1. Let $A$ be a $4 \times 4$ matrix, and suppose that $v_1$, $v_2$, $v_3$ and $v_4$ are column vectors satisfying $Av_1 = 2v_1$, $Av_2 = 2v_2 + v_1$, $Av_3 = 3v_3$ and $Av_4 = 3v_4 + v_3$. Let $T$ be the matrix whose columns are $v_1$, $v_2$, $v_3$ and $v_4$ (in that order). Prove that $AT = T\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

2. For each of the following matrices $A$ find a nonsingular matrix $T$ such that $T^{-1}AT$ is diagonal.
   
   (a) $A = \begin{pmatrix} 9 & -2 & 7 \\ 4 & -1 & 4 \\ -4 & 2 & -2 \end{pmatrix}$
   
   (b) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
   
   Check that it is possible in part (b) to choose $T$ in such a way that the sum of the squares of the entries in each column of $T$ is 1, and that if this is done then $T^{-1} = ^tT$.

3. Prove that if $A$ and $B$ are matrices such that $AB$ is defined then $^tB^tA$ is defined, and $^tB^tA = (^tA)^t(AB)$.

4. Let $A$ be a matrix satisfying $^tA = A$ and let $u$ and $v$ be eigenvectors of $A$ with corresponding eigenvalues $\lambda$ and $\mu$. (That is, $u$ and $v$ are nonzero and $Au = \lambda u$ and $Av = \mu v$.) Prove that if $\lambda \neq \mu$ then $(^tA)v = 0$. (Hint: Show that $(^tA)u = \lambda (^tA)u$, and then expand $(^tA)Av$ in two ways.)

   Investigate the connection between this exercise and 2 (b).

5. Show that if $\alpha$ and $\beta$ are arbitrary complex numbers then $(\alpha + \beta) = \overline{\alpha + \overline{\beta}}$ and $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$, where the overline denotes complex conjugation (defined by $(x + iy) = x - iy$ for all $x, y \in \mathbb{R}$, where $i = \sqrt{-1}$).

   If $A$ is a complex matrix let $\overline{A}$ be the matrix whose entries are the complex conjugates of the entries of $A$. Use the previous part to show that $\overline{AB} = \overline{A}\overline{B}$ for all complex matrices $A$ and $B$ such that $AB$ exists.