1. This question is about the group of symmetries of the tetrahedron with vertices labelled 1, 2, 3 and 4 as shown below. Use MAGMA to set up the group $G := \text{Sym}(4)$ of all permutations of \{1, 2, 3, 4\}.

Every rotational symmetry of the tetrahedron corresponds to a permutation of \{1, 2, 3, 4\}.

(i) Find a (nontrivial) rotational symmetry that fixes the vertex 1 and another that fixes the vertex 2, and find the corresponding permutations.

(ii) Let $H$ be the subgroup of $G$ generated by the permutations you found in Part (i). Get MAGMA to print out all the elements of $H$, and show that the order of $H$ is 12.

(iii) Describe each element of $H$ geometrically (e.g. as a rotation about an axis).

(iv) List the order of each element of $H$.

(v) Find a subgroup of $H$ of order 4.

Solution.

Let $\ell_1$ be the line through vertex 1 and the central point of the face 234. The rotations about the axis $\ell_1$ through 120° and 240° are symmetries of the tetrahedron fixing vertex 1. The corresponding permutations are (2, 3, 4) and (2, 4, 3). Similarly, if $\ell_2$ is the line through vertex 2 and the centroid of the face 134 then rotations about $\ell_2$ through 120° and 240° are symmetries of the tetrahedron fixing vertex 2. The corresponding permutations are (1, 3, 4) and (1, 4, 3). For Part (i) of the question I chose (2, 3, 4) and (1, 3, 4). There are three other possible choices that would be equally valid.

Solution.

Let $\ell_1$ be the line through vertex 1 and the central point of the face 234. The rotations about the axis $\ell_1$ through 120° and 240° are symmetries of the tetrahedron fixing vertex 1. The corresponding permutations are (2, 3, 4) and (2, 4, 3). Similarly, if $\ell_2$ is the line through vertex 2 and the centroid of the face 134 then rotations about $\ell_2$ through 120° and 240° are symmetries of the tetrahedron fixing vertex 2. The corresponding permutations are (1, 3, 4) and (1, 4, 3). For Part (i) of the question I chose (2, 3, 4) and (1, 3, 4). There are three other possible choices that would be equally valid.

Sure enough, $H$ has 12 elements. Four of them have been described above. The permutations (1, 2, 4) and (1, 4, 2) correspond to rotations through 120° and 240° about $\ell_3$, the line joining vertex 3 to the centroid of 124. Similarly, (1, 2, 3) and (1, 3, 2) correspond to rotations through 120° and 240° about $\ell_4$, the line joining vertex 4 to the centroid of 123. The identity is a rotation through 0° (about any axis). The remaining three elements of $H$ are all half-turns: rotations through 180°. For the permutation (1, 2)(3, 4) the axis is the line joining the mid-point of 12 to the mid-point of 34. Similarly, for (1, 3)(2, 4) the axis is the line joining the mid-point of 13 to the mid-point of 24, and for (1, 4)(2, 3) the axis is the line joining the mid-point of 14 to the mid-point of 23.

By Sylow’s Theorem $H$ must have a subgroup of order 4, since 4 is the largest
power of the prime 2 that is a divisor of 12, the order of $H$. An element of order $k$ generates a cyclic subgroup of order $k$, and by Lagrange’s Theorem the order of a subgroup has to be a divisor of the order of the group. So the order of any element of a group of order 4 must be a divisor of 4. Now in $H$ there are only four elements whose orders are divisors of 4: the three elements of order 2 and the identity (of order 1). So these four elements are the only ones that can possibly be contained in a group of order 4. But $H$ does have a subgroup of order 4, which certainly contains four elements of $H$. So it must be these four. So

$$\{\text{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

is a subgroup of $H$ of order 4.

2. In MAGMA, define $G$ to be the group $\text{Sym}(4)$, and define $p1:=\{(1,4),(2,3)\}$.

(i) How many elements does the set $p1$ have? Check your answer with MAGMA. (Use $\#p1$.)

(ii) Define $P := p1^{-G}$; and then get MAGMA to print $P$. (Here $p1^{-G}$ means the set of everything that $p1$ can be changed into by applying a permutation of $\{1,2,3,4\}$. This same example will be discussed in Q1 of Tutorial 10.

(iii) How many elements does $P$ have? Check your answer with MAGMA (via the command $\#P$).

(iv) Each element of $P$ corresponds to a partitioning of the set $\{1,2,3,4\}$ into two subsets of size 2. (Each such partitioning corresponds to a way of pairing up four tennis players for a game of doubles. Thus $p1$ above corresponds to players 1 and 4 teaming up against players 2 and 3.) Define now $p2 := \{(2,4),(1,3)\}$; and $p3 := \{(3,4),(1,2)\}$; so that $P$ is $\{p1, p2, p3\}$. Observe that $p1$ is a set with two elements, both of which are themselves sets. And $P$ is a set whose elements are sets whose elements are sets.

(v) Put $x:=G!(1,4,3,2)$; and get MAGMA to print $p1^x$, $p2^x$ and $p3^x$. Hence find the permutation of $\{p1, p2, p3\}$ derived from the permutation $x$ of $\{1,2,3,4\}$.

(vi) Each permutation of $\{1,2,3,4\}$ gives rise to a permutation of $\{p1, p2, p3\}$: so we have a function $f$ from the group of all permutations of $\{1,2,3,4\}$ to the group of all permutations of $\{p1, p2, p3\}$. This function is, in fact, a homomorphism. The MAGMA command $f,L,K := \text{Action}(G,P)$; defines $f$ to be this homomorphism, $L$ to be the image of $f$, and $K$ to be the kernel of $f$. After typing this command, get MAGMA to print $f$, $L$ and $K$.

(vii) Type the MAGMA command $f(x)$; The response should agree with your answer to Part (v).

(viii) Find the permutations of $\{p1, p2, p3\}$ corresponding to each of the permutations $(1,4), (1,3,2), (1,2,3,4), (1,3), (2,4,3)$, by using commands such as $f(G!(1,4))$.

(ix) Find the permutations of $\{p1, p2, p3\}$ corresponding to each of the permutations $(1,2)(3,4), (1,3)(2,4)$ and $(1,3)(4,2)$. Note that these three permutations are all in the group $K$. Print Set($K$) to confirm this.

(x) Put $A := \{x*k : k \text{ in } k\}$; and then do the following loop:

for $t$ in $A$ do
  $f(t)$;
end for;

What do you notice about the answer? Put $B := \{G!(1,4)*k : k \text{ in } K\}$, and do a similar for loop. Observe that you again get the same answer four times. Do some more similar loops.

Solution.

#p1 is 2. The two elements of p1 are the sets $\{1,4\}$ and $\{2,3\}$.

```magma
> p1:=[(1,4),(2,3)];
> #p1;
2
> P:=p1^G;
> P;
GSet{
  { 1, 4 },
  { 2, 3 }
},
  { 1, 2 },
  { 3, 4 }
},
  { 1, 3 },
  { 2, 4 }
}
```

The set $P$ has three elements; they are $p1$, $p2$ and $p3$, where $p2$ and $p3$ are $\{2,4\}, \{1,3\}$ and $\{3,4\}, \{1,2\}$.

```magma
> #P;
3
> p2:=[(2,4),(1,3)];
> p3:=[(3,4),(1,2)];
> P eq {p1,p2,p3};
true
> x:=G!(1,4,3,2);
```
Thus x gives rise to the permutation \((p_1, p_3)\) of \(\{p_1, p_2, p_3\}\).

Thus the permutations \((1, 4)\), \((2, 4)\), \((1, 2, 3)\) and \((1, 2, 4)\) of \(\{1, 2, 3, 4\}\) give
rise (respectively) to the permutations \((p_2, p_3), (p_1, p_3), (p_1, p_2, p_3)\) and \((p_1, p_3, p_2)\) of \(\{p_1, p_2, p_3\}\).

\[
\begin{align*}
> \text{Set}(K); \\
> \{ \text{Id}(K), \\
> \quad (1, 3)(2, 4), \\
> \quad (1, 2)(3, 4), \\
> \quad (1, 4)(2, 3) \\
> \} \\
> f(G!(1,2)(3,4)); \\
> \text{Id}(L) \\
> f(G!(1,3)(2,4)); \\
> \text{Id}(L) \\
> f(G!(1,4)(2,3)); \\
> \text{Id}(L) \\
> A:=\{x*k : k in K\}; \\
> \text{for } t \text{ in } A \text{ do } f(t); \text{ end for;}
\end{align*}
\]

So all the elements in the coset \(xK\) give rise to the same permutation of \(\{p_1, p_2, p_3\}\), namely \((p_1, p_3)\).

\[
\begin{align*}
> B:=\{G!(1,4)*k : k in K\}; \\
> \text{for } t \text{ in } B \text{ do } f(t); \text{ end for;}
\end{align*}
\]

All the elements in the coset \((1,4)K\) give rise to the same permutation of \(\{p_1, p_2, p_3\}\), namely \((p_2, p_3)\).

\[
\begin{align*}
> C:=\{G!(2,4)*k : k in K\}; \\
> \text{for } t \text{ in } C \text{ do } f(t); \text{ end for;}
\end{align*}
\]

All elements of \((1,2,4)K\) give rise to \((p_1, p_3, p_2)\), and all elements of \((2,4)K\) give rise to \((p_1, p_3)\).

Why do elements of \((2,4)K\) give rise to the same permutation of \(\{p_1, p_2, p_3\}\) as do elements of \(xK = (1,4,3,2)K\)? Because \((2,4)K = (1,4,3,2)K\):

\[
\begin{align*}
> G!(2,4)*K \text{ eq } x*K; \\
> \text{true}
\end{align*}
\]