1. Determine which of the following subsets \( A \) are compact subsets in the appropriate \( \mathbb{R}^n \).
   
   (i) \( A = [0, 2) \)
   
   (ii) \( A = \mathbb{Q} \cap [0, 1] \)
   
   (iii) \( A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \} \)
   
   (iv) \( A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4 \} \cup \{(1, 2)\} \).

Solution.

(i) Compact sets are closed (in any Hausdorff space). Since \( [0, 2) \) is not closed it is not compact. (The open covering \( [0, 2) \subseteq \bigcup_{n=1}^{\infty} (-1, 2 - \frac{1}{n}) \) has no finite subcovering.)

(ii) This set is not closed, and so it is not compact. (And, for example \( A \subseteq \bigcup_{n=1}^{\infty} \left( (-1, \frac{(n-1)\sqrt{2}}{2n}) \cup \left( \frac{(n+1)\sqrt{2}}{2n}, 2 \right) \right) \) is an open covering for which there is no finite subcovering.)

(iii) To be compact a set must be bounded; this set is not.

(iv) This set is closed and bounded; so it is compact (by Heine-Borel).

2. Let \( A \) and \( B \) be compact subsets of a topological space \( X \). Show that \( A \cap B \) and \( A \cup B \) are also compact.

Solution.

The question is wrong as stated: to prove that \( A \cap B \) is compact an extra assumption is needed. However, \( A \cup B \) is necessarily compact. For suppose that \( (V_i)_{i \in I} \) is a family of open sets with \( A \cup B \subseteq \bigcup_{i \in I} V_i \). Since \( A \subseteq A \cup B \) it follows that \( A \subseteq \bigcup_{i \in I} V_i \), and since \( A \) is compact there is a finite subset \( J \) of \( I \) with \( A \subseteq \bigcup_{j \in J} V_j \). Similarly there is a finite subset \( K \) of \( I \) with \( B \subseteq \bigcup_{k \in K} V_k \). So \( \bigcup_{j \in J} V_j \cup \bigcup_{k \in K} V_k = \bigcup_{j \in J \cup K} V_j \). Since \( J \) and \( K \) are finite, so is \( J \cup K \), and we have produced a finite subcovering from an arbitrary open covering of \( A \cup B \). So \( A \cup B \) is compact.

If the space \( X \) is assumed to be Hausdorff then the set \( A \) is closed, since compact implies closed in Hausdorff spaces (as we proved in lectures). It then follows from Question 4 below that \( A \cap B \) is compact. (Note that the result in Question 4 is valid for all topological spaces, not just Hausdorff ones.)

3. Let \( (A_i)_{i \in I} \) be any family of compact subsets of a metric space \( (X, d) \). Prove that \( B = \bigcap_{i \in I} A_i \) is compact, while \( \bigcup_i A_i \) is not necessarily compact.

Solution.

An infinite union obviously need not be bounded. For example, in \( \mathbb{R} \) let \( A_i = [-i, i] \), for each positive integer \( i \). Each \( A_i \) is compact, by Heine-Borel, but \( \bigcup_{i=1}^\infty A_i = \mathbb{R} \) is not bounded, so not compact.

For the other part it suffices to assume that \( X \) is a Hausdorff space. (This is weaker than assuming that \( X \) is a metric space: all metric spaces are Hausdorff, but there are Hausdorff spaces that are not metrizable.) Again, we can use Question 4: choose a fixed \( i_0 \in I \), put \( A = \bigcap_{k \neq i_0} A_i \) and put \( B = A_{i_0} \). Since \( X \) is Hausdorff the sets \( A_i \) are all closed, and so \( A \) is closed (being an intersection of closed sets). Since \( B \) is compact it follows that \( A \cap B \) is compact. But \( A \cap B = \bigcap_{i \in I} A_i \).

4. Let \( A \) and \( B \) be subsets of a topological space \( X \) such that \( A \) is closed and \( B \) is compact. Show that \( A \cap B \) is compact.

Solution.

Let \( (V_i)_{i \in I} \) be a family of open sets such that \( A \cap B \subseteq \bigcup_{i \in I} V_i \). Let \( J \) be a set obtained by adding one more element to \( I \): say \( J = I \cup \{ j \} \). Define \( V_j = X \setminus A \), and observe that \( V_j \) is open since \( A \) is closed. Now for all \( b \in B \), we have either that \( b \in V_j \) (if \( b \notin A \)), or else \( b \in A \cap B \subseteq \bigcup_{i \in I} V_i \), giving \( b \in V_i \) for some \( i \in I \). In either case \( b \in V_i \) for some \( i \in J \). So the family of sets \( (V_i)_{i \in J} \) form an open covering of \( B \), and since \( B \) is compact there exists a finite subset \( L \) of \( J \) with \( B \subseteq \bigcup_{i \in L} V_i \). Now the set \( L \setminus \{ j \} \) is a finite subset of \( J \setminus \{ j \} \). So we can show that \( A \cap B \subseteq \bigcup_{i \in L \setminus \{ j \}} V_i \). For suppose that \( b \in A \cap B \). Then \( b \in B \subseteq \bigcup_{i \in L} V_i \), and so \( b \in V_i \) for some \( i \in L \). But \( b \in A \), and so \( b \notin X \setminus A = V_j \). So \( b \in V_i \) for some \( i \in L \setminus \{ j \} \), as required. Thus the arbitrarily chosen open covering \( (V_i)_{i \in I} \) of the set \( A \cap B \) has a finite subcovering, namely \( (V_i)_{i \in L \setminus \{ j \}} \). Hence \( A \cap B \) is compact.

5. Let \( X \) be a non-empty set with \( d \) the standard discrete metric, and \( A \) any subset of \( X \). Show that \( A \) is compact if and only if \( A \) is finite.

Solution.

Recall that this metric satisfies \( d(x, y) = 1 \) whenever \( x \neq y \). It follows that for every \( x \in X \) the open ball \( B(x, \frac{1}{2}) \) is just the singleton set \( \{ x \} \). Now every subset \( A \) of \( X \) can be expressed as a union of open balls; specifically, \( A = \bigcup_{a \in A} \{ x \} \). So all subsets of \( X \) are open. This condition implies that the compact sets are precisely the finite sets.

Firstly, suppose that \( A \) is a finite set, and suppose that \( A \subseteq \bigcup_{i \in I} V_i \), where the \( V_i \) are any subsets of \( X \). For each \( a \in A \) we have \( a \in \bigcup_{i \in I} V_i \), and so we may choose an element \( i_a \in I \) such that \( a \in V_{i_a} \). Then \( A \subseteq \bigcup_{a \in A} V_{i_a} \), and since \( A \) is a finite set there are only finitely many terms in this union. So
we have shown that every covering of $A$ has a finite subcovering, and so $A$ is compact.

Conversely, suppose that $A$ is a compact subset of $X$. Then the singleton sets $(\{x\})_{x \in X}$ form an open covering of $A$, since $A \subseteq X = \bigcup_{x \in X} \{x\}$, and since the singleton sets are open. By the compactness of $A$ there is a finite subcovering; that is, there is a finite subset $B$ of $X$ such that $A \subseteq \bigcup_{x \in B} \{x\}$ = $B$. Since $A$ is a subset of the finite set $B$, it too is finite.

6. Let $X$ be a compact metric space (or topological space), and $A$ any infinite subset of $X$. Show that $A$ has an accumulation point in $X$. (That is, show that $A' \neq \emptyset$).

Solution.
Let $A$ be a subset of $X$ with no accumulation points in $X$. We shall show that $A$ is finite.

Let $x \in X$ be arbitrary. Since $x$ is not an accumulation point of $A$ there is an open neighbourhood $U_x$ of $x$ such that $U_x \cap A \setminus \{x\} = \emptyset$. Equivalently, $U_x \cap A \subseteq \{x\}$. Since $x \in U_x$ it follows that $X \subseteq \bigcup_{x \in X} U_x$. That is, the family $C = (U_x)_{x \in X}$ is an open covering of $X$. Since $X$ is compact, $C$ has a finite subcovering: there exists a finite subset $\{x_1, x_2, \ldots, x_m\}$ of $X$ with $X \subseteq \bigcup_{i=1}^m U_{x_i}$. (Indeed, this union equals $X$ since $X$ is the whole space.) It follows that $A = X \cap A = \bigcup_{i=1}^m U_{x_i} \cap A \subseteq \bigcup_{i=1}^m \{x_i\}$. So $A$ has at most $m$ points, and is therefore finite.

Consequently, any infinite subset $A$ of $X$ must have an accumulation point in $X$.

7. Let $X$ be a topological space and $A$ a subspace of $X$. Prove that a set $B \subseteq A$ is compact in $X$ if and only if $B$ is compact in $A$ (with respect to the subspace topology on $A$).

Solution.
Suppose that $A$ is compact in $X$. Let $C$ be a covering of $B$ by the subsets of $A$ which are open in the subspace topology on $A$. Then $C = (V_i \cap A)_{i \in I}$, where each $V_i$ is an open subset of $X$ (and $I$ is an indexing set). Thus $B \subseteq \bigcup_{i \in I} (V_i \cap A) \subseteq \bigcup_{i \in I} V_i$, so that $D = (V_i)_{i \in I}$ is a covering of $B$ by the open sets in $X$. Since $A$ is compact in $X$ there is a finite subfamily $J$ of $I$ such that $B \subseteq \bigcup_{i \in J} V_i$, and since $B \subseteq A$ it follows that $B = B \cap A \subseteq \bigcup_{i \in J} (V_i \cap A)$, showing that $(V_i \cap A)_{i \in J}$ is a finite subcovering of $C$. Since $C$ was arbitrary, this shows that $B$ is compact as a subset of $A$.

Conversely, suppose that $B$ is compact in the subspace topology, and let $(V_i)_{i \in I}$ be a covering of $B$ by open sets of $X$. Since $B \subseteq A$ we see that $(V_i \cap A)_{i \in I}$ is a covering of $B$ by subsets of $A$, and furthermore these subsets are open in the subspace topology. So there is a finite subfamily $J$ of $I$ such that $(V_i \cap A)_{i \in J}$ is a covering of $B$, and it follows that $(V_i)_{i \in J}$ is a finite subcovering of the original open covering of $B$. Thus $B$ is compact in $X$.

8. Let $X$ be a metric space (or topological space). Prove that $X$ is compact if and only if for every family $(F_i)_{i \in I}$ of closed subsets of $X$, if $\bigcap_{i \in I} F_i = \emptyset$ then there is a finite subset $\{i_1, i_2, \ldots, i_m\}$ of $I$ such that $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m} = \emptyset$.

Solution.
Suppose that $X$ is compact and $(F_i)_{i \in I}$ is a family of closed subsets of $X$ with $\bigcap_{i \in I} F_i = \emptyset$. By De Morgan’s Law, $X = X \setminus \emptyset = X \setminus \bigcap_{i \in I} F_i = \bigcup_{i \in I} (X \setminus F_i)$, showing that $(X \setminus F_i)_{i \in I}$ is an open covering of $X$, so each $F_i$ is closed. Since $X$ is compact, there is a finite subfamily $J$ of $I$ such that $(X \setminus F_i)_{i \in J}$ is a covering of $X$. That is, $X = \bigcup_{i \in J} (X \setminus F_i)$. Thus, by De Morgan’s Law,

$$0 = X \setminus X = X \setminus \left( \bigcup_{i \in J} (X \setminus F_i) \right) = \bigcap_{i \in J} (X \setminus (X \setminus F_i)) = \bigcap_{i \in J} F_i,$$

showing, as desired, that there is a finite subset of $I$ such that the intersection of the corresponding sets $F_i$ is empty.

Conversely, suppose the condition holds. Let $(V_i)_{i \in I}$ be an open covering of $X$; that is, $X = \bigcup_{i \in I} V_i$. By De Morgan, $\emptyset = X \setminus \bigcup_{i \in I} V_i = \bigcap_{i \in I} (X \setminus V_i)$. Let $F_i = X \setminus V_i$. Since each $V_i$ is open, each $F_i$ is closed. So $(F_i)_{i \in I}$ is a family of closed sets with empty intersection. By the hypothesis there is a finite subfamily $J$ of $I$ such that the subfamily $(F_i)_{i \in J}$ also has empty intersection. By De Morgan,

$$X = X \setminus \emptyset = X \setminus \left( \bigcap_{i \in J} F_i \right) = X \setminus \left( \bigcap_{i \in J} (X \setminus V_i) \right) = \bigcup_{i \in J} \left( X \setminus (X \setminus V_i) \right) = \bigcup_{i \in J} V_i.$$

Hence an arbitrary open covering of $X$ has a finite subcovering, and therefore $X$ is compact.

9. Let $X$ be a metric space (or topological space). Prove that $X$ is compact if and only if for every family $(F_i)_{i \in I}$ of closed subsets of $X$ with the property that every finite subfamily $(F_{i_1}, F_{i_2}, \ldots, F_{i_m})$ has a non-empty intersection has itself, a non-empty intersection.

Solution.
This is just the contrapositive of the result proved in the previous question.