Solution.

1. Prove that isomorphic vector spaces have the same dimension.

(Hint: Use Theorem 4.17. This was proved in Exercise 5 of Tutorial 4.)

Solution.

Let $V$ and $W$ be isomorphic vector spaces and let $\theta: V \rightarrow W$ be an isomorphism. That is, $\theta$ is a bijective linear transformation. Let $v_1, v_2, \ldots, v_n$ be a basis for $V$. By 4.17 (ii) the elements $\theta(v_1), \theta(v_2), \ldots, \theta(v_n)$ span $W$ (since $\theta$ is surjective), and by 4.17 (i) they are linearly independent (since $\theta$ is injective). So these elements form a basis for $W$, and we see that bases of $W$ have the same number of elements as do bases of $V$.

2. Is it possible to find subspaces $U$, $V$, and $W$ of $\mathbb{R}^4$ such that

$$\mathbb{R}^4 = U \oplus V = V \oplus W = W \oplus U?$$

Solution.

Yes; for instance, define $U$, $V$, and $W$ to be (respectively)

$$\left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}, \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}, \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$  

Each of these is a subspace of dimension two: it can be seen that

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

are bases of $U$, $V$, and $W$, respectively. Now since $U \cap V = \{0\}$ the sum $U + V$ is direct, and its dimension is therefore equal to $\dim U + \dim V = 4$. The only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ itself; so we conclude that $U \oplus V = \mathbb{R}^4$. (Indeed, combining the bases $b$ of $U$ and $c$ of $V$ gives the standard basis of $\mathbb{R}^4$.) Since it is also true that $U \cap W = \{0\}$ and $V \cap W = \{0\}$ it follows that $U \oplus W = V \oplus W = \mathbb{R}^4$ as well.

3. (i) Let $V$ and $W$ be vector spaces over $F$. Show that the Cartesian product of $V$ and $W$ (see §1b) becomes a vector space if addition and scalar multiplication are defined in the natural way. (This space is called the external direct sum of $V$ and $W$, and is sometimes denoted by $V + W'$.)

(ii) Show that $V' = \{ (v, 0) \mid v \in V \}$ and $W' = \{ (0, w) \mid w \in W \}$ are subspaces of $V + W$ with $V' \cong V$ and $W' \cong W$, and that $V + W = V' \oplus W'$.

(iii) Prove that $\dim(V + W) = \dim V + \dim W$.

Solution.

(i) Elements of $V + W$ are ordered pairs $(v, w)$ with $v \in V$ and $w \in W$. Addition and scalar multiplication are defined by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad \lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$ and all $\lambda \in F$. To prove that this gives a vector space simply a matter of checking the axioms. The zero element of $V + W$ is the ordered pair $(0, 0)$ (where the first 0 is the zero of $V$ and the second the zero of $W$). The negative of $(v, w)$ is $(-v, -w)$.

For all $\lambda, \mu \in F$ and all $v \in V$ and $w \in W$ we have

$$(\lambda + \mu)(v, w) = ((\lambda + \mu)v, (\lambda + \mu)w) \quad \text{(definition of scalar multiplication)}$$

$$= (\lambda v + \mu v, \lambda w + \mu w) \quad \text{(vector space axioms in } V, W)$$

$$= (\lambda v, \mu w) + (\mu v, \lambda w) \quad \text{(definition of addition)}$$

$$= \lambda(v, w) + \mu(v, w) \quad \text{(definition of scalar multiplication)}$$

proving Axiom (vii) of Definition 2.3. The other axioms can be done similarly, in each case making use of the fact that the axiom in question is satisfied in $V$ and in $W$ (since it is given that $V$ and $W$ are vector spaces).

(ii) Define $\theta: V \rightarrow V + W$ by $\theta(v) = (v, 0)$ for all $v \in V$. Then for all $u, v \in V$ and $\lambda, \mu \in F$ we have

$$\theta(\lambda u + \mu v) = (\lambda u + \mu v, 0) = \lambda(u, 0) + \mu(v, 0) = \lambda \theta(u) + \mu \theta(v).$$

Hence $\theta$ is a linear transformation. The kernel of $\theta$ consists of all $v \in V$ such that $(v, 0)$ is the zero element of $V + W$. Hence ker $\theta = \{0\}$, and it follows that $\theta$ is injective. The image of $\theta$ is the subset of $V + W$ consisting of all elements of the form $\theta(v)$ for $v \in V$; thus im $\theta = V'$. By 3.14 we deduce that $V'$ is a subspace of $V + W$.

Define $\theta': V' \rightarrow V'$ by $\theta'(v) = \theta(v)$ for all $v$. That is, $\theta'$ is just $\theta$ with its codomain cut down to coincide with its image. This makes $\theta'$ surjective, and it is also injective (since $\theta$ is). Hence $\theta'$ is an isomorphism, and $V' \cong V$. 
Virtually identical arguments using the map \( w \mapsto (0, w) \) show that \( W' \) is a subspace and isomorphic to \( W \). Since an arbitrary element of \( V + W \) has the form \( (v, w) = (v, 0) + (0, w) \in V' + W' \) we see that \( V + W = V' + W' \), and since \( (v, 0) = (0, w) \) implies \( v = w = 0 \) we see that \( V' \cap W' = \{0\} \). Hence \( V + W = V' \oplus W' \).

(iii) Since \( V' \cong V \) and \( W' \cong W \) we deduce that \( \dim V' = \dim V \) and \( \dim W' = \dim W \) (by Exercise 1). But since \( V + W = V' \oplus W' \) Theorem 6.9 gives \( \dim(V + W) = \dim V' + \dim W' \), whence the result.

4. Let \( S \) and \( T \) be subspaces of a vector space \( V \) and let \( U \) be a subspace of \( T \) such that \( T = (S \cap T) \oplus U \). Prove that \( S + T = S \oplus U \) (see Tutorial 3 for the definition of \( S + T \)), and hence deduce that

\[
\dim(S + T) = \dim S + \dim T - \dim(S \cap T).
\]

Solution.

From an earlier tutorial we know that \( S + T \) is a subspace of \( V \). If \( s \in S \) then \( s = s + 0 \in S + T \); so \( S \subseteq S + T \). Similarly \( T \subseteq S + T \), and since \( U \subseteq T \) we have \( U \subseteq S + T \). So \( S \) and \( U \) are subspaces of \( S + T \), and we must show that \( S + U = S + T \) and \( S \cap U = \{0\} \).

Let \( x \in S + T \). Then \( x = s + t \) for some \( s \in S \), \( t \in T \). Since \( T = (S \cap T) \oplus U \) there exist \( r \in S \cap T \), \( u \in U \) with \( t = r + u \). Since \( r \in S \cap T \subseteq S \) and \( s \in S \) we have \( s + r \in S \), and therefore

\[
x = s + (r + u) = (s + r) + u \in S + U.
\]

Since \( x \) was arbitrary we have shown that all elements of \( S + T \) lie in the subspace \( S + U \) of \( S + T \); thus \( S + U = S + T \).

Let \( a \in S \cap U \). Then \( a \in S \) and \( a \in U \subseteq T \); so \( a \in S \cap T \). But \( a \in U \); so \( a \in (S \cap T) \cap U \). Because the sum of \( S \cap T \) and \( U \) is direct we have that \( (S \cap T) \cap U = \{0\} \), and therefore \( a = 0 \). But \( a \) was an arbitrary element of \( S \cap U \), and so we have shown that \( S \cap U = \{0\} \), as required.

Alternatively, making use of some easily proved facts about adding subspaces, we have

\[
S + T = S + ((S \cap T) + U) = (S + (S \cap T)) + U = S + U
\]

(where \( S + (S \cap T) = S \) holds since \( S \cap T \subseteq S \)) and

\[
S \cap U = S \cap (T \cap U) = (S \cap T) \cap U = \{0\}
\]

(where \( U = T \cap U \) holds since \( U \subseteq T \)).

Since \( T = (S \cap T) \oplus U \) we have

\[
(1) \quad \dim T = \dim(S \cap T) + \dim U.
\]

Since \( S + T = S \oplus U \) we have

\[
(2) \quad \dim(S + T) = \dim S + \dim U.
\]

Eliminating \( \dim U \) from equations (1) and (2) gives

\[
\dim(S + T) = \dim S + \dim T - \dim(S \cap T).
\]

5. (i) Let \( S \) and \( T \) be subspaces of a vector space \( V \). Prove that \( (s, t) \mapsto s + t \) defines a linear transformation from \( S + T \) to \( V \) which has image \( S + T \) and kernel isomorphic to \( S \cap T \).

(ii) The Main Theorem on Linear Transformations (see p. 158 of the book) asserts that if \( V \) is a finitely generated vector space and \( \theta \) a linear transformation from \( V \) to another space \( W \), then the sum of the dimensions of \( \ker \theta \) and \( \im \theta \) equals the dimension of \( V \). Use this and Part (i) to give another proof that \( \dim(S + T) + \dim(S \cap T) = \dim S + \dim T \).

Solution.

Since every element of \( S + T \) is uniquely expressible in the form \( (s, t) \) with \( s \in S \) and \( t \in T \), and since \( S \) and \( T \) are subspaces of the vector space \( V \), the formula \( \theta(s, t) = s + t \) defines a function from \( S + T \) to \( V \). Now if \( (s, t), (s', t') \in S + T \) and \( \lambda \) is a scalar then

\[
\theta((s, t) + (s', t')) = \theta(s + s', t + t') = (s + s') + (t + t') = (s + t) + (s' + t') = \theta(s, t) + \theta(s', t')
\]

(by definition of \( \theta \), definition of addition in \( S + T \) and properties of addition in the vector space \( V \)), and

\[
\theta(\lambda(s, t)) = \theta(\lambda s, \lambda t) = \lambda s + \lambda t = \lambda(s + t) = \lambda\theta(s, t)
\]

similarly. Hence \( \theta \) is linear.

The image of \( \theta \) is the set of all elements of \( V \) of the form \( \theta(s, t) = s + t \) with \( s \in S \) and \( t \in T \); that is, \( \im \theta = S + T \). The kernel of \( \theta \) consists of all \( (s, t) \) such that \( s \in S \), \( t \in T \) and \( s + t = 0 \). For these conditions to be satisfied we must have \( s = -t \in T \), and hence \( s \in S \cap T \). Conversely, if \( x \in S \cap T \) then \( (x, -x) \) is in the kernel. So the kernel \( \ker \theta = \{(x, -x) \mid x \in S \cap T \} \). Hence the mapping \( \phi: S \cap T \to \ker \theta \) defined by \( \phi(x) = (x, -x) \) is surjective. It is also injective, since \( (x, -x) = (y, -y) \) implies \( x = y \). Finally, \( \phi \) is linear since

\[
\phi(\lambda x + \mu y) = (\lambda x + \mu y, -(\lambda x + \mu y)) = (\lambda x, -\lambda x) + (\mu y, -\mu y)
\]

\[
= \lambda(x, -x) + \mu(y, -y) = \lambda\phi(x) + \mu\phi(y)
\]

for all \( x, y \in S \cap T \) and all scalars \( \lambda \) and \( \mu \). Hence \( \ker \theta \cong S \cap T \).

By the Main Theorem, \( \dim \ker \theta + \dim \im \theta = \dim(S + T) \). Since \( \ker \theta \cong S \cap T \) we know (by Exercise 1) that \( \dim \ker \theta = \dim(S \cap T) \), and by Exercise 3 we know that \( \dim(S + T) = \dim S + \dim T \). Combining all this with \( \im \theta = S + T \) gives \( \dim(S + T) + \dim(S \cap T) = \dim S + \dim T \), as required.