Tutorial 3

1. Which of the following functions are linear transformations?
   (i) \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)
   (ii) \( S: \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)
   (iii) \( g: \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y \\ x - y \end{pmatrix} \)
   (iv) \( f: \mathbb{R} \to \mathbb{R}^2 \) defined by \( f(x) = \begin{pmatrix} x \\ x + 1 \end{pmatrix} \)

2. Let \( \mathcal{A} \) be the set of all 2-component column vectors whose entries are differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \). Thus, for example, if \( h \) and \( k \) are the functions defined by \( h(t) = \cos t \) and \( k(t) = t^2 + 1 \) for all \( x \in \mathbb{R} \) then \( \begin{pmatrix} h \\ k \end{pmatrix} \) is an element of \( \mathcal{A} \).
   (i) How should addition and scalar multiplication be defined so that \( \mathcal{A} \) becomes a vector space over \( \mathbb{R} \)?
   (ii) If \( f \) and \( g \) are real-valued functions on \( \mathbb{R} \) then their pointwise product is the function \( f \cdot g \) defined by \( (f \cdot g)(t) = f(t)g(t) \) for all \( t \in \mathbb{R} \). Prove that \( \begin{pmatrix} f \\ g \end{pmatrix} \mapsto h \cdot f + g' \) (where \( h \) is as above and \( g' \) is the derivative of \( g \)) defines a linear transformation from \( \mathcal{A} \) to the space of all real-valued functions on \( \mathbb{R} \).

3. Let \( V \) be a vector space and let \( S \) and \( T \) be subspaces of \( V \).
   (i) Prove that \( S \cap T \) is a subspace of \( V \).
   (ii) Let \( S + T = \{ x + y \mid x \in S \text{ and } y \in T \} \). Prove that \( S + T \) is a subspace of \( V \).

4. Let \( V \) be a vector space over the field \( F \) and let \( v_1, v_2, \ldots, v_n \) be arbitrary elements of \( V \). Prove that the span of \( \{ v_1, v_2, \ldots, v_n \} \)
   \( \text{Span}(v_1, v_2, \ldots, v_n) = \{ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in F \} \)
   is a subspace of \( V \).

5. Let \( A \) and \( B \) be \( n \times n \) matrices over the field \( F \). We say that \( B \) is similar to \( A \) if there exists a nonsingular matrix \( T \) such that \( B = T^{-1} AT \). Prove
   (i) every \( n \times n \) matrix is similar to itself,
   (ii) if \( B \) is similar to \( A \) then \( A \) is similar to \( B \),
   (iii) if \( C \) is similar to \( B \) and \( B \) is similar to \( A \) then \( C \) is similar to \( A \).