Tutorial 12

1. Let $A$ be an $n \times n$ matrix whose rank is less than $n$. Prove that 0 is an eigenvalue of $A$.

2. Let $V$ be a vector space and $S$ and $T$ subspaces of $V$ such that $V = S \oplus T$. Prove or disprove the following assertion:
   
   If $U$ is any subspace of $V$ then $U = (U \cap S) \oplus (U \cap T)$.

3. (i) Let $A$, $B$ and $C$ be $n \times n$ matrices, and suppose that the column space of $B$ equals the column space of $C$. Prove that the column space of $AB$ equals that of $AC$. (Hint: Use Proposition 7.16 of the text.)

   (ii) Let $A$ be an $n \times n$ matrix and suppose that the rank of $A^4$ is the same as the rank of $A^3$. Prove that $A^5$ and all higher powers of $A$ also have this same rank. (Hint: Apply Part (i) with $B = A^3$ and $C = A^4$.)

4. Let $V$ and $W$ be vector spaces over the field $F$ and let $b = (v_1, v_2, \ldots, v_n)$ and $c = (w_1, w_2, \ldots, w_m)$ be bases of $V$ and $W$ respectively. Let $L(V, W)$ be the set of all linear transformations from $V$ to $W$, and let $\text{Mat}(m \times n, F)$ be the set of all $m \times n$ matrices over $F$. We know that $\text{Mat}(m \times n, F)$ is a vector space over $F$, and we have seen in Question 3 of Tutorial 5 that $L(V, W)$ is too. Let $\Omega: L(V, W) \rightarrow \text{Mat}(m \times n, F)$ be the function defined by $\Omega(\theta) = \text{M}_{cb}(\theta)$ for all $\theta \in L(V, W)$.

   (i) Prove that $\Omega$ is a linear transformation. (Hint: The task is to prove that $\text{M}_{cb}(\phi + \theta) = \text{M}_{cb}(\phi) + \text{M}_{cb}(\theta)$ and $\text{M}_{cb}(\lambda \phi) = \lambda \text{M}_{cb}(\phi)$. Now the $j^{th}$ column of $\text{M}_{cb}(\phi + \theta)$ is $cv_c((\phi + \theta)(v_j))$ while the $j^{th}$ columns of $\text{M}_{cb}(\phi)$ and $\text{M}_{cb}(\theta)$ are $cv_c(\phi(v_j))$ and $cv_c(\theta(v_j))$. Use the definition of $\phi + \theta$ and fact that $x \mapsto cv_c(x)$ is linear to prove that the $j^{th}$ column of $\text{M}_{cb}(\phi + \theta)$ is the sum of the $j^{th}$ columns of $\text{M}_{cb}(\phi)$ and $\text{M}_{cb}(\theta)$.)

   (ii) Prove that the kernel of $\Omega$ is $\{z\}$, where $z: V \rightarrow W$ is the zero function.

   (iii) Prove that $\Omega$ is a vector space isomorphism. (Hint: By the first two parts we know that $\Omega$ is linear and injective; so surjectivity is all that remains. That is, given a $m \times n$ matrix $M$ we must show that there is a linear transformation $\theta$ from $V$ to $W$ having $M$ as its matrix. Now the coefficients of $M$ determine what $\theta(v_i)$ has to be for each $i$, and Theorem 4.18 guarantees that such a linear transformation exists.)

   (iv) Find a basis for $L(V, W)$. (Hint: (Find a basis of $\text{Mat}(m \times n, F)$ first. The corresponding linear transformations will give the desired basis of $L(V, W)$.)