x - 1$;
  \item [(ii)] Print out the values of \texttt{P}, \texttt{f} and \texttt{g}.
  \item [(iii)] Next, print the sum and product of \texttt{f} and \texttt{g}.
  \item [(iv)] \texttt{MAGMA} has a function \texttt{Eltsseq} which returns the sequence of coefficients of a polynomial. To see how it works, type \texttt{Eltsseq(f);}. Note that the constant term comes first, then the coefficient of $x^1$, and so on. You can also define a polynomial by entering its sequence of coefficients, and “coercing” this sequence into the polynomial algebra. Test this by typing \texttt{h:=P![-1,1,-2,3];} and then \texttt{print h;}.
\end{itemize}

\textbf{Solution.}

\begin{verbatim}
> R := RealField();
> P<x> := PolynomialAlgebra(R);
> f := x^7 - 5*x^4 + 2*x - 1;
> g := 3*x^3 - 2*x^2 + x - 1;

> Evaluate(f,0); Evaluate(g,0);
-1
-1

> Evaluate(f+g,0); Evaluate(f*g,0);
-2
1

> Eltsseq(f);
[ -1, 2, 0, 0, -5, 0, 0, 1 ]

> Eltsseq(g);
[ -1, 1, -2, 3 ]

> h := P([-1,1,-2,3]);
true
\end{verbatim}

2. (i) If $f(x)$ is a polynomial then the mathematical notation for the number obtained by putting $x = 3$ (say) is $f(3)$. \texttt{MAGMA} has a function \texttt{Evaluate} for this: in \texttt{MAGMA}, if \texttt{f} is a polynomial then \texttt{Evaluate(f,3)} evaluates \texttt{f} at $x = 3$. Evaluate \texttt{f}, \texttt{g}, \texttt{f-g} and \texttt{f*g} at $x = 0$, $1$ and $10$.

(ii) \texttt{MAGMA} can integrate and differentiate polynomials with the commands \texttt{Integral} and \texttt{Derivative}. We can use \texttt{Integral} to define a \texttt{MAGMA} function that returns the inner product of two polynomial functions on the interval $[-1,1]$, using the inner product defined in lectures, namely $(f,g) = \int_{-1}^{1} f(x)g(x) \, dx$. The \texttt{MAGMA} code you need is

\texttt{polyip := func< a,b| Evaluate(int,1) - Evaluate(int,-1) >;}

where \texttt{int:=Integral(a*b) >;}

Use \texttt{polyip} to find the inner product of the polynomials $f$ and $g$ of the previous exercise.

(iii) Calculate the \textbf{length} of $f$ and the \textbf{angle} between $f$ and $g$. (Recall that in any inner product space the length of an element $v$ is defined to be $\sqrt{\langle v, v \rangle}$, and the angle between $v$ and $w$ is $\arccos((\langle v, w \rangle)/\|v\|\|w\|)$. Start by defining \texttt{Length:=func< v | Sqrt(polyip(v,v)) >;}).

\textbf{Solution.}

\begin{verbatim}
> P,f,g;
Univariate Polynomial Ring in x over Real Field
x^7 - 5*x^4 + 2*x - 1
3*x^3 - 2*x^2 + x - 1

> f+g, f*g;

> Evaluate(f,0); Evaluate(g,0);
-1
-1

> Evaluate(f+g,0); Evaluate(f*g,0);
-2
1

> polyip := func< a,b| Evaluate(int,1) - Evaluate(int,-1) >;
> int:=Integral(a*b) >;
> polyip(f,g);
55/8

> Length := func< v | Sqrt(polyip(v,v)) >;
> Evaluate(f,1); Evaluate(g,1);  
> Evaluate(f+g,1); Evaluate(f*g,1);  
> Evaluate(f,10); Evaluate(g,10);  
> Evaluate(f+g,10); Evaluate(f*g,10);  
> \text{polyip}:=\text{func<a,b| Evaluate(int,1) - Evaluate(int,-1) where}  
\text{int}:=\text{Integral(a*b);}  
> \text{polyip}(f,g);  
\begin{align*}  
\int_{-1}^{1} x^{k+m} \, dx &= \frac{1}{k+m+1} \left( (k+m+1) - (-1)^{k+m+1} \right),  
\end{align*}  
which is zero if \( k + m + 1 \) is even and is \( 2/(k + m + 1) \) if \( k + m + 1 \) is odd.

5. Check that \( 1 + 2x^2 \) and \( x^3 \) are orthogonal and then find the projections of \( x^4 \) and \( x^5 \) onto the subspace spanned by \( 1 + 2x^2 \) and \( x^3 \). Explain why in both cases you get a multiple of one of the original vectors.
Hint. Remember that the projection $p$ of the vector $v$ onto the subspace with the orthogonal basis $a_1, a_2, \ldots, a_k$ is

$$p = \frac{(v, a_1)}{(a_1, a_1)}a_1 + \frac{(v, a_2)}{(a_2, a_2)}a_2 + \cdots + \frac{(v, a_k)}{(a_k, a_k)}a_k.$$

In this question, $k$ is 2, $a_1 = 1 + 2x^2$ and $a_2 = x^3$.

Solution.

```plaintext
> p := 1 + 2*x^2;
> q := x^3;
> polyip(p,q);
0
> (polyip(x^4,p)/polyip(p,p))*p+(polyip(x^4,q)/polyip(q,q))*q;
102/329*x^2 + 51/329
> (polyip(x^5,p)/polyip(p,p))*p+(polyip(x^5,q)/polyip(q,q))*q;
7/9*x^3
```

The projection of $x^4$ is a multiple of $p$ since $(x^4, q) = 0$. And the projection of $x^5$ is a multiple of $q$ since $x^5$ is orthogonal to $p$.

6. (i) The aim of this question is to apply the Gram-Schmidt process to the polynomials $1, x, x^2, x^3, x^4, x^5$ and $x^6$, and hence obtain the first seven Legendre polynomials.

A MAGMA input file t4defs.m has been prepared containing the following MAGMA code (which you are not required to understand):

```plaintext
polyProjection := func< Q,g | #Q eq 0 select 0*g else &+[polyip(f,g)/polyip(f,f))*f : f in Q] >;
polyGS := func< Q | [f-polyProjection(Self(),f) : f in Q]>
```

The function polyGS does this: if $B$ is a sequence of polynomials then polyGS($B$) is the orthogonal sequence produced by Gram-Schmidt.

Type `load "t4defs.m";` to get MAGMA to read the input file. Next, define $B$ to be the sequence of polynomials consisting of the powers of $x$ from $x^0$ up to $x^6$. You can do this by

```plaintext
B := [x^0, x^1, x^2, x^3, x^4, x^5, x^6];
```

or more sneakily by

```plaintext
B := [x^k : k in [0..6]];  
```

Then apply polyGS to $B$.

(ii) Actually, Legendre polynomials are usually scaled so that their value at 1 is 1. If we are given a polynomial $f$ and we define a new polynomial $g$ by $g(x) = cf(x)$, where $c$ is a scalar, then of course $g(1) = cf(1)$. So we can make $g(1) = 1$ by putting $c = 1/f(1)$. In MAGMA the corresponding command is $g := (1/Evaluate(f,1))*f$; Do this scaling on the polynomials you obtained in Part (i).